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THE HERZOG–SCHÖNHEIM CONJECTURE FOR FINITE GROUPS OF ORDER LESS THAN 2000

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Abstract. The Herzog–Schönheim conjecture asserts that, in any partition of a group into cosets of proper subgroups, two subgroups have the same index. We prove the conjecture for every finite group of order less than 2000 by combining arithmetic restrictions on subgroup indices with an exhaustive computation in GAP.

1. Introduction

The Herzog–Schönheim conjecture [3] states that if a group G admits a finite coset partition

$$G = \bigsqcup_{i=1}^{n} g_i U_i, \qquad U_i < G, \tag{1}$$

then $[G : U_i] = [G : U_j]$ for some $i \neq j$.

The conjecture is known for finite pyramidal groups, that is, groups with an ordered Sylow tower, and hence for nilpotent and supersolvable groups [1]. Garonzi and Margolis established it for simple and symmetric groups [2]. Margolis and Schnabel proved it for all finite groups of order less than 1440 [4]. We extend this order bound.

Theorem 1.1. *Every finite group of order less than* 2000 *satisfies the Herzog–Schönheim conjecture.*

The proof enumerates the possible index sets of a smallest counterexample and applies four obstructions of Margolis and Schnabel. Every order in the range $1440 < |G| < 2000$ is eliminated in this way. At order 1440, all surviving index sets contain the same nine indices. Sun's theorem excludes a smallest counterexample with a normal Sylow 5-subgroup, and a GAP [6] computation shows that every remaining group lacks at least one of these required indices. Thus subgroup-index enumeration suffices; no search over coset representatives is needed. Throughout the paper, groups are finite. We write $\mathrm{Div}(N)$ for the positive divisors of N and $\mathrm{Ind}(G) = \{[G : H] : H \leq G\}$, with repetitions omitted.

2. Necessary conditions for a counterexample

In this section, we give some necessary conditions for a counterexample for the Herzog–Schönheim conjecture.

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Lemma 2.1. *Let G be a counterexample of smallest order, with partition (1), and put $a_i = [G : U_i]$. Then*

(i) *the a_i are distinct divisors of $|G|$ and $a_i \geq 3$;*
(ii) $\sum_i 1/a_i = 1$;
(iii) $\gcd(a_i, a_j) > 1$ *for* $i \neq j$;
(iv) $\bigcap_i \mathrm{Core}_G(U_i) = 1$, *where* $\mathrm{Core}_G(U) = \bigcap_{g \in G} gUg^{-1}$.

Proof. Distinctness is the counterexample assumption, divisibility follows from Lagrange's theorem, and (ii) follows by counting elements. If two indices a, b are coprime, the index of the intersection of the corresponding subgroups is divisible by ab and at most ab. It is therefore ab, so their product is G. Every pair of their cosets intersects, proving (iii).

Suppose one index is 2. Translate the partition so that this coset is a subgroup U, which is normal in G. Every other subgroup V lies in U: otherwise V maps onto G/U, and each of its cosets meets U. All remaining cosets lie in $G \setminus U$. Translating them onto U yields a partition with distinct indices $[U : V] = [G : V]/2$. Its subgroups are proper in U, since $V = U$ would repeat index 2 in G. This contradicts minimality and proves (i).

Finally, let $K = \bigcap_i \mathrm{Core}_G(U_i)$. Each coset in (1) is a union of K-cosets, so the partition descends to G/K with the same indices. If $K \neq 1$, this gives a smaller counterexample, proving (iv). □

A set $S \subseteq \mathrm{Div}(N) \setminus \{1, 2\}$ is called *admissible for N* if

$$\sum_{a \in S} \frac{1}{a} = 1, \qquad \gcd(a, b) > 1 \quad (a \neq b \in S). \tag{2}$$

By Lemma 2.1, the index set of a smallest counterexample is admissible for its order.

Lemma 2.2. *A smallest counterexample of order 1440 cannot have a normal Sylow 5-subgroup.*

Proof. Sun's theorem [5, Theorem 1.1] applies to a coset partition with $K = \bigcap_i \mathrm{Core}_G(U_i)$ if G/K is solvable and its Sylow subgroup for the largest prime divisor of $|G/K|$ is normal. It then forces repeated indices. For a smallest counterexample, $K = 1$ by Lemma 2.1(iv).

Now $1440 = 2^5 \cdot 3^2 \cdot 5$. If its Sylow 5-subgroup P is normal, then G/P has order 288 and is solvable by Burnside's $p^a q^b$ theorem. Since P is cyclic of order 5, G is solvable as well. Sun's theorem, with largest prime 5, gives the required contradiction. □

We also use the following universal obstructions to disjoint coset families.

Proposition 2.3 ([4, Propositions 4.2, 4.3, 4.5, 4.7]). *None of the following tuples can occur as the subgroup indices of pairwise disjoint cosets in any group:*

(F1) $(2r_1, 2r_2, 2r_3)$;
(F2) $(3r_1, 3r_2, 3r_3, 3r_4)$;
(F3) $(2r_1, 4r_2, 4r_3, 4r_4)$, *where r_1 is odd;*
(F4) $(3, 3r_2, 6r_3, 6r_4, 6r_5)$, *where r_2 is odd.*

In each line the parameters r_i are positive and pairwise coprime.

An admissible set is called *clean* if it contains none of these tuples. The index set of a counterexample must be clean.

3. Arithmetic reduction

For a fixed order N, put $m = N/a$ for each divisor $a \geq 3$. Then the reciprocal identity in (2) becomes the integer subset-sum equation

$$\sum_{a \in S} \frac{N}{a} = N. \tag{3}$$

We enumerate increasing subsets of the divisors, rejecting an index coprime to an already selected one. A branch is also discarded if the sum of all unused terms N/a is smaller than the remaining target. The procedure uses exact integer arithmetic.

Proposition 3.1. *For* $1440 < N < 2000$*, admissible sets occur only at the seven orders in Table 1, and none is clean. At order* 1440*, there are* 71262 *admissible sets, of which* 223 *are clean. The intersection of these* 223 *sets is*

$$C = \{4, 6, 8, 12, 16, 18, 24, 30, 32\}. \tag{4}$$

Exact enumeration. The subset-sum search exhausts the admissible sets: each is reached exactly once in increasing order, and the pruning rules exclude only subsets that violate necessary conditions. Applying the four predicates from Proposition 2.3 gives the following tables and the intersection (4). Appendix A gives the complete procedure. □

Table 1. Admissible sets for $1440 < N < 2000$.

N	Admissible sets	Clean sets
1512	147	0
1560	7	0
1620	233	0
1680	331327	0
1800	1603	0
1920	4212	0
1980	59	0

Table 2. Filtering at order 1440, using the first applicable obstruction.

First obstruction	Number of sets
F1	65860
F2	430
F3	4749
F4	0
Clean	223

Consequently, a smallest counterexample of order less than 2000 must have order 1440 and must satisfy $C \subseteq \mathrm{Ind}(G)$. In the following, we deal with groups of order 1440.

Proposition 3.2. *Every group* G *of order* 1440 *satisfies at least one of the following:*

(i) *a Sylow* 5*-subgroup of* G *is normal;*

(ii) $C \not\subseteq \mathrm{Ind}(G)$.

Finite verification. The SmallGroups library contains 5958 groups of order 1440. Of these, 5847 have a normal Sylow 5-subgroup. For each of the remaining 111 groups, compute

$$\mathrm{Ind}(G) = \{[G : \mathrm{Representative}(c)] : c \in \mathrm{ConjugacyClassesSubgroups}(G)\}.$$

Conjugate subgroups have the same index, so one representative from each class suffices. In every case $C \setminus \mathrm{Ind}(G)$ is nonempty. The counts are

Groups of order 1440	5958
Groups with a normal Sylow 5-subgroup	5847
Remaining groups	111
Remaining groups lacking an index in C	111
Unresolved groups	0

The verification was completed with GAP 4.12.1 and SmallGroups 1.5.3, retaining one verdict for every group. Appendix B supplies the complete GAP program. □

Proof of Theorem 1.1. Suppose a counterexample of order less than 2000 exists and choose one, G, of smallest order. By [4, Theorem A], $|G| \geq 1440$. Its index set S is admissible and clean by Lemma 2.1 and Proposition 2.3. Proposition 3.1 therefore gives $|G| = 1440$ and $C \subseteq S \subseteq \mathrm{Ind}(G)$. By Proposition 3.2, a Sylow 5-subgroup of G must be normal. This contradicts Lemma 2.2. Hence no counterexample of order less than 2000 exists. □

Appendix A. Arithmetic enumeration

The following Python program implements the exact search. It precomputes the forbidden subsets as bit masks on the divisor list; a set contains an obstruction precisely when its mask contains one of these masks.

```
from math import gcd
from itertools import combinations

def coprime(xs):
    return all(gcd(a,b)==1 for a,b in combinations(xs,2))

def forbidden(S,k):
    if k==1:
        return any(coprime([a//2 for a in q])
          for q in combinations([a for a in S if a%2==0],3))
    if k==2:
        return any(coprime([a//3 for a in q])
          for q in combinations([a for a in S if a%3==0],4))
    if k==3:
        return any(coprime([a//2]+[b//4 for b in q])
          for a in S if a%4==2
          for q in combinations([b for b in S if b%4==0],3))
    return 3 in S and any(coprime([a//3]+[b//6 for b in q])
        for a in S if a!=3 and a%6==3
```

```
        for q in combinations([b for b in S if b%6==0],3))

def enumerate_sets(N,A):
    M=[N//a for a in A]; suffix=[0]*(len(A)+1)
    for i in reversed(range(len(A))):
        suffix[i]=suffix[i+1]+M[i]
    def dfs(i,r,S):
        if r==0:
            yield tuple(S); return
        if i==len(A) or suffix[i]<r: return
        for j in range(i,len(A)):
            if M[j]<=r and all(gcd(A[j],b)>1 for b in S):
                yield from dfs(j+1,r-M[j],S+[A[j]])
    return dfs(0,N,[])

for N in range(1440,2000):
    A=[a for a in range(3,N+1) if N%a==0]
    pos={a:i for i,a in enumerate(A)}
    mask=lambda S: sum(1<<pos[a] for a in S)
    patterns=[[mask(q) for q in combinations(A,size)
               if forbidden(q,k)]
              for k,size in enumerate([3,4,4,5],1)]
    count=0; hits=[0]*4; clean=[]
    for S in enumerate_sets(N,A):
        count+=1; m=mask(S)
        for k,P in enumerate(patterns):
            if any(m&p==p for p in P): hits[k]+=1; break
        else: clean.append(set(S))
    if count: print(N,count,hits,len(clean))
    if N==1440:
        assert count==71262 and hits==[65860,430,4749,0]
        assert len(clean)==223
        assert set.intersection(*clean)=={4,6,8,12,16,18,24,30,32}
    else: assert not clean
```

Appendix B. The GAP verification

The SmallGroups, TransGrp and PrimGrp packages must be available. No candidate files or coset-search input files are required. The program checks the entire library at order 1440 and prints the missing indices for each group without a normal Sylow 5-subgroup.

```
if LoadPackage("smallgrp") <> true then Error("SmallGroups required"); fi;
if LoadPackage("transgrp") <> true then Error("TransGrp required"); fi;
if LoadPackage("primgrp") <> true then Error("PrimGrp required"); fi;
CheckOrder1440 := function()
  local core,total,normal5,excluded,id,G,ind,missing;
  core := [4,6,8,12,16,18,24,30,32];
  total := NrSmallGroups(1440);
```

```
  normal5 := 0; excluded := 0;
  Print("GAP ",GAPInfo.Version," SmallGroups ",
        PackageInfo("smallgrp")[1].Version,"\n");
  for id in [1..total] do
    G := SmallGroup(1440,id);
    if IsNormal(G,SylowSubgroup(G,5)) then
      normal5 := normal5+1;
      Print("GROUP ",id," NORMAL5\n");
    else
      ind := Set(List(ConjugacyClassesSubgroups(G),
                    c -> Index(G,Representative(c))));
      missing := Difference(core,ind);
      if IsEmpty(missing) then Error("Unresolved group: ",id); fi;
      excluded := excluded+1;
      Print("GROUP ",id," MISSING ",missing,"\n");
    fi;
  od;
  if total<>5958 or normal5<>5847 or excluded<>111 then
    Error("Unexpected totals");
  fi;
  Print("SUCCESS TOTAL ",total," NORMAL5 ",normal5,
        " MISSING_CORE ",excluded,"\n");
end;;
CheckOrder1440();
QUIT_GAP(0);
```

Conflicts of Interest: There is no conflicts of interest.

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