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STRONG CONVERGENCE OF ANCHORED GENERAL PICARD–MANN AND HALPERN ITERATIONS FOR NONEXPANSIVE MAPPINGS

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Abstract. We study an anchored general Picard–Mann method for nonexpansive mappings with fixed points on nonempty closed convex subsets of real Hilbert spaces. The method combines an anchor, a Mann average, and a fixed number of Picard steps. We prove that, when the anchor weights lie strictly between zero and one, tend to zero, and have a divergent sum, all iterates and intermediate sequences converge strongly to the projection of the anchor onto the fixed-point set, provided the Mann parameters stay uniformly away from zero and one. This result requires no compactness assumption or bounded-variation condition on the controls. If the anchor weights are summable, the iterates converge weakly; they converge strongly if the image set is relatively compact, if the iterates approach the fixed-point set along a subsequence, or if the mapping satisfies condition (I). We also study a reduced method that omits the Mann step. Its auxiliary sequence follows Halpern iteration for a positive integer power of the mapping. When the anchor weights tend to zero, have a divergent sum, and have bounded variation, both sequences converge strongly to the projection onto that power's fixed-point set, which may differ from the original fixed-point set. Equality of the two fixed-point sets is necessary and sufficient for convergence to the projection onto the original set for every anchor; this equality holds when there is one Picard step or when the mapping is averaged. Examples and a higher-dimensional numerical comparison illustrate fixed-point selection, periodic-point obstructions, and the distinction between iteration counts and operator-evaluation costs.

1. Introduction

Let C be a nonempty closed convex subset of a Banach or Hilbert space and let $T : C \to C$ be nonexpansive. The approximation of points in $\mathrm{Fix}(T)$ is a standard problem in nonlinear functional analysis and arises in optimization, feasibility problems, inverse problems, and evolution equations. The Krasnosel'skiĭ–Mann method, associated with the work of Krasnosel'skiĭ [7] and Mann [9], is a standard fixed-point iteration. For nonexpansive mappings, its convergence is typically weak, even in Hilbert spaces. Halpern [4] incorporated a fixed anchor to obtain a strong-convergence mechanism, and Ishikawa [5] later introduced a two-step iteration. Wittmann [19] established a classical strong-convergence theorem for Halpern iterations under regularity conditions on the control sequence. Related fixed-point selection procedures include the viscosity method of Moudafi [10].

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Yao et al. [22] introduced a modified Krasnosel'skiĭ–Mann iteration by inserting a vanishing nonsummable anchor before a Mann step. Shukla et al. [16] subsequently introduced the general Picard–Mann (GPM) method, consisting of a Mann average followed by a fixed number of Picard evaluations. The former construction provides a strong-convergence mechanism, whereas the latter supplies a unified multi-step structure.

Motivated by these two approaches, we consider the anchored general Picard–Mann iteration

$$y_n = \alpha_n u + (1-\alpha_n)x_n, \qquad z_n = (1-\beta_n)y_n + \beta_n T y_n, \qquad x_{n+1} = T^k z_n.$$

The anchor u selects a distinguished fixed point, while the Mann stage provides a residual descent term that remains valid after the k Picard evaluations.

The analysis has three principal components. First, the zero anchor in the modified Krasnosel'skiĭ–Mann method is replaced by an arbitrary point $u \in C$, while the Mann step is followed by the k Picard evaluations of GPM. Under a vanishing nonsummable anchor and uniformly interior Mann parameters, the resulting sequence converges strongly to the explicitly identified point $P_{\mathrm{Fix}(T)}u$, without a compactness assumption. Thus the anchor supplies fixed-point selection and strong convergence, whereas the Picard steps preserve the residual descent generated by the Mann average. Second, for summable anchoring, the method is treated as a summably perturbed GPM iteration; a quasi-Fejér argument yields weak convergence and gives additional sufficient conditions for strong convergence. Third, the reduced rule $x_{n+1} = T^k y_n$ is distinguished from the three-stage method. Its auxiliary sequence (y_n) is a Halpern iteration for T^k, and both sequences converge to $P_{\mathrm{Fix}(T^k)}u$ under a bounded-variation assumption. The equality $\mathrm{Fix}(T^k) = \mathrm{Fix}(T)$ guarantees that the limit is $P_{\mathrm{Fix}(T)}u$ for every anchor; for a specified anchor, it suffices that $P_{\mathrm{Fix}(T^k)}u \in \mathrm{Fix}(T)$. These conclusions are supplemented by exact reductions to several established iterations and by examples illustrating the limiting behaviour.

The paper is organized as follows. Section 2 introduces the notation and background results. Section 3 recalls the two motivating iterative methods. Section 4 defines the anchored GPM method. Sections 5–7 contain the principal convergence theorems. Section 8 records special cases, Section 9 presents illustrative examples, and the final section contains concluding remarks.

2. Preliminaries

Let H be a real Hilbert space, let $C \subset H$ be nonempty, closed and convex, and let $T : C \to C$ be nonexpansive:

$$\|Tx - Ty\| \le \|x - y\| \qquad (x, y \in C).$$

Write

$$F := \mathrm{Fix}(T) = \{x \in C : Tx = x\},$$

and assume $F \neq \varnothing$. The set F is closed and convex, hence the metric projection $P_F u$ is well defined for every $u \in H$. Its characteristic variational inequality is

$$q = P_F u \iff q \in F \quad \text{and} \quad \langle u - q, p - q\rangle \le 0 \quad (p \in F). \tag{2.1}$$

Weak and strong convergence are denoted by $\rightharpoonup$ and $\to$, respectively. We take $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and interpret $T^0 = I$. Standard background on nonexpansive mappings and metric fixed-point theory may be found in Goebel and Kirk [3].

2.1. **Preliminary lemmas.**

Lemma 2.1 (Demiclosedness; cf. [1]). *If $v_j \rightharpoonup v$ in C and $v_j - Tv_j \to 0$, then $v \in F$.*

Proof. Pass to a subsequence along which $\|v_j - v\|$ has a limit. Since $Tv_j - v_j \to 0$ and T is nonexpansive,

$$\|v_j - Tv\| \le \|v_j - Tv_j\| + \|Tv_j - Tv\| \le \|v_j - Tv_j\| + \|v_j - v\|.$$

Also,

$$\|v_j - Tv\|^2 = \|v_j - v\|^2 + \|v - Tv\|^2 + 2\langle v_j - v, v - Tv\rangle.$$

The last inner product tends to zero. Comparing the last two displays gives $\|v - Tv\|^2 \le 0$, so $Tv = v$. □

Lemma 2.2 (Scalar convergence; cf. [21]). *Let $a_n \ge 0$, $\lambda_n \in [0,1]$, $\sum_n \lambda_n = \infty$, and*

$$a_{n+1} \le (1 - \lambda_n)a_n + \lambda_n\delta_n.$$

If $\limsup_n \delta_n \le 0$, then $a_n \to 0$.

Lemma 2.3 ([8]). *Let $a_n \ge 0$ and suppose it is not eventually nonincreasing. For all sufficiently large n, set*

$$\tau(n) = \max\{j \le n : a_j \le a_{j+1}\}.$$

Then $\tau(n) \to \infty$, $a_{\tau(n)} \le a_{\tau(n)+1}$, and $a_n \le a_{\tau(n)+1}$ for all sufficiently large n.

3. Background iterative methods

3.1. **The modified Krasnosel'skiĭ–Mann method.**

The classical Krasnosel'skiĭ–Mann (KM) iteration has the form

$$x_{n+1} = \gamma_n x_n + (1 - \gamma_n)Tx_n, \qquad \gamma_n \in [0,1]. \tag{3.1}$$

In a Hilbert space, this sequence converges weakly to a fixed point under the standard divergence condition $\sum_n \gamma_n(1-\gamma_n) = \infty$; strong convergence cannot be expected in general.

Yao et al. [22] replaced (3.1) by

$$y_n = (1 - \alpha_n)x_n, \qquad x_{n+1} = (1 - \beta_n)y_n + \beta_n Ty_n. \tag{3.2}$$

Their assumptions are that H is a real Hilbert space, $T : H \to H$ is nonexpansive, $F \ne \varnothing$, $\alpha_n, \beta_n \in [0,1]$,

$$\alpha_n \to 0, \qquad \sum_{n=0}^{\infty} \alpha_n = \infty, \qquad \beta_n \in [a,b] \subset (0,1). \tag{3.3}$$

Yao et al. prove strong convergence under these conditions. In the Hilbert-space setting considered here, the zero anchor identifies the selected limit as $P_F 0$; see Remark 3.1.

For $p \in F$, nonexpansiveness and the Hilbert-space norm identity yield

$$\|Tx - x\|^2 \le 2\langle x - p, x - Tx\rangle. \tag{3.4}$$

Consequently, the averaged step satisfies

$$\|(1-\beta)x + \beta Tx - p\|^2 \le \|x - p\|^2 - \beta(1-\beta)\|x - Tx\|^2. \tag{3.5}$$

The remaining ingredients are the demiclosedness principle, a scalar convergence lemma of the form $a_{n+1} \le (1-\lambda_n)a_n + \lambda_n\delta_n$, and a last-increase subsequence for the treatment of a nonmonotone distance sequence. Precise versions of these tools are stated in Section 2.1.

Remark 3.1 (Identification of the limit)**.** The anchoring step in (3.2) selects a distinguished element of the fixed-point set. Indeed, the appropriate limit is

$$q = P_F 0,$$

because (3.2) contains the anchor 0 through $y_n = \alpha_n 0 + (1-\alpha_n)x_n$. Demiclosedness alone only shows that weak cluster points belong to F; the projection inequality $\langle -q, z-q\rangle \le 0$ for $z \in F$ is what identifies the selected point. Corollary 5.2 gives the corresponding strong-convergence statement.

3.2. **The general Picard–Mann method.** For a Banach space X, a nonempty closed convex set $C \subset X$, a mapping $T : C \to C$, a fixed integer $k \in \mathbb{N}$, and $\theta_n \in [0,1]$, the GPM method of Shukla et al. [16] is

$$x_{n+1} = T^k\big((1-\theta_n)x_n + \theta_n T x_n\big). \tag{3.6}$$

Thus GPM performs one Mann averaging step and then k Picard steps. For $k = 1$ it is the normal S-iteration of Sahu [12], also known as the Picard–Mann hybrid of Khan [6]. Shukla et al. [16] note that $k = 2$ and $k = 3$ give iterations previously called M (or F^*) and F, respectively. The motivation is to retain the stabilising convex average while adding Picard evaluations and encompassing several multi-step schemes. Related modified Picard–Mann constructions were studied by Deng [2], while Shukla [14] considered GPM for quasi-nonexpansive mappings. Shukla and Panicker [15] established convergence results for GPM under nonexpansive-type assumptions. For nonexpansive mappings in a uniformly convex Banach space, it establishes weak convergence under

$$\sum_{n=1}^{\infty} \theta_n(1-\theta_n) = \infty \tag{3.7}$$

and either the Opial property or the Kadec–Klee property of the dual. The paper also gives strong convergence if $T(C)$ is relatively compact, if $\liminf_n \operatorname{dist}(x_n, F) = 0$, or if the Senter–Dotson condition (I) holds [13].

The basic Fejér estimate is

$$\|x_{n+1} - p\| \le \|(1-\theta_n)x_n + \theta_n T x_n - p\| \le \|x_n - p\|, \qquad p \in F. \tag{3.8}$$

Uniform convexity strengthens this to a residual-descent estimate: the decrease depends on $\theta_n(1-\theta_n)$ and on the modulus of convexity evaluated at the separation between x_n and Tx_n. Together with the control condition, this yields asymptotic regularity; demiclosedness then places weak cluster points in the fixed-point set. Opial's property [11], or a dual Kadec–Klee argument, gives uniqueness of the weak limit. The uniform-convexity inequality underlying this estimate is discussed by Xu [20].

Several related multi-step iterations have also been investigated. Sahu [12] studied the S-iteration process and its applications to constrained minimization and split feasibility problems. Thakur et al. [17] and Ullah and Arshad [18] analysed further multi-step methods for generalized nonexpansive mappings.

We record two structural distinctions. First, (3.6) is Fejér monotone but generally only weakly convergent for nonexpansive mappings. Second, the strong-convergence mechanism in (3.2) is supplied by the vanishing nonsummable anchor, rather than merely by the presence of an additional parameter.

4. The anchored general Picard–Mann iteration

We now combine the anchoring mechanism in (3.2) with the GPM structure in (3.6).

Algorithm 4.1 (Anchored GPM method)**.** Choose $x_0, u \in C$, a fixed integer $k \in \mathbb{N}_0$, and controls $\alpha_n, \beta_n \in [0,1]$. Define, for $n \geq 0$,

$$\begin{aligned} y_n &= \alpha_n u + (1-\alpha_n)x_n, \\ z_n &= (1-\beta_n)y_n + \beta_n T y_n, \\ x_{n+1} &= T^k z_n. \end{aligned} \tag{4.1}$$

We call (4.1) the anchored general Picard–Mann method, denoted MGPM(u,k).

The first line introduces a general anchor u. The second and third lines form a GPM step evaluated at y_n rather than at x_n. For $k \geq 1$, the mapping T is evaluated once in the Mann stage and then k additional times after averaging.

The central observation is that the additional Picard steps preserve the descent term generated by the Mann average. For every $p \in F$,

$$\begin{aligned} \|x_{n+1}-p\|^2 &= \|T^k z_n - T^k p\|^2 \\ &\leq \|z_n - p\|^2 \\ &\leq \|y_n - p\|^2 - \beta_n(1-\beta_n)\|y_n - Ty_n\|^2. \end{aligned} \tag{4.2}$$

Estimate (4.2) is the master inequality for the method.

5. Strong convergence of the anchored GPM method

Theorem 5.1 (Strong convergence to the selected fixed point)**.** *Let H, C, T, F be as in Section 2. Fix $u, x_0 \in C$ and $k \in \mathbb{N}_0$. Suppose*

$$0 < \alpha_n < 1, \qquad \alpha_n \to 0, \qquad \sum_{n=0}^{\infty} \alpha_n = \infty, \qquad 0 < \beta_{\min} \leq \beta_n \leq \beta_{\max} < 1. \tag{5.1}$$

Then the sequence generated by (4.1) converges strongly to $q = P_F u$. Moreover $y_n \to q$, $z_n \to q$, and

$$\|y_n - Ty_n\| \to 0, \qquad \|x_n - Tx_n\| \to 0. \tag{5.2}$$

Proof. Put $q = P_F u$ and $\mu = \beta_{\min}(1-\beta_{\max}) > 0$. From (4.2),

$$\|x_{n+1} - q\|^2 \leq \|y_n - q\|^2 - \mu\|y_n - Ty_n\|^2. \tag{5.3}$$

Step 1: boundedness. Dropping the negative term and taking square roots gives

$$\begin{aligned} \|x_{n+1} - q\| &\leq \|y_n - q\| \\ &\leq (1-\alpha_n)\|x_n - q\| + \alpha_n\|u - q\| \\ &\leq \max\{\|x_n - q\|, \|u - q\|\}. \end{aligned}$$

Induction yields

$$\|x_n - q\| \le R := \max\{\|x_0 - q\|, \|u - q\|\}.$$

Thus (x_n) and (y_n) are bounded; nonexpansiveness and $Tq = q$ also bound (Ty_n), (z_n), and (x_{n+1}). In addition,

$$\|y_n - x_n\| = \alpha_n \|u - x_n\| \longrightarrow 0. \tag{5.4}$$

Set $d_n = \|x_n - q\|^2$. The identity

$$\begin{aligned} \|y_n - q\|^2 &= (1-\alpha_n)d_n + 2\alpha_n\langle u - q, y_n - q\rangle \\ &\quad - \alpha_n(1-\alpha_n)d_n - \alpha_n^2\|u - q\|^2 \end{aligned} \tag{5.5}$$

implies, after dropping the last two nonpositive terms,

$$d_{n+1} \le (1-\alpha_n)d_n + 2\alpha_n\langle u - q, y_n - q\rangle. \tag{5.6}$$

Step 2: the eventually nonincreasing case. Suppose (d_n) is eventually nonincreasing. Then d_n has a limit and $d_n - d_{n+1} \to 0$. Boundedness and $\alpha_n \to 0$ imply

$$\big|\|y_n - q\|^2 - d_n\big| \to 0. \tag{5.7}$$

Consequently (5.3) gives

$$0 \le \mu\|y_n - Ty_n\|^2 \le \|y_n - q\|^2 - d_{n+1} \longrightarrow 0.$$

Hence $\|y_n - Ty_n\| \to 0$. By (5.4) and nonexpansiveness,

$$\begin{aligned} \|x_n - Tx_n\| &\le \|x_n - y_n\| + \|y_n - Ty_n\| + \|Ty_n - Tx_n\| \\ &\le 2\|x_n - y_n\| + \|y_n - Ty_n\| \longrightarrow 0. \end{aligned} \tag{5.8}$$

We next prove

$$\limsup_{n\to\infty}\langle u - q, y_n - q\rangle \le 0. \tag{5.9}$$

Choose a subsequence along which the scalar product attains its limsup. Boundedness and weak sequential compactness of bounded Hilbert-space sets give a further subsequence, still indexed by n_j, with $x_{n_j} \rightharpoonup w \in C$. Equations (5.8) and Lemma 2.1 imply $w \in F$. Equation (5.4) gives $y_{n_j} \rightharpoonup w$. Therefore

$$\limsup_{n}\langle u - q, y_n - q\rangle = \langle u - q, w - q\rangle \le 0$$

by (2.1). Apply Lemma 2.2 to (5.6), with $a_n = d_n$, $\lambda_n = \alpha_n$ and $\delta_n = 2\langle u - q, y_n - q\rangle$. We obtain $d_n \to 0$.

Step 3: the nonmonotone case. Suppose (d_n) is not eventually nonincreasing and let $\tau(n)$ be given by Lemma 2.3. At an index $j = \tau(n)$, $d_j \le d_{j+1}$, and (5.3) yields

$$\mu\|y_j - Ty_j\|^2 \le \|y_j - q\|^2 - d_{j+1} \le \|y_j - q\|^2 - d_j \longrightarrow 0,$$

where the last limit follows from boundedness, $j \to \infty$, and $\alpha_j \to 0$. Thus $\|y_{\tau(n)} - Ty_{\tau(n)}\| \to 0$, and the analogue of (5.8) holds along $\tau(n)$. Repeating the weak-subsequence, demiclosedness, and projection argument gives

$$\limsup_{n\to\infty}\langle u - q, y_{\tau(n)} - q\rangle \le 0. \tag{5.10}$$

Since $d_j \le d_{j+1}$ and $\alpha_j > 0$, (5.6) gives

$$d_j \le 2\langle u - q, y_j - q\rangle.$$

Taking $j = \tau(n)$ and using (5.10), while $d_j \ge 0$, proves $d_{\tau(n)} \to 0$. Moreover, the scalar products in (5.6) are bounded and $\alpha_{\tau(n)} \to 0$; hence $d_{\tau(n)+1} \to 0$. Lemma 2.3 now gives

$$0 \le d_n \le d_{\tau(n)+1} \longrightarrow 0.$$

Thus $x_n \to q$ in both cases.

Finally, (5.4) gives $y_n \to q$; nonexpansiveness gives $Ty_n \to q$; convexity gives $z_n \to q$; and $x_{n+1} = T^k z_n \to q$. The residual conclusions follow from the proof above (or directly from strong convergence and continuity of T). □

Corollary 5.2 (Zero-anchor specialization)**.** *Under the control conditions in* (3.3) *and the additional assumption* $0 < \alpha_n < 1$ *for every* n*, the iteration* (3.2) *converges strongly to* $P_F 0$.

Proof. Take $C = H$, $u = 0$, and $k = 0$ in Theorem 5.1. □

Remark 5.3. The strict positivity of α_n is used at last-increase indices when dividing by $\alpha_{\tau(n)}$. Allowing arbitrary zero terms under only $\alpha_n \to 0$ and $\sum_n \alpha_n = \infty$ leaves a gap in this particular proof. One convenient sufficient assumption is $0 < \alpha_n < 1$ for every n (eventual strict positivity would also suffice). No bounded-variation condition such as $\sum_n |\alpha_{n+1} - \alpha_n| < \infty$ is required.

6. A reduced Picard-after-anchor iteration

Consider also the reduced iteration obtained by omitting the Mann point z_n:

$$y_n = \alpha_n u + (1 - \alpha_n)x_n, \qquad x_{n+1} = T^k y_n. \tag{6.1}$$

This is a valid method, but it is not governed by the descent estimate (4.2): the coercive term $\beta_n(1 - \beta_n)\|y_n - Ty_n\|^2$ has disappeared. Consequently, the assumptions of Theorem 5.1 alone do not provide the same proof. The correct observation is that the auxiliary sequence (y_n) in (6.1) satisfies a Halpern recurrence for the nonexpansive map $S = T^k$, while $x_{n+1} = Sy_n$.

Theorem 6.1 (Strong convergence of the reduced method)**.** *Let* H *be a real Hilbert space, let* $C \subset H$ *be nonempty, closed and convex, and let* $T : C \to C$ *be nonexpansive with* $\mathrm{Fix}(T) \neq \varnothing$. *Fix* $u, x_0 \in C$ *and* $k \in \mathbb{N}$. *Suppose*

$$0 < \alpha_n < 1, \qquad \alpha_n \to 0, \qquad \sum_{n=0}^{\infty} \alpha_n = \infty, \qquad \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty. \tag{6.2}$$

Let (x_n) *and* (y_n) *be generated by* (6.1). *Then*

$$y_n \longrightarrow q_k := P_{\mathrm{Fix}(T^k)} u, \qquad x_n \longrightarrow q_k$$

strongly. In particular, if

$$\mathrm{Fix}(T^k) = \mathrm{Fix}(T), \tag{6.3}$$

then both sequences converge strongly to $P_{\mathrm{Fix}(T)} u$.

Proof. Put $S = T^k$ and $Q = \mathrm{Fix}(S)$. Since T is nonexpansive, so is S. Moreover $\mathrm{Fix}(T) \subseteq Q$, so Q is nonempty. The set Q is closed and convex, and hence $q_k = P_Q u$ is well defined.

The two equations in (6.1) give

$$y_{n+1} = \alpha_{n+1}u + (1-\alpha_{n+1})Sy_n. \tag{6.4}$$

Thus (y_n) is the Halpern sequence associated with S, up to its initial indexing.

Step 1: boundedness. Since $Sq_k = q_k$, nonexpansiveness gives

$$\begin{aligned}\|y_{n+1} - q_k\| &\le \alpha_{n+1}\|u - q_k\| + (1-\alpha_{n+1})\|Sy_n - q_k\| \\ &\le \alpha_{n+1}\|u - q_k\| + (1-\alpha_{n+1})\|y_n - q_k\| \\ &\le \max\{\|u - q_k\|, \|y_n - q_k\|\}.\end{aligned}$$

Induction proves that (y_n) is bounded. Therefore (Sy_n) and $x_{n+1} = Sy_n$ are also bounded.

Step 2: successive differences and asymptotic regularity. Subtracting two successive instances of (6.4), we obtain

$$\begin{aligned}y_{n+1} - y_n = (1-\alpha_{n+1})(Sy_n - Sy_{n-1}) \\ + (\alpha_{n+1} - \alpha_n)(u - Sy_{n-1}).\end{aligned}$$

Boundedness supplies a constant $M > 0$ such that

$$\|y_{n+1} - y_n\| \le (1-\alpha_{n+1})\|y_n - y_{n-1}\| + M|\alpha_{n+1} - \alpha_n|. \tag{6.5}$$

Iterating (6.5) from an index N gives

$$\begin{aligned}\|y_{n+1} - y_n\| \le \|y_N - y_{N-1}\| \prod_{j=N+1}^{n+1}(1-\alpha_j) \\ + M\sum_{i=N}^{n}|\alpha_{i+1} - \alpha_i| \prod_{j=i+2}^{n+1}(1-\alpha_j).\end{aligned}$$

The first term tends to zero because $\sum_j \alpha_j = \infty$ and $\prod_j(1-\alpha_j) \le \exp(-\sum_j \alpha_j)$. The second term has limsup at most $M\sum_{i=N}^{\infty}|\alpha_{i+1} - \alpha_i|$. Letting $N \to \infty$ and using bounded variation yields

$$\|y_{n+1} - y_n\| \longrightarrow 0. \tag{6.6}$$

Furthermore, (6.4) gives

$$\|y_{n+1} - Sy_n\| = \alpha_{n+1}\|u - Sy_n\| \longrightarrow 0.$$

Together with (6.6), this proves

$$\|y_n - Sy_n\| \longrightarrow 0. \tag{6.7}$$

Step 3: identification of weak cluster points. Every bounded subsequence of (y_n) has a weakly convergent further subsequence. If $y_{n_j} \rightharpoonup w$, then (6.7) and the demiclosedness principle applied to S imply $w \in Q$. Consequently,

$$\limsup_{n\to\infty}\langle u - q_k, y_{n+1} - q_k\rangle \le 0. \tag{6.8}$$

Indeed, take a subsequence attaining the limsup and then a weakly convergent subsequence; its weak limit lies in Q, where the projection inequality gives $\langle u - q_k, w - q_k\rangle \le 0$.

Step 4: strong convergence. From (6.4), nonexpansiveness, and $Sq_k = q_k$,

$$\begin{aligned}\|y_{n+1} - q_k\|^2 \leq (1-\alpha_{n+1})\|y_n - q_k\|^2 \\ + 2\alpha_{n+1}\langle u - q_k, y_{n+1} - q_k\rangle.\end{aligned} \tag{6.9}$$

Apply Lemma 2.2 to (6.9), using (6.8) and $\sum_n \alpha_n = \infty$. It follows that $y_n \to q_k$. Finally,

$$\|x_{n+1} - q_k\| = \|Sy_n - Sq_k\| \leq \|y_n - q_k\| \longrightarrow 0.$$

If (6.3) holds, then $q_k = P_{\mathrm{Fix}(T)}u$, as asserted. □

Corollary 6.2. *For $k = 1$, the reduced iteration*

$$y_n = \alpha_n u + (1-\alpha_n)x_n, \qquad x_{n+1} = Ty_n$$

converges strongly to $P_{\mathrm{Fix}(T)}u$ under (6.2).

Proof. For $k = 1$, $\mathrm{Fix}(T^k) = \mathrm{Fix}(T)$ trivially, so the conclusion follows from Theorem 6.1. □

Corollary 6.3. *The conclusion $x_n, y_n \to P_{\mathrm{Fix}(T)}u$ also holds for every $k \geq 1$ if T is an averaged mapping.*

Proof. Write $T = (1-\lambda)I + \lambda R$, where $0 < \lambda < 1$ and R is nonexpansive. For $p \in \mathrm{Fix}(T)$, the averaged-map estimate is

$$\|Tx - p\|^2 \leq \|x - p\|^2 - \frac{1-\lambda}{\lambda}\|x - Tx\|^2.$$

If $T^k x = x$, apply this inequality successively to $x, Tx, \ldots, T^{k-1}x$ and add the results. The distance terms telescope, so every nonnegative residual term is zero; in particular $Tx = x$. Hence $\mathrm{Fix}(T^k) = \mathrm{Fix}(T)$, and Theorem 6.1 applies. □

Remark 6.4 (The fixed-set condition and the choice of anchor)**.** For a general nonexpansive mapping, $\mathrm{Fix}(T^k)$ may strictly contain $\mathrm{Fix}(T)$. Let $H = C = \mathbb{R}$, $Tx = -x$, $k = 2$, and $u \neq 0$. Then $T^2 = I$, so (6.1) becomes

$$x_{n+1} = y_n = \alpha_n u + (1-\alpha_n)x_n.$$

Under (6.2), $x_n \to u$. But $\mathrm{Fix}(T) = \{0\}$ and $u \notin \mathrm{Fix}(T)$. Thus the unconditional conclusion is convergence to $P_{\mathrm{Fix}(T^k)}u$, not to a fixed point of T.

For a specified anchor u, the limit belongs to $\mathrm{Fix}(T)$ if and only if $P_{\mathrm{Fix}(T^k)}u \in \mathrm{Fix}(T)$. In that case it equals $P_{\mathrm{Fix}(T)}u$, since $\mathrm{Fix}(T) \subseteq \mathrm{Fix}(T^k)$ and the metric projection is unique. Thus equality of the two fixed-point sets is not necessary for an individual anchor: in the example above, $u = 0$ gives the limit 0 despite the unequal fixed-point sets. However, equality is necessary and sufficient to obtain $P_{\mathrm{Fix}(T)}u$ for every anchor $u \in C$. Indeed, if $v \in \mathrm{Fix}(T^k)\setminus\mathrm{Fix}(T)$, choosing $u = v$ gives $P_{\mathrm{Fix}(T^k)}u = v \notin \mathrm{Fix}(T)$.

Remark 6.5 (Relation to Halpern iteration)**.** Identity (6.4) shows that (y_n) is exactly a Halpern sequence for the nonexpansive mapping T^k, with shifted indexing; the primary iterates satisfy $x_{n+1} = T^k y_n$. This identifies the reduced method with a Halpern scheme through its auxiliary sequence, rather than asserting that (x_n) itself satisfies the classical Halpern recurrence.

7. Weak convergence under summable anchoring

When the anchor weights are summable, they need not select $P_F u$. Under the same uniform interior bounds on the Mann parameters, however, the iterates still converge weakly.

Theorem 7.1 (Weak convergence under summable anchoring)**.** *Let H, C, T, F, u, x_0 and $k \in \mathbb{N}_0$ be as above. Assume*

$$\alpha_n \in [0,1], \qquad \sum_{n=0}^{\infty} \alpha_n < \infty, \qquad 0 < \beta_{\min} \le \beta_n \le \beta_{\max} < 1. \tag{7.1}$$

Then the sequence generated by (4.1) converges weakly to a point of F. It converges strongly if, in addition, $T(C)$ is relatively compact or $\liminf_n \operatorname{dist}(x_n, F) = 0$. In particular, the latter condition follows if T satisfies condition (I): there is a nondecreasing $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$, $g(t) > 0$ for $t > 0$, and $\|x - Tx\| \ge g(\operatorname{dist}(x, F))$ for every $x \in C$.

Proof. Fix $p \in F$ and put

$$\mu = \beta_{\min}(1 - \beta_{\max}) > 0.$$

The same estimate as in the strong proof gives

$$\|x_{n+1} - p\| \le (1 - \alpha_n)\|x_n - p\| + \alpha_n \|u - p\|,$$

so (x_n) is bounded, and hence so are (y_n) and (Ty_n). Put $L = \sup_n \|u - x_n\| < \infty$. Dropping the negative term in (4.2) gives the uniform estimate

$$\|x_{n+1} - p\| \le \|y_n - p\| \le \|x_n - p\| + L\alpha_n \qquad (p \in F). \tag{7.2}$$

For the fixed point p chosen above, boundedness also supplies a constant $M_p > 0$, which may depend on p, such that

$$\|y_n - p\|^2 \le \|x_n - p\|^2 + M_p \alpha_n. \tag{7.3}$$

By (4.2) and the definition of μ,

$$\|x_{n+1} - p\|^2 \le \|y_n - p\|^2 - \mu \|y_n - Ty_n\|^2.$$

Combining this inequality with (7.3) gives

$$\|x_{n+1} - p\|^2 \le \|x_n - p\|^2 + M_p \alpha_n - \mu \|y_n - Ty_n\|^2. \tag{7.4}$$

Estimate (7.2) makes (x_n) quasi-Fejér monotone with respect to F, with the summable error $L\alpha_n$ independent of p. Since $\sum_n \alpha_n < \infty$, the standard tail-correction argument shows that $\|x_n - p\|$ has a limit for every $p \in F$: indeed,

$$b_n := \|x_n - p\|^2 + M_p \sum_{j=n}^{\infty} \alpha_j$$

is nonincreasing.

Summing (7.4) from 0 to N gives

$$\mu \sum_{n=0}^{N} \|y_n - Ty_n\|^2 \le \|x_0 - p\|^2 + M_p \sum_{n=0}^{\infty} \alpha_n,$$

and hence $\|y_n - Ty_n\| \to 0$. Summability implies $\alpha_n \to 0$. Since (x_n) is bounded,

$$\|y_n - x_n\| = \alpha_n \|u - x_n\| \longrightarrow 0.$$

The triangle inequality and nonexpansiveness give

$$\|x_n - Tx_n\| \le \|x_n - y_n\| + \|y_n - Ty_n\| + \|Ty_n - Tx_n\| \le 2\|x_n - y_n\| + \|y_n - Ty_n\| \longrightarrow 0.$$

Every weak cluster point therefore belongs to F by Lemma 2.1.

There can be at most one weak cluster point. Otherwise take subsequences weakly converging to distinct $p_1, p_2 \in F$. Opial's property of Hilbert spaces and the existence of both distance limits give

$$\begin{aligned}\lim_n \|x_n - p_1\| &< \lim_n \|x_n - p_2\| \\ &< \lim_n \|x_n - p_1\|,\end{aligned}$$

a contradiction. Boundedness and weak sequential compactness now imply $x_n \rightharpoonup p_\infty \in F$.

If $T(C)$ is relatively compact, choose a subsequence for which $Tx_{n_j} \to r$. Since $x_{n_j} - Tx_{n_j} \to 0$, it follows that $x_{n_j} \to r$. By continuity of T, we have $Tx_{n_j} \to Tr$; on the other hand, $Tx_{n_j} \to r$. Hence $Tr = r$, so $r \in F$. The already established existence of $\|x_n - r\|$ forces the entire sequence to converge strongly to r.

If $\liminf_n \operatorname{dist}(x_n, F) = 0$, take the infimum over $p \in F$ in (7.2) to obtain

$$\operatorname{dist}(x_{n+1}, F) \le \operatorname{dist}(x_n, F) + L\alpha_n.$$

Hence $\operatorname{dist}(x_n, F) + L\sum_{i=n}^{\infty} \alpha_i$ is nonincreasing, so $\operatorname{dist}(x_n, F)$ has a limit. The assumed liminf makes that limit zero. Iterating (7.2), uniformly for $p \in F$ and $m > n$, gives

$$\|x_m - p\| \le \|x_n - p\| + L\sum_{i=n}^{m-1} \alpha_i.$$

Given $\varepsilon > 0$, choose n so large that $\operatorname{dist}(x_n, F) < \varepsilon$ and $L\sum_{i=n}^{\infty} \alpha_i < \varepsilon$. Choose $p_n \in F$ with $\|x_n - p_n\| < 2\varepsilon$. Then, for every $m \ge n$, $\|x_m - p_n\| < 3\varepsilon$. Therefore, for $m, \ell \ge n$, $\|x_m - x_\ell\| < 6\varepsilon$. Thus (x_n) is Cauchy and converges strongly to some p. Since $\operatorname{dist}(x_n, F) \to 0$ and F is closed, $p \in F$. Finally, condition (I) and $\|x_n - Tx_n\| \to 0$ imply $g(\operatorname{dist}(x_n, F)) \to 0$, hence $\operatorname{dist}(x_n, F) \to 0$. □

8. Relations with existing iterative methods

The following are algebraic special cases; the convergence assumptions of the theorems above need not carry over to them.

Proposition 8.1. *For the method (4.1):*

(i) If $\alpha_n = 0$ for all n, then $y_n = x_n$ and

$$x_{n+1} = T^k\big((1 - \beta_n)x_n + \beta_n Tx_n\big),$$

which is precisely GPM with control $\theta_n = \beta_n$.

(ii) If $k = 0$, $u = 0$, and $0 \in C$, then

$$y_n = (1 - \alpha_n)x_n, \qquad x_{n+1} = (1 - \beta_n)y_n + \beta_n Ty_n,$$

which is the 2009 modified KM method.

(iii) If $k = 0$ and $\alpha_n = 0$, then

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n Tx_n,$$

the classical KM iteration.

(iv) If $\alpha_n = 0$ and $k = 1$, one obtains the normal S-iteration of Sahu, also known as the Picard–Mann hybrid of Khan [6, 12]. The cases $k = 2$ and $k = 3$ give the M/F^* and F iterations, respectively, as listed by Shukla et al. [16].

The reductions do not mean that the hypotheses of Theorem 5.1 reduce with them. In particular, $\alpha_n \equiv 0$ violates $\sum_n \alpha_n = \infty$, so the strong-convergence theorem does not cover unanchored GPM. This is consistent with standard GPM results for nonexpansive mappings, which generally yield weak convergence unless additional assumptions are imposed. Also, $\beta_n = 0$ or $\beta_n = 1$ destroys the uniform residual term in (4.2); those endpoint cases require different assumptions and are not covered by the strong theorem.

9. Illustrative examples

We present four examples. The first illustrates the metric-projection selection mechanism, the second exhibits the influence of the Picard depth k, and the third shows that the fixed-set condition in Theorem 6.1 cannot be omitted in a guarantee valid for every anchor. The fourth gives a higher-dimensional comparison between the anchored GPM and Halpern iterations.

Example 9.1 (Finite convergence to the selected fixed point)**.** Let $H = C = \mathbb{R}$ and define

$$Tx = P_{[0,\infty)}x = \max\{x, 0\}.$$

Then T is firmly nonexpansive and $F = \mathrm{Fix}(T) = [0, \infty)$. Choose

$$u = -2, \qquad x_0 = 4, \qquad \alpha_n = \frac{1}{n+2}, \qquad \beta_n = \frac{1}{2}, \qquad k = 1.$$

Since $P_F u = 0$, Theorem 5.1 predicts convergence to zero.

At $n = 0$,

$$y_0 = \frac{1}{2}(-2) + \frac{1}{2}(4) = 1, \qquad z_0 = \frac{1}{2}y_0 + \frac{1}{2}Ty_0 = 1,$$

and therefore $x_1 = Tz_0 = 1$. At $n = 1$,

$$y_1 = \frac{1}{3}(-2) + \frac{2}{3}(1) = 0,$$

so $z_1 = 0$ and $x_2 = 0$. For every $n \geq 2$, if $x_n = 0$, then $y_n = -2/(n+2) < 0$, whence

$$z_n = \frac{1}{2}y_n + \frac{1}{2}Ty_n = \frac{1}{2}y_n < 0 \quad \text{and} \quad x_{n+1} = Tz_n = 0.$$

Thus the sequence reaches the selected fixed point $P_F u = 0$ after two iterations. Notice that F contains infinitely many points, so the example also displays the role of the anchor in selecting one particular fixed point.

Example 9.2 (Effect of the Picard depth)**.** Let $H = C = \mathbb{R}$ and

$$Tx = \rho x, \qquad 0 < \rho < 1.$$

Then T is a contraction and $F = \{0\}$. For a constant $\beta_n = \beta \in (0, 1)$, the anchored GPM iteration reduces to

$$x_{n+1} = c_k\big(\alpha_n u + (1 - \alpha_n)x_n\big), \qquad c_k = \rho^k(1 - \beta + \beta\rho). \tag{9.1}$$

Since $0 < c_k < 1$, formula (9.1) makes the damping effect of the Picard depth explicit.

For illustration, take

$$\rho = 0.8, \qquad \beta = \frac{1}{2}, \qquad \alpha_n = \frac{1}{n+2}, \qquad u = 1, \qquad x_0 = 3.$$

The corresponding coefficients are $c_0 = 0.9$, $c_1 = 0.72$, and $c_2 = 0.576$. The iterates are displayed below.

n	$k=0$	$k=1$	$k=2$
0	3.000000	3.000000	3.000000
1	1.800000	1.440000	1.152000
2	1.380000	0.931200	0.634368
5	0.909510	0.442392	0.243761
10	0.627994	0.235225	0.124099
20	0.393835	0.122478	0.064691

In all three cases $x_n \to 0 = P_F u$, in agreement with Theorem 5.1. For this scalar contraction, increasing k decreases the error at a fixed iteration index. This observation is illustrative only: each increase of k also requires an additional evaluation of T.

Example 9.3 (The reduced iteration may converge outside $\mathrm{Fix}(T)$)**.** Let $H = C = \mathbb{R}$, $Tx = -x$, $k = 2$, $u = 1$, $x_0 = 0$, and

$$\alpha_n = \frac{1}{n+2}.$$

The mapping T is nonexpansive, $\mathrm{Fix}(T) = \{0\}$, but $T^2 = I$ and hence $\mathrm{Fix}(T^2) = \mathbb{R}$. The reduced iteration becomes

$$x_{n+1} = \frac{1}{n+2} + \frac{n+1}{n+2} x_n.$$

An induction gives

$$x_n = \frac{n}{n+1} \qquad (n \geq 0),$$

and consequently $x_n \to 1$. This is precisely $P_{\mathrm{Fix}(T^2)} u$, as asserted by Theorem 6.1, but it is not a fixed point of T. Hence the equality $\mathrm{Fix}(T^k) = \mathrm{Fix}(T)$ cannot be suppressed from a guarantee of convergence to $P_{\mathrm{Fix}(T)} u$ valid for every anchor. For an individual anchor, the weaker condition described in Remark 6.4 is sufficient.

Figure 1 provides graphical representations of Examples 9.2 and 9.3. The left panel plots the error $|x_n - P_F u|$ on a logarithmic scale for the three Picard depths considered in Example 9.2. The right panel displays the reduced Halpern sequence from Example 9.3, thereby illustrating convergence to a point in $\mathrm{Fix}(T^2)$ that does not belong to $\mathrm{Fix}(T)$.

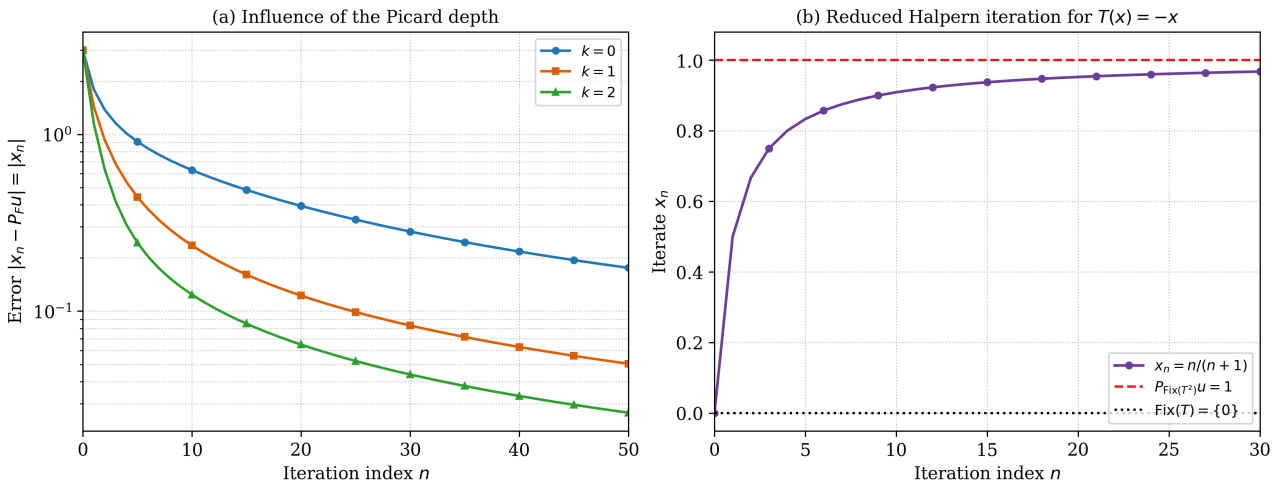


FIGURE 1. Numerical convergence behaviour. Panel (a) shows the error in the anchored GPM iteration for $k = 0, 1, 2$ with $T(x) = 0.8x$, $\beta = 1/2$, $\alpha_n = 1/(n+2)$, $u = 1$, and $x_0 = 3$. Panel (b) shows the reduced Halpern iteration for $T(x) = -x$ and $k = 2$, for which $x_n = n/(n+1) \to 1$ although $\mathrm{Fix}(T) = \{0\}$.

Example 9.4 (A higher-dimensional comparison with the Halpern method)**.** Let $H = C = \mathbb{R}^5$ with the Euclidean norm, and define the linear mapping $T : \mathbb{R}^5 \to \mathbb{R}^5$ by

$$Tx = Dx, \quad D = \mathrm{diag}(1, 0.9, 0.75, 0.5, 0.2).$$

Since $\|D\|_2 = 1$, the mapping T is nonexpansive. Its fixed-point set is

$$F = \mathrm{Fix}(T) = \{(t, 0, 0, 0, 0) : t \in \mathbb{R}\}.$$

Choose

$$u = (2, -1, 3, -2, 1), \qquad x_0 = (-3, 4, -1, 5, -2),$$

and set

$$\alpha_n = \frac{1}{n+2}, \qquad \beta_n = \frac{1}{2}, \quad k = 2.$$

The selected fixed point is therefore

$$q = P_F u = (2, 0, 0, 0, 0).$$

Let (x_n^{G}) denote the anchored GPM sequence

$$\begin{aligned} y_n^{\mathrm{G}} &= \alpha_n u + (1 - \alpha_n) x_n^{\mathrm{G}}, \\ z_n^{\mathrm{G}} &= (1 - \beta_n) y_n^{\mathrm{G}} + \beta_n T y_n^{\mathrm{G}}, \\ x_{n+1}^{\mathrm{G}} &= T^2 z_n^{\mathrm{G}}, \end{aligned}$$

and let (x_n^{H}) be the classical Halpern sequence generated from the same initial point and anchor:

$$x_{n+1}^{\mathrm{H}} = \alpha_n u + (1 - \alpha_n) T x_n^{\mathrm{H}}.$$

Both sequences converge strongly to q. The following values give the Euclidean errors $e_n^{\mathrm{G}} = \|x_n^{\mathrm{G}} - q\|$ and $e_n^{\mathrm{H}} = \|x_n^{\mathrm{H}} - q\|$:

n	e_n^{G}	e_n^{H}
0	8.426150	8.426150
1	2.811367	3.059105
2	1.823695	2.428652
5	0.984166	1.847451
10	0.585732	1.271615
20	0.317855	0.759565
50	0.131260	0.331091
100	0.066280	0.167603

Thus the anchored GPM sequence has a smaller error at the displayed iteration indices. This comparison does not by itself imply lower computational cost: in the direct implementation above, one anchored GPM iteration with $k = 2$ uses three evaluations of T, whereas one Halpern iteration uses one. Figure 2 therefore reports the errors both against iteration count and against the cumulative number of operator evaluations. At the common budget of 300 evaluations, anchored GPM has completed 100 iterations and has error 0.066280, whereas Halpern has completed 300 iterations and has error 0.056240. Thus, in this example, anchored GPM gives the smaller error per iteration, while Halpern gives the smaller error at the stated equal evaluation budget.

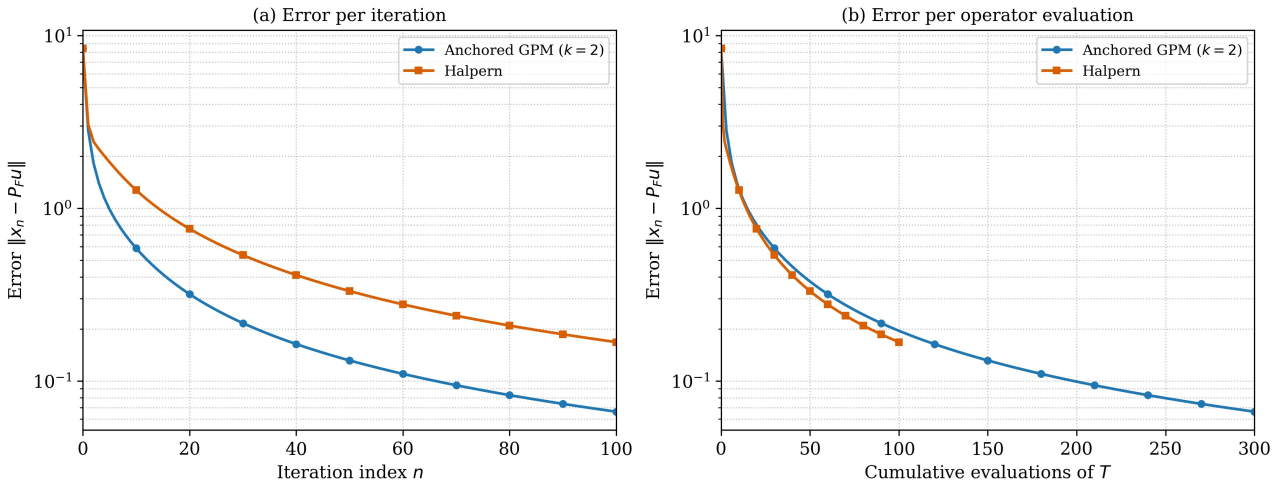


FIGURE 2. Comparison in $\mathbb{R}^5$ for the diagonal nonexpansive mapping in Example 9.4. Panel (a) plots the error against the iteration index through $n = 100$. Panel (b) compares the methods over the same budget of 300 evaluations of T, counting three evaluations per anchored GPM iteration for $k = 2$ and one evaluation per Halpern iteration. The plots are numerical illustrations and do not assert a general rate ordering between the methods.

10. Comparison and concluding remarks

10.1. **Comparison of the iterative methods.** The classical Krasnosel'skiĭ–Mann iteration of Mann [9]

$$x_{n+1} = \gamma_n x_n + (1-\gamma_n)Tx_n$$

is applicable to nonexpansive mappings in Hilbert spaces and, more generally, in suitable uniformly convex Banach spaces. Under the usual divergence condition on the averaging parameters, its convergence is generally weak; strong convergence cannot be guaranteed without additional hypotheses.

The modified Krasnosel'skiĭ–Mann method of Yao et al. [22] introduces the preliminary zero-anchor step

$$y_n = (1-\alpha_n)x_n, \qquad x_{n+1} = (1-\beta_n)y_n + \beta_n T y_n.$$

In a real Hilbert space, Corollary 5.2 gives strong convergence to $P_F 0$ under the Yao control conditions and the additional assumption $0 < \alpha_n < 1$ for every n. Thus its strong-convergence mechanism is the vanishing nonsummable anchor rather than the Mann averaging step alone.

The general Picard–Mann method of Shukla et al. [16]

$$x_{n+1} = T^k\big((1-\theta_n)x_n + \theta_n T x_n\big)$$

retains the Mann average and follows it by k Picard evaluations. For nonexpansive mappings in uniformly convex Banach spaces, its basic conclusion is weak convergence under the geometric assumptions described in Section 3.2. Strong convergence in the corresponding results requires an additional compactness, distance, or condition (I) hypothesis.

The anchored GPM iteration (4.1) incorporates both mechanisms. It uses a general anchor u, performs the Mann average at the anchored point y_n, and then applies k Picard steps. Theorem 5.1 proves strong convergence to the explicitly selected point $P_F u$ for an arbitrary nonexpansive mapping on a real Hilbert space, without compactness of C or $T(C)$. When the anchor sequence is summable, Theorem 7.1 instead gives weak convergence, with strong convergence available under the additional conditions stated there.

The reduced iteration (6.1) omits the Mann stage and is therefore structurally different. Its auxiliary sequence (y_n) satisfies a Halpern recurrence for T^k and, under (6.2), both (x_n) and (y_n) converge strongly to $P_{\mathrm{Fix}(T^k)}u$. The equality $\mathrm{Fix}(T^k) = \mathrm{Fix}(T)$ guarantees the limit $P_{\mathrm{Fix}(T)}u$ for every anchor; it holds, for example, when $k = 1$ or when T is averaged and $k \geq 1$. For a specified anchor, it suffices that $P_{\mathrm{Fix}(T^k)}u \in \mathrm{Fix}(T)$.

The anchored method recovers the modified Krasnosel'skiĭ–Mann method when $k = 0$, $u = 0$, and $0 \in C$, and it recovers unanchored GPM when $\alpha_n = 0$. The main result adds strong convergence and an explicit description of the limit under the stated Hilbert-space assumptions. It does not provide a general comparison of convergence rates.

10.2. **Discussion.** The anchored GPM method combines two distinct mechanisms. The vanishing, nonsummable anchor is responsible for selecting the metric projection $P_F u$, while the GPM component contributes the Mann residual estimate followed by k Picard evaluations. The master inequality (4.2) shows that the additional Picard steps preserve the descent produced by the Mann average. Consequently, strong convergence follows without compactness of C or $T(C)$.

The parameter restrictions in Theorem 5.1 have specific roles. The condition $\alpha_n \to 0$ makes the individual anchor contribution vanish, whereas $\sum_n \alpha_n = \infty$ prevents its cumulative influence from being summable. The bounds $0 < \beta_{\min} \leq \beta_n \leq \beta_{\max} < 1$ guarantee the uniform coercivity

$$\beta_n(1-\beta_n) \geq \beta_{\min}(1-\beta_{\max}) > 0.$$

If β_n approaches either endpoint, the present residual argument no longer applies. Similarly, the proof is genuinely Hilbertian: it uses metric projection geometry, weak sequential compactness, demiclosedness, and inner product identities. An extension to general Banach spaces would require a duality-map or sunny-retraction formulation.

The reduced method (6.1) has a different structure. Once the Mann stage is deleted, its auxiliary sequence (y_n) satisfies a Halpern recurrence for T^k, with $x_{n+1} = T^k y_n$. Its natural solution set is therefore $\mathrm{Fix}(T^k)$ rather than $\mathrm{Fix}(T)$. The counterexample in Remark 6.4 demonstrates that this distinction is essential. Thus the reduced method is a related auxiliary scheme and is not an equivalent form of the anchored GPM iteration.

No convergence-rate or computational-efficiency conclusion is asserted here. Although extra Picard evaluations may reduce iteration counts in particular examples, a meaningful comparison must account for the number and cost of operator evaluations.

10.3. **Conclusion.** An anchored general Picard–Mann method for nonexpansive mappings has been introduced and analysed in a real Hilbert space. The principal result proves strong convergence to the selected fixed point $P_{\mathrm{Fix}(T)}u$ under a vanishing nonsummable anchor and uniformly interior Mann parameters. A summable-anchor regime yields weak convergence, with strong convergence under additional compactness, distance, or condition (I) assumptions.

The reduced iteration $x_{n+1} = T^k y_n$ was also studied. Under bounded variation of the anchor sequence, it converges strongly to $P_{\mathrm{Fix}(T^k)}u$; convergence to $P_{\mathrm{Fix}(T)}u$ follows when $\mathrm{Fix}(T^k) = \mathrm{Fix}(T)$, in particular for $k = 1$ or for averaged mappings.

The examples confirm the projection-selection mechanism and show explicitly how the Picard depth modifies the damping factor in a scalar contractive setting. They also demonstrate that periodic points of T obstruct an unconditional fixed-point conclusion for the reduced method. The higher-dimensional example further shows that a method may have a smaller error per iteration without having a smaller error for a fixed number of operator evaluations; consequently, iteration counts and evaluation costs must be distinguished in numerical comparisons.

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