

GENERALIZED GENERAL-TERM FORMULAS FOR k -TH ORDER LINEAR RECURRENCE RELATIONS

K. L. VERMA

ABSTRACT. This paper investigates the k -th order generalized linear recurrence relation $V_n = \sum_{j=1}^k p_j V_{n-j}$, where $p_k \neq 0$ and the coefficients p_j and initial values $V_j = a_j$ are arbitrary integers. By constructing a generalized generating function, we derive a closed-form explicit formula that establishes a unifying framework, generalizing several existing results in the literature. We demonstrate that specializing the order k , the initial conditions, and the recurrence coefficients naturally reduces our formulation to classical sequences, such as the Fibonacci, Tribonacci, and Tetranacci sequences. Furthermore, by framing these higher-order sequences through a $k \times k$ state-transition matrix M , we extend this theoretical groundwork to demonstrate its utility in advanced applied domains, including cryptography, cyber security, and network vulnerability modeling.

1. INTRODUCTION

A recurrence relation is an equation that defines a sequence dynamically in terms of its own preceding elements. A cornerstone of discrete mathematics, the classical Fibonacci sequence serves as a primary example of such a relation, and its fundamental properties have been documented extensively across numerous monographs ([5, 13, 17]). In the literature, the Fibonacci sequence has been generalized through various mathematical frameworks ([2–4, 7–9, 18]). A prominent approach involves extending the order of the underlying linear recurrence relation by summing a larger set of preceding terms ([1, 12]). This extension yields the k -generalized Fibonacci sequence, alternatively referred to as the k -bonacci, k -step Fibonacci, or k -th order Fibonacci sequence. Extensive theoretical results exist for classical Fibonacci and Lucas numbers and polynomials. In particular, [14–16] investigated their fundamental properties and relationships with other established polynomial families. To bypass the computational inefficiencies of step-by-step recursion when calculating large values of n , an explicit, closed-form expression is highly desirable. For the standard Fibonacci sequence, this is achieved via Binet’s classical formula, which computes the n -th term directly utilizing the roots of the sequence’s characteristic algebraic equation. For the higher-order sequences investigated in this paper, we consider the generalized linear recurrence relation

$$V_n(a_j, p_j) = \sum_{j=1}^k p_j V_{n-j}$$

DEPARTMENT OF MATHEMATICS, CAREER POINT UNIVERSITY, (HP), 178041, INDIA

E-mail addresses: klverma@netscape.net, klverma@cpuh.edu.in.

Key words and phrases. Generalized recurrence relations; k^{th} order; General term; Binet type formula; $k \times k$ matrix representation.

where $p_k \neq 0$, and the initial states $V_j = a_j$ and coefficients p_j are treated as arbitrary constants. Directly computing terms for $n \geq k + 1$ using the recurrence definition requires calculating all intermediate values, which becomes computationally intensive. While explicit, Binet-type formulas have been well-established for lower-order linear recurrences featuring fixed initial terms and specific coefficients, unified formulations handling entirely arbitrary initial conditions and coefficients are less common. To fill this gap, this paper establishes a generalized, closed-form Binet-type formula valid for any index $n \geq k + 1$. We demonstrate that this overarching framework is highly comprehensive, naturally encompassing the classical Binet formulas for the Fibonacci, Tribonacci, Tetranacci, and higher-order sequences as special parametric cases.

2. MATHEMATICAL BACKGROUND

Definition 2.1. We define the Generalized k^{th} – order sequence $\{V_n\}$ by the following generalized linear recurrence relation:

$$(1) \quad V_n(a_j, p_j) = \sum_{j=1}^k p_j V_{n-j}, n \geq k + 1,$$

with initial terms $a_j = V_j$, a_j, p_j , $j = 1, 2, \dots, k$, $p_k \neq 0$ are arbitrary non-zero real numbers.

Theorem 2.2. For $n \geq k + 1$

$$(2) \quad V_n = (1 + p_1) V_{n-1} + \sum_{j=2}^k (p_j - p_{j-1}) V_{n-j} - p_k V_{n-k-1}.$$

Proof. Using definition (1), we have

$$\begin{aligned} V_n &= p_1 V_{n-1} + p_2 V_{n-2} + p_3 V_{n-3} + p_3 V_{n-3} + \dots + p_k V_{n-k} \\ V_n &= p_1 V_{n-1} + p_2 V_{n-2} + p_3 V_{n-3} + p_3 V_{n-3} + \dots + p_k V_{n-k} + (V_{n-1} - V_{n-1}) \\ V_n &= p_1 V_{n-1} + p_2 V_{n-2} + p_3 V_{n-3} + p_3 V_{n-3} + \dots + p_k V_{n-k} + V_{n-1} \\ &\quad - (p_1 V_{n-2} + p_2 V_{n-3} + p_3 V_{n-4} + p_3 V_{n-5} + \dots + p_k V_{n-k-1}) \\ V_n &= (1 + p_1) V_{n-1} + (p_2 - p_1) V_{n-2} + (p_3 - p_2) V_{n-3} + \dots + (p_k - p_{k-1}) V_{n-k} - p_k V_{n-k-1} \\ V_n &= (1 + p_1) V_{n-1} + \sum_{j=2}^k (p_j - p_{j-1}) V_{n-j} - p_k V_{n-k-1}. \end{aligned}$$

which completes the proof of the theorem. □

Corollary 2.3. If $p_j = 1, j = 1, 2, 3 \dots k$, then

$$V_n = 2V_{n-1} - V_{n-k-1}.$$

which is the result obtained by Howard and Cooper ([6]).

Theorem 2.4. For $n \geq k + 1$

(3)

$$\begin{aligned} V_n &= (1 + p_1)^k V_k + (1 + p_1)^{k-1} \left[\sum_{j=2}^k (p_j - p_{j-1}) V_{k-j-(k-1)} - p_k V_{n-2k+2} \right] \\ &+ (1 + p_1)^{k-2} \left[\sum_{j=2}^k (p_j - p_{j-1}) V_{k-j-(k-2)} - p_k V_{n-2k+3} \right] \\ &+ (1 + p_1)^{k-3} \left[\sum_{j=2}^k (p_j - p_{j-1}) V_{k-j-(k-3)} - p_k V_{n-2k+4} \right] + \dots + \sum_{j=2}^k (p_j - p_{j-1}) V_{n-j} - p_k V_{n-k-1}. \end{aligned}$$

Proof. From the previous Theorem 2.2 we have

$$\begin{aligned} V_n &= (1 + p_1) V_{n-1} + \sum_{j=2}^k (p_j - p_{j-1}) V_{n-j} - p_k V_{n-k-1} \\ (1 + p_1) V_{n-1} &= (1 + p_1)^2 V_{n-2} + (1 + p_1) \sum_{j=2}^k (p_j - p_{j-1}) V_{n-j-1} - (1 + p_1) p_k V_{n-k-2} \\ (1 + p_1)^2 V_{n-2} &= (1 + p_1)^3 V_{n-3} + (1 + p_1)^2 \sum_{j=2}^k (p_j - p_{j-1}) V_{n-j-2} - (1 + p_1)^2 p_k V_{n-k-3} \\ &\dots\dots\dots \\ (1 + p_1)^{n-(k+1)} V_{(k+1)} &= (1 + p_1)^{n-k} V_k + (1 + p_1)^{n-k-1} \sum_{j=2}^k (p_j - p_{j-1}) V_{k+1-j} - (1 + p_1)^{n-k-1} p_k V_0 \end{aligned}$$

After $(k - 1)$ steps, summed these equations side by side, we have

$$\begin{aligned} V_n &= (1 + p_1)^k V_k + (1 + p_1)^{k-1} \left[\sum_{j=2}^k (p_j - p_{j-1}) V_{k-j-(k-1)} - p_k V_{n-2k+2} \right] \\ &+ (1 + p_1)^{k-2} \left[\sum_{j=2}^k (p_j - p_{j-1}) V_{k-j-(k-2)} - p_k V_{n-2k+3} \right] \\ &+ (1 + p_1)^{k-3} \left[\sum_{j=2}^k (p_j - p_{j-1}) V_{k-j-(k-3)} - p_k V_{n-2k+4} \right] + \dots + \sum_{j=2}^k (p_j - p_{j-1}) V_{n-j} - p_k V_{n-k-1}. \end{aligned}$$

which completes the proof of the theorem. $\square$

2.1. Few Initial terms of the Generalized k^{th} – order Sequence.

$$(4) \quad V_n = \left\{ \begin{aligned} &a_1, a_2, a_3 \dots, a_{k-1}, a_k, p_1 a_k + p_2 a_{k-1} + p_3 a_{k-2} + \dots + p_{k-1} a_2 + p_k a_1, \\ &(p_1^2 + p_2) a_k + (p_1 p_2 + p_3) a_{k-1} + (p_1 p_3 + p_4) a_{k-2} + \dots \\ &+ (p_1 p_{k-1} + p_k) a_{k-1} + p_1 p_k a_1, \\ &[p_1 (p_1^2 + p_2) + p_1 p_2 + p_3] a_k + (p_1 (p_1 p_2 + p_3) + p_4 + p_2^2) a_{k-1} \\ &+ (p_2 p_3 + p_1 (p_1 p_3 + p_4) + \dots + p_{k-1}) a_{k-2} + (p_2 p_4 + p_1 (p_1 p_4 + p_5) + p_6) a_3 \\ &+ (p_2 p_{k-1} + p_1 (p_1 p_{k-1} + p_k)) a_2 + (p_2 + p_1^2) p_k a_1 + \dots \end{aligned} \right\}.$$

2.2. First 15 terms of the k^{th} , $k = 8, 7, 6, 5, 4$ orders sequences.

- The first few terms of the 8th-order sequence (OEIS A079262), deduced from the generalization in (1), with initial conditions $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = 0, a_8 = 1$ and $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = p_7 = p_8 = 1$, are given by:

$$\{V_n\}_{n \geq 0} = 0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 4, 8, 16, 32, 64, 128, 255, 509, 1016, 2028, 4048, 8080, \dots$$

- The first few terms of the 7th order sequence (OEIS A122189 [10]) deduced from the generalization in (1), with initial conditions $a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0, a_6 = 0, a_7 = 1$, and $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = p_7 = 1$, deduced from the above generalization are:

$$\{V_n\}_{n \geq 0} = 0, 0, 0, 0, 0, 0, 1, 1, 2, 4, 8, 16, 32, 64, 127, 253, 504, 1004, 2000, 3984, 7936, \dots$$

- First 15 terms of the 6th order sequence (OEIS A001592 [10]) deduced from the generalization in (1), with initial conditions $a_1 = 0, a_2 = 1, a_3 = 1, a_4 = 2, a_5 = 4, a_6 = 8$, and $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = 1$, deduced from the above generalization are:

$$\{V_n\}_{n \geq 0} = 0, 1, 1, 2, 4, 8, 16, 32, 63, 125, 248, 492, 976, 1936, 3880, \dots$$

- First 15 terms of the 6th order sequence (A074584 [10]) deduced from the generalization in (1), with initial conditions $V_1 = 6, V_2 = 1, V_3 = 3, V_4 = 7, V_5 = 15, V_6 = 31$, and $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = 1$, deduced from the above generalization are:

$$\{V_n\}_{n \geq 0} = 6, 1, 3, 7, 15, 31, 63, 120, 239, 475, 943, 1871, 3711, 7359, 14598, \dots$$

- First 15 terms of the sequence, deduced from the generalization in (1), with initial conditions $V_1 = 0, V_2 = 0, V_3 = 0, V_4 = 0, V_5 = 1, V_6 = 1$, and $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11, p_6 = 13$ deduced from the above generalization are:

$$\{V_n\}_{n \geq 0} = 0, 0, 0, 0, 1, 1, 5, 18, 63, 223, 782, 2740, 9635, 33852, 118929, \dots$$

- First 15 terms of the 5th order sequence, deduced from the generalization in (1), with initial conditions $V_1 = 0, V_2 = 1, V_3 = 1, V_4 = 1, V_5 = 2$, and $p_1 = p_2 = p_3 = p_4 = p_5 = 1$, see [11], recurrence relation (1) is known as the Pentanacci sequences. The first few terms of this sequence deduced from the above generalization are:

$$\{V_n\}_{n \geq 0} = 0, 1, 1, 1, 2, 10, 19, 37, 73, 144, 283, 556, 1093, 2149, 4225, \dots$$

- First 15 terms of the 5th order sequence deduced from the generalization in (1), with initial conditions $a_1 = 0, a_2 = 1, a_3 = 1, a_4 = 3, a_5 = 8$, and $p_1 = 2, p_2 = p_3 = p_4 = p_5 = 1$ in (1), it reduces to pentanacci sequence which is also discussed in [11]. The first few terms of the sequence are:

$$\{V_n\}_{n \geq 0} = 0, 1, 1, 3, 8, 21, 55, 143, 373, 973, 2538, 6620, 17267, 45038, 117474, \dots$$

- First 15 terms of the 5th order sequence If we substitute the initial conditions $a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 1, a_5 = 1$, and $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11$ in (1), it reduces to Pentanacci sequence which is also discussed in [11]. The first few terms of the sequence are:

$$\{V_n\}_{n \geq 0} = 0, 1, 1, 3, 8, 21, 55, 143, 373, 973, 2538, 6620, 17267, 45038, 117474, \dots$$

- First 15 terms of the 4th order sequence deduced from the generalization in (1), with initial conditions $a_1 = 0, a_2 = 1, a_3 = 1, a_4 = 2$, and $p_1 = p_2 = p_3 = p_4 = 1$, see [11], recurrence relation (1) is known as the Tetranacci sequences. The first few terms of this sequence deduced from the above generalization are:

$$\{V_n\}_{n \geq 0} = \{V_n\} = 0, 1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, 2872, \dots$$

- First 15 terms of the 4th order sequence deduced from the generalization in (1), with initial conditions $a_1 = 4, a_2 = 1, a_3 = 3, a_4 = 7$, and $p_1 = p_2 = p_3 = p_4 = 1$, reduces to tetranacci sequence which is also discussed in [11]. The terms of the sequence are:

$$\{V_n\}_{n \geq 0} = \{K_n\} = 4, 1, 3, 7, 15, 26, 51, 99, 191, 367, 708, 1365, 2631, 5071, 9775, \dots$$

3. GENERATING FUNCTION

Theorem 3.1. If $V(x)$ is generating function of the sequence defined in the recursion relation (1)

$$V_n(a_j, p_j) = \sum_{j=1}^k p_j V_{n-j}, n \geq k+1,$$

then

$$(5) \quad V(x) = \frac{f(x)}{1 - p_1x - p_2x^2 - p_3x^3 - \dots - p_kx^k},$$

where

$$f(x) = A_1 + A_2x + A_3x^2 + \dots + A_{k-1}x^{k-2} + A_kx^{k-1},$$

$$A_1 = V_1, A_2 = V_2 - p_1V_1, A_3 = V_3 - p_1V_2 - p_2V_1, A_4 = V_4 - p_1V_3 - p_2V_2 - p_3V_1,$$

$$A_5 = V_5 - p_1V_4 - p_2V_3 - p_3V_2 - p_4V_1, A_6 = V_6 - p_1V_5 - p_2V_4 - p_3V_3 - p_4V_2 - p_5V_1$$

$$A_k = V_k - p_1V_{k-1} - p_2V_{k-2} - p_3V_{k-3} - \dots - p_{k-1}V_1.$$

Proof. Since

$$(6) \quad V(x) = \sum_{n=0}^{\infty} V_n x^n,$$

$$(7) \quad p_j x^j V(x) = p_j \sum_{n=j}^{\infty} V_{n-j} x^n, \quad j = 1, 2, \dots, k.$$

Now employing (6)-(7), we obtain generating function

$$V(x) [1 - p_1x - p_2x^2 - p_3x^3 - \dots - p_{k-1}x^{k-1} - p_kx^k] = f(x).$$

Therefore generating function has the rational form:

$$V(x) = \frac{A_1 + A_2x + A_3x^2 + \dots + A_{k-1}x^{k-2} + A_kx^{k-1}}{1 - p_1x - p_2x^2 - \dots - p_{k-1}x^{k-1} - p_kx^k},$$

Hence $V(x)$ is the generating function of the sequence $\{V_n\}$. □

3.1. Odd and Even terms Generating Functions.

Theorem 3.2. *The generating functions of even $V_{2n}(x)$ and odd $V_{2n+1}(x)$ terms of the Generalized k^{th} order generalized hexanacci sequences are*

$$(8) \quad V_{2n}(x) = \frac{NumE}{Denom},$$

and

$$(9) \quad V_{2n}(x) = \frac{NumO}{D}$$

where

$$\begin{aligned} NumE = & A_1 + (A_3 - p_1A_2 - p_2A_3)x + (A_5 - p_1A_4 - p_2A_3 - p_3A_2 + p_4A_1)x^2 + \dots \\ & + (-p_kA_{k-5} + p_{k-1}A_{k-4} - p_{k-2}A_{k-3} + p_{k-3}A_{k-2} - p_{k-4}A_{k-1} + p_{k-5}A_k)x^{k-3} \\ & + (p_{k-3}A_k - p_{k-2}A_{k-1} + p_{k-1}A_{k-2} - p_kA_{k-3})x^{k-2} + (p_{k-1}A_k - p_kA_{k-1})x^{k-1} \end{aligned}$$

$$\begin{aligned} NumO = & (A_2 + p_1A_1) + (A_4 + p_1A_3 - p_2A_2 + p_3A_1)x \\ & + (A_6 + p_1A_5 - p_2A_4 + p_3A_3 - p_4A_2 + p_5A_1)x^2 + \dots \\ & + (-p_kA_{k-4} + p_{k-1}A_{k-3} - p_{k-2}A_{k-2} + p_{k-3}A_{k-1} - p_{k-4}A_k)x^{k-3} \\ & + (-p_{k-2}A_k + p_{k-1}A_{k-1} - p_kA_{k-2})x^{k-2} - p_kA_kx^{k-1} \end{aligned}$$

$$\begin{aligned} D = & 1 - (p_1^2 + 2p_2)x - (2p_1p_3 - p_2^2 + 2p_4)x^2 + \dots \\ & + (p_{k-2}^2 + 2p_{k-4}p_k - 2p_{k-2}p_{k-1})x^{k-2} + (2p_{k-2}p_k - p_{k-1}^2)x^{k-1} + p_k^2x^k \end{aligned}$$

Remark 1. *The generating functions for even $V_{2n}(x)$ and odd $V_{2n+1}(x)$ terms for the Generalized lower orders sequences 4^{th} , 5^{th} , 6^{th} orders sequences can be obtained from Theorem (3.2).*

3.2. Generating function of the lower orders sequences k^{th} , $k = 4, 5, 6 \dots$

- Generating function of the k^{th} , $k = 4$ (fourth) order sequence is

$$V_n(x) = \frac{f(x)}{1 - p_1x - p_2x^2 - p_3x^3 - p_4x^4}$$

where $f(x) = V_1 + (V_2 - p_1V_1)x + (V_3 - p_1V_2 - p_2V_1)x^2 + (V_4 - p_1V_3 - p_2V_2 - p_3V_1)x^3$.

- Generating function of the k^{th} , $k = 5$ (fifth) order sequence is

$$V(x) = \frac{f(x)}{1 - p_1x - p_2x^2 - p_3x^3 - p_4x^4 - p_5x^5}$$

where

$$f(x) = A_1 + A_2x + A_3x^2 + A_4x^3 + A_5x^4.$$

- Generating function of the k^{th} , $k = 6$ (sixth) order sequence is

$$V(x) = \frac{f(x)}{1 - p_1x - p_2x^2 - p_3x^3 - p_4x^4 - p_5x^5 - p_6x^6},$$

where

$$f(x) = A_1 + A_2x + A_3x^2 + A_4x^3 + A_5x^4 + A_6x^5,$$

$$A_1 = V_1, \quad A_2 = V_2 - p_1V_1, \quad A_3 = V_3 - p_1V_2 - p_2V_1, \quad A_4 = V_4 - p_1V_3 - p_2V_2 - p_3V_1,$$

$$A_5 = V_5 - p_1V_4 - p_2V_3 - p_3V_2 - p_4V_1, \quad A_6 = V_6 - p_1V_5 - p_2V_4 - p_3V_3 - p_4V_2 - p_5V_1$$

4. MAIN RESULTS

Theorem 4.1. General form of the Binet's formula for the generalized k sequence defined

$$V_n = \sum_{j=1}^k p_j V_{n-j} \text{ is}$$

$$(10) \quad V_n(x) = \sum_{j=1}^k \left[\frac{A_1 \alpha_j^{k-1} + A_2 \alpha_j^{k-2} + A_3 \alpha_j^{k-3} + \cdots + A_k}{\prod_{\substack{1 \leq i \leq k \\ i \neq j}} (\alpha_j - \alpha_i)} \right] \alpha_j^n.$$

Proof. Consider partial fraction decomposition of the right-hand side of the generating function of the sequence defined as

$$\begin{aligned} V(x) &= \frac{A_1 + A_2 x + A_3 x^2 + A_4 x^3 + \cdots + A_k x^{k-1}}{1 - p_1 x - p_2 x^2 - p_3 x^3 - p_4 x^4 - \cdots - p_k x^k} \\ &= \frac{A_1 + A_2 x + A_3 x^2 + A_4 x^3 + \cdots + A_k x^{k-1}}{(1 - \alpha_1 x)(1 - \alpha_2 x)(1 - \alpha_3 x)(1 - \alpha_4 x) \cdots (1 - \alpha_k x)}, \end{aligned}$$

where

$$\begin{aligned} A_1 &= V_1, \quad A_2 = V_2 - p_1 V_1, \quad A_3 = V_3 - p_1 V_2 - p_2 V_1, \quad A_4 = V_4 - p_1 V_3 - p_2 V_2 - p_3 V_1, \\ A_5 &= V_5 - p_1 V_4 - p_2 V_3 - p_3 V_2 - p_4 V_1, \quad A_6 = V_6 - p_1 V_5 - p_2 V_4 - p_3 V_3 - p_4 V_2 - p_5 V_1, \end{aligned}$$

and $\alpha_i, i = 1, 2, \dots, k$ are the roots of the equation $1 - p_1 x - p_2 x^2 - p_3 x^3 - \cdots - p_k x^k = 0$.

On simplifying the above equation, we obtained the following formula

$$V_n(x) = \sum_{j=1}^k \left[\frac{A_1 \alpha_j^{k-1} + A_2 \alpha_j^{k-2} + A_3 \alpha_j^{k-3} + \cdots + A_k}{\prod_{\substack{1 \leq i \leq k \\ i \neq j}} (\alpha_j - \alpha_i)} \right] \alpha_j^n$$

□

Theorem 4.2. For the k^{th} order generalized linear recurrence relation (1), the n^{th} k -generalized number is

$$(11) \quad V_n = \sum_{j=0}^{k-1} \left[\frac{\left[\sum_{m=1}^k A_m \alpha_j^{k-m} \right] (\alpha_j^{k+1} - \alpha_j^k)}{(1 + p_1) \alpha_j^k + \sum_{m=2}^k m (p_m - p_{m-1}) \alpha_j^{k-(m-1)} - p_k (k+1)} \right] \alpha_j^{n-1}.$$

Proof. The generating function (3.2) of linear recurrence relation (1), can be expressed as:

$$(12) \quad V(x) = \frac{\sum_{j=0}^{k-1} A_j x^j}{1 - \sum_{j=1}^k p_j x^j} = \frac{\sum_{j=0}^{k-1} A_j x^j}{h_k(x)},$$

where $h_k(x) = 1 - \sum_{j=1}^k p_j x^j$. Since α_j are roots of $x^k - \sum_{j=1}^k p_j x^{k-j} = 0$, So $\frac{1}{\alpha_j}$ are roots of

$$1 - \sum_{j=1}^k p_j x^j = 0.$$

(Miller [9]) shown that the zeros of $h_k\left(\frac{1}{x}\right)$ are simple, consequently the zeros of $h_k(x)$ are simple. Therefore,

$$(13) \quad V(x) = \frac{\sum_{j=0}^{k-1} A_j x^j}{\prod_{j=0}^{k-1} (1 - \alpha_j x)}.$$

Expressing right-hand side of (13) by using partial fraction decomposition, we have

$$(14) \quad V(x) = \sum_{j=0}^{k-1} \frac{R_j}{(1 - \alpha_j x)},$$

where

$$R_j = \frac{(A_1 \alpha_j^{k-1} + A_2 \alpha_j^{k-2} + A_3 \alpha_j^{k-3} + \cdots + A_{k-1} \alpha_j + A_k)}{\prod_{\substack{m=0 \\ m \neq j}}^{k-1} (\alpha_j - \alpha_m)}$$

Further,

$$\begin{aligned} R_j &= \frac{\left(A_1 \alpha_j^{k-2} + A_2 \alpha_j^{k-3} + A_3 \alpha_j^{k-4} + \cdots + A_{k-1} + \frac{A_k}{\alpha_j}\right)}{\prod_{\substack{m=0 \\ m \neq j}}^{k-1} \left(1 - \alpha_m \left(\frac{1}{\alpha_j}\right)\right)} \\ &= \frac{\left(A_1 \alpha_j^{k-2} + A_2 \alpha_j^{k-3} + A_3 \alpha_j^{k-4} + \cdots + A_{k-1} + \frac{A_k}{\alpha_j}\right) (-\alpha_j)}{h_k' \left(\frac{1}{\alpha_j}\right)} \end{aligned}$$

Now (14) become

$$V(x) = \frac{\sum_{j=0}^{k-1} A_j x^j}{\prod_{j=0}^{k-1} (1 - \alpha_j x)} = \sum_{j=0}^{k-1} \frac{\left(A_1 \alpha_j^{k-2} + A_2 \alpha_j^{k-3} + A_3 \alpha_j^{k-4} + \cdots + A_{k-1} + \frac{A_k}{\alpha_j}\right) (-\alpha_j)}{h_k' \left(\frac{1}{\alpha_j}\right)} \frac{1}{(1 - \alpha_j x)}$$

This implies

$$V(x) = \sum_{i=0}^{\infty} \left[\sum_{j=0}^{k-1} \frac{\left(A_1 \alpha_j^{k-2} + A_2 \alpha_j^{k-3} + A_3 \alpha_j^{k-4} + \cdots + A_{k-1} + \frac{A_k}{\alpha_j}\right) (-\alpha_j)}{h_k' \left(\frac{1}{\alpha_j}\right)} \alpha_j^i x^i \right]$$

We have

$$V_n = \sum_{j=0}^{k-1} \frac{\left(A_1 \alpha_j^{k-2} + A_2 \alpha_j^{k-3} + A_3 \alpha_j^{k-4} + \cdots + A_{k-1} + \frac{A_k}{\alpha_j}\right) (-\alpha_j) (\alpha_j)^n}{h_k' \left(\frac{1}{\alpha_j}\right)},$$

Since

$$\begin{aligned} h_k' \left(\frac{1}{\alpha_j}\right) &= - \sum_{m=1}^k m p_m \left(\frac{1}{\alpha_j}\right)^{m-1} \\ 1 - \sum_{j=1}^k p_j \left(\frac{1}{\alpha_j}\right)^{m-1} &= 0, \quad 1 \leq j \leq k \end{aligned}$$

Therefore

$$(15) \quad V_n = \sum_{j=0}^{k-1} \frac{\left(A_1 \alpha_j^{k-2} + A_2 \alpha_j^{k-3} + A_3 \alpha_j^{k-4} + \cdots + A_{k-1} + \frac{A_k}{\alpha_j} \right) (\alpha_j) (\alpha_j)^n \left(1 - \frac{1}{\alpha_j} \right)}{(1 + p_1) + 2(p_2 - p_1) \left(\frac{1}{\alpha_j} \right) + 3(p_3 - p_2) \left(\frac{1}{\alpha_j} \right)^2 + \cdots - (k+1)p_k \left(\frac{1}{\alpha_j} \right)^k}.$$

On further simplification we have

$$(16) \quad V_n = \sum_{j=0}^{k-1} \frac{\left(A_1 \alpha_j^{k-1} + A_2 \alpha_j^{k-2} + A_3 \alpha_j^{k-3} + \cdots + A_{k-1} \alpha_j + A_k \right) \left(\alpha_j^{k+1} - \alpha_j^k \right)}{(1 + p_1) \alpha_j^k + 2(p_2 - p_1) \alpha_j^{k-1} + 3(p_3 - p_2) \alpha_j^{k-2} + \cdots + k(p_k - p_{k-1}) \alpha_j - (k+1)p_k} \alpha_j^{n-1}.$$

$$(16) \quad V_n = \sum_{j=0}^{k-1} \left[\frac{\sum_{m=1}^k A_m \alpha_j^{k-m} (\alpha_j^{k+1} - \alpha_j^k)}{(1 + p_1) \alpha_j^k + \sum_{m=2}^k m(p_m - p_{m-1}) \alpha_j^{k-(m-1)} - p_k(k+1)} \right] \alpha_j^{n-1}.$$

Hence equation (16) is the generalized Binet type formula for the k^{th} order generalized recurrence relation. $\square$

Corollary 4.3. If $p_i = 1$, for all i , $V_i = 0$ ($0 \leq i \leq k-1$) and $V_k = 1$, then generalized Binet formula (16) reduces to

$$(17) \quad V_n = \sum_{j=0}^{k-1} \frac{(\alpha_j^{k+1} - \alpha_j^k)}{2\alpha_j^k - (k+1)} \alpha_j^n,$$

which is the result obtained by Spickerman and Joyner ([12]).

Corollary 4.4. When $k = 2$, then generalized Binet formula reduces to (16),

$$(18) \quad V_n = \frac{(A_1 \alpha + A_2) (\alpha^3 - \alpha^2)}{(1 + p_1) \alpha + 2(p_2 - p_1) - 3p_2} \alpha^{n-1} + \frac{(A_1 \beta + A_2) (\beta^3 - \beta^2)}{(1 + p_1) \beta + 2(p_2 - p_1) - 3p_2} \beta^{n-1},$$

α and β are roots of the $x^2 - p_1 x - p_2 = 0$.

Corollary 4.5. When $k = 2$, $p_1 = p_2 = 1$ and $V_1 = 0$, $V_2 = 1$ implies $A_1 = 0$, $A_2 = 1$, then from generalized Binet formula (16), we obtain

$$(19) \quad V_n = \frac{(\alpha - 1)}{2\alpha - 3} \alpha^{n+1} + \frac{(\beta - 1)}{2\beta - 3} \beta^{n+1},$$

where α and β are roots of the equation $x^2 - x - 1 = 0$.

Theorem 4.6. The generalized Binet formula V_n obtained in theorem (4.2) can be expressed as

$$(20) \quad V_n = \sum_{j=0}^{k-1} \left[\frac{\left(\sum_{m=1}^k A_m \alpha_j^{k-m} \right) (\alpha_j - 1)}{(1 + p_1) + \sum_{m=2}^k m (p_m - p_{m-1}) \alpha_j^{(m-1)} + (k+1) \left[\alpha_j - (p_1 + 1) - \sum_{m=2}^k (p_m - p_{m-1}) \alpha_j^{-(m-1)} \right]} \right] \alpha_j^{n-1}.$$

Proof. Since we have the result from Theorem (4.2)

$$V_n = \sum_{j=0}^{k-1} \frac{\left(A_1 \alpha_j^{k-1} + A_2 \alpha_j^{k-2} + A_3 \alpha_j^{k-3} + \dots + A_{k-1} \alpha_j + A_k \right) (\alpha_j^{k+1} - \alpha_j^k)}{(1 + p_1) \alpha_j^k + 2 (p_2 - p_1) \alpha_j^{k-1} + 3 (p_3 - p_2) \alpha_j^{k-2} + \dots + k (p_k - p_{k-1}) \alpha_j - (k+1) p_k} \alpha_j^{n-1}.$$

Rewrite above equation in short form, we have

$$(21) \quad V_n = \sum_{j=0}^{k-1} \left[\frac{\sum_{m=1}^k A_m \alpha_j^{k-m} (\alpha_j^{k+1} - \alpha_j^k)}{(1 + p_1) \alpha_j^k + \sum_{m=2}^k m (p_m - p_{m-1}) \alpha_j^{k-(m-1)} - p_k (k+1)} \right] \alpha_j^{n-1}.$$

This equation (21) can be written as

$$(22) \quad V_n = \sum_{j=0}^{k-1} \left[\frac{\left(\sum_{m=1}^k A_m \alpha_j^{k-m} \right) (\alpha_j - 1)}{(1 + p_1) + \sum_{m=2}^k m (p_m - p_{m-1}) \alpha_j^{(m-1)} - (k+1) p_k \alpha_j^{-k}} \right] \alpha_j^{n-1}.$$

Now the from characteristic equation $x^k - \sum_{j=1}^k p_j x^{k-j} = 0$ of the recurrence relation we have

$\alpha_j^k - \sum_{j=1}^k p_j \alpha_j^{k-j} = 0$, where α_j , $(j = 1, 2 \dots k)$ are roots of the characteristic equation.

$$\left(\alpha_j^k - \sum_{j=1}^k p_j \alpha_j^{k-j} \right) (1 - \alpha_j) = 0.$$

We obtain

$$\alpha_j^k \left[\alpha_j - (p_1 + 1) - \sum_{m=2}^k (p_m - p_{m-1}) \alpha_j^{-(m-1)} \right] = -p_k.$$

This equation can be expressed as

$$(23) \quad \left[\alpha_j - (p_1 + 1) - \sum_{m=2}^k (p_m - p_{m-1}) \alpha_j^{-(m-1)} \right] = -p_k \alpha_j^{-k}.$$

From (21)- (23), we obtain

$$(24) \quad V_n = \sum_{j=0}^{k-1} \left[\frac{\left(\sum_{m=1}^k A_m \alpha_j^{k-m} \right) (\alpha_j - 1)}{(1 + p_1) + \sum_{m=2}^k m (p_m - p_{m-1}) \alpha_j^{(m-1)} + (k+1) \left[\alpha_j - (p_1 + 1) - \sum_{m=2}^k (p_m - p_{m-1}) \alpha_j^{-(m-1)} \right]} \right] \alpha_j^{n-1}.$$

which is another form of generalized Binet formula for generalized kth order recurrence relations. Hence the theorem. $\square$

Corollary 4.7. If $p_i = 1$, for all i , $V_i = 0$ ($0 \leq i \leq k-1$) and $V_k = 1$, then the generalized Binet formula obtained in Theorem (4.2) become

$$(25) \quad V_n = \sum_{j=0}^{k-1} \left[\frac{\alpha_j - 1}{2 + (k+1)(\alpha_j - 2)} \right] \alpha_j^{n-1},$$

which is the same result obtained by Dresden and Du ([1]).

Corollary 4.8. When $k = 2$, then the generalized Binet formula obtained in Theorem (4.2) become

$$(26) \quad V_n = \sum_{j=1}^2 \left[\frac{(A_1 \alpha_j + A_2) (\alpha_j - 1)}{(1 + p_1) + 2(p_2 - p_2) \alpha_j + 3[\alpha_j - (p_1 + 1) - 2(p_2 - p_2) \alpha_j^{-1}]} \right] \alpha_j^{n-1}.$$

α_1 and α_2 are roots of the $x^2 - p_1 x - p_2 = 0$.

Corollary 4.9. When $k = 2$ and if $p_i = 1$, for all i , $V_i = 0$ ($0 \leq i \leq k-1$), $V_k = 1$, then the generalized Binet formula obtained in Theorem (4.2) become

$$(27) \quad V_n = \frac{(\alpha - 1)}{2 + 3(\alpha - 2)} \alpha^{n-1} + \frac{(\beta - 1)}{2 + 3(\beta - 2)} \beta^{n-1},$$

α and β are roots of the $x^2 - x - 1 = 0$.

5. MATRIX REPRESENTATION OF k -TH ORDER GENERALIZED RECURRENCE

For $V_n = \sum_{j=1}^k p_j V_{n-j}$, where $p_k \neq 0$,

$$(28) \quad M = \begin{pmatrix} p_1 & p_2 & \cdots & p_{k-1} & p_k \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

then

$$(29) \quad M^n = \begin{pmatrix} p_2 T_{n+(k-2)} + p_3 T_{n+(k-3)} & p_{k-1} T_{n+(k-2)} & & & \\ T_{n+(k-1)} & + \cdots + p_k T_{n+(k-k)} & \cdots & + p_k T_{n+(k-3)} & p_k T_{n+(k-2)} \\ & p_2 T_{n+(k-3)} + p_3 T_{n+(k-4)} & & p_{k-1} T_{n+(k-3)} & \\ T_{n+(k-2)} & + \cdots + p_k T_{n+(k-(k-1))} & \cdots & + p_k T_{n+(k-4)} & p_k T_{n+(k-3)} \\ & p_2 T_{n+(k-4)} + p_3 T_{n+(k-5)} & & p_{k-1} T_{n+(k-4)} & \\ T_{n+(k-3)} & + \cdots + p_k T_{n+(k-(k-2))} & \cdots & + p_k T_{n+(k-5)} & p_k T_{n+(k-4)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & p_k T_{n+(k-(k-1))} + p_3 T_{n+(k-2)} & & p_{k-1} T_{n+(k-(k-1))} & \\ T_{n+(k-k)} & + \cdots + p_k T_{n+(k-(k-4))} & \cdots & + p_k T_{n+(k-(k-2))} & p_k T_{n+(k-(k-1))} \end{pmatrix},$$

$$\text{where } \begin{pmatrix} T_{n+(k-1)} \\ T_{n+(k-2)} \\ T_{n+(k-3)} \\ \cdots \\ T_{n+(k-k)} \end{pmatrix} = \begin{pmatrix} p_1 & p_2 & \cdots & p_{k-1} & p_k \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}^{n-k} \begin{pmatrix} p_1^{k-1} + (k-3)p_1^{k-4}p_2 + \cdots \\ \cdots \\ p_1 \\ 1 \\ 0 \end{pmatrix}, \quad n \geq k.$$

5.1. Properties of M^n and M . For

$$M = \begin{pmatrix} p_1 & p_2 & \cdots & p_{k-1} & p_k \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

- The Characteristic equation of matrix M is

$$(30) \quad f(\lambda) = \lambda^k - p_1 \lambda^{k-1} - p_2 \lambda^{k-2} - p_3 \lambda^{k-3} - \cdots - p_{k-1} \lambda - p_k.$$

- For matrix M and all integers $n \geq 1$, the i -th row of M^n is equal to the $(i+1)$ -th row of M^{n+1} . If

$$M^n = \left(m_{i,j}^{(n)} \right)_{k \times k},$$

then

$$(31) \quad m_{i,j}^{(n)} = p_1 m_{i,j}^{(n-1)} + p_2 m_{i,j}^{(n-2)} + p_3 m_{i,j}^{(n-3)} + \cdots + p_k m_{i,j}^{(n-k)}, \quad 1 \leq i, j \leq k.$$

•

$$(32) \quad \text{Trace}(M^n) = \sum_{i=1}^k p_i m_{i,i}^{(n-i)}.$$

$$(33) \quad \det(M^n) = (-1)^{k-1} (p_k)^n.$$

• Inverse of M

$$(34) \quad M^{-1} = \frac{1}{p_k} \begin{pmatrix} 0 & -p_k & 0 & \cdots & 0 \\ 0 & 0 & -p_k & \cdots & 0 \\ \vdots & \vdots & \vdots & & \cdots \\ 0 & 0 & 0 & \vdots & -p_k \\ -1 & p_1 & p_2 & \cdots & p_k \end{pmatrix}.$$

Remark 2. *The properties of a square matrix M (of size $k \times k$) and its powers M^n form the bedrock of several heavy-hitting applications like cybersecurity and cryptography and various other applications.*

6. CONCLUSION

In this paper, we have delved into the properties of a k -th order generalized recurrence relation. By introducing a generalized generating function, we have successfully derived a generalized explicit Binet formula. This formula serves as a unifying framework, encompassing various existing Binet formulas as special cases. The flexibility of our approach is evident in its ability to generate a wide range of sequences, including the well-known Fibonacci, Tribonacci, and Tetranacci sequences, by adjusting the parameters of the recurrence relation. This underscores the potential of our method to explore and analyze a vast array of integer sequences. Given that these higher-order linear recurrences can be compactly represented via a $k \times k$ state-transition matrix M , the foundational properties of M and its powers M^n open valuable avenues for practical applications. Future research directions may involve investigating the asymptotic behavior of these sequences, exploring their combinatorial interpretations, and applying their matrix algebraic properties to advanced fields such as cybersecurity, cryptographic key-exchange protocols, and network vulnerability modeling.

Conflicts of Interest: There is no conflicts of interest.

REFERENCES

- [1] G. P. B. Dresden, Z. Du, A Simplified Binet Formula for k -Generalized Fibonacci Numbers, J. Integer Seq. 17 (2014) Article 14.4.7.
- [2] D. E. Ferguson, An expression for generalized Fibonacci numbers, Fibonacci Quart. 4 (1966) 270–273.
- [3] I. Flores, Direct calculation of k -generalized Fibonacci numbers, Fibonacci Quart. 5 (1967) 259–266.
- [4] H. Gaba, Generalized Fibonacci k -sequences, Fibonacci Quart. 8 (1970) 31–38.
- [5] V. E. Hoggat, Fibonacci and Lucas Numbers, Mifflin, California, 1969.
- [6] F. T. Howard, C. Cooper, Some identities for r -Fibonacci numbers, Fibonacci Quart. 49 (2011) 231–242.
- [7] R. Kahkeshani, M. Arab, A Generalized Binet Formula That Counts the Tilings of a $(2 \times n)$ -board, Integer 18 (2018) 17.5.4.
- [8] G.-Y. Lee, S.-G. Lee, J.-S. Kim, H.-K. Shin, The Binet formula and representations of k -generalized Fibonacci numbers, Fibonacci Quart. 39 (2001) 158–164.
- [9] M. D. Miller, On generalized Fibonacci numbers, Am. Math. Mon. 78 (1971) 1108–1109.
- [10] N. J. A. Sloane, The on-line encyclopedia of integer sequences, <http://oeis.org/>.
- [11] Y. Soykan, Binomial Transform of the Generalized Hexanacci Sequence, Asian Basic and Applied Research Journal Volume 15 (2022), 1–20.
- [12] W. R. Spickerman, R. N. Joyner, Binet's formula for the recursive sequence of order k , Fibonacci Quart. 22 (1984) 327–331.

- [13] S. Vajda, Fibonacci and Lucas numbers, and the Golden Section. Theory and applications, Ellis Horwood limited, 1989.
- [14] K. L. Verma, A comprehensive generalization of classical Fibonacci sequences, Binet formula and Identities, J. Appl. Pure Math. 6 (2024) 283–299.
- [15] K. L. Verma, The mathematics of generalized Fibonacci sequences: Binet’s formula and identities, Math. Morav. 29 (2025) 113–124.
- [16] K. L. Verma, On the matrix representation of generalized Fibonacci sequences and applications, J. Math. Sci. Comput. Math. 6 (2025) 134–144.
- [17] N. N. Vorobiev, Fibonacci Numbers, Birkhäuser, Basel, 2003.
- [18] A. Wolfram, Solving generalized Fibonacci recurrences, Fibonacci Quart. 36 (1998) 129–145.