

# STRONG CONVERGENCE OF A VISCOSITY METHOD FOR RELAXED COCOERCIVE VARIATIONAL INEQUALITIES AND A FINITE FAMILY OF NONEXPANSIVE MAPPINGS

QING YUAN

**ABSTRACT.** In this paper, we introduce a viscosity-type iterative algorithm with two projection steps for finding a common element of the solution set of a variational inequality involving a relaxed  $(u, v)$ -cocoercive and  $\mu$ -Lipschitz continuous mapping and the common fixed point set of a finite family of nonexpansive mappings in a real Hilbert space. The finite family of nonexpansive mappings is treated by means of the  $W$ -mapping. Under suitable conditions on the control sequences, we prove that the sequence generated by the algorithm converges strongly to a common element  $q$ , which is the unique solution of the variational inequality  $\langle (I - f)q, p - q \rangle \geq 0$  for all  $p$  in the common solution set, where  $f$  is a contraction.

## 1. INTRODUCTION

Let  $H$  be a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\| \cdot \|$ , let  $C$  be a nonempty closed convex subset of  $H$ , and let  $P_C$  denote the metric projection of  $H$  onto  $C$ . Given a mapping  $A : C \rightarrow H$ , the classical variational inequality problem, denoted by  $VI(C, A)$ , is to find  $u \in C$  such that

$$(1.1) \quad \langle Au, v - u \rangle \geq 0, \quad \forall v \in C.$$

We also use  $VI(C, A)$  for the set of solutions of (1.1).

Variational inequalities provide a unified and powerful framework for a wide class of problems arising in economics, finance, image reconstruction, ecology, transportation and networks; see [1–8] and the references therein. Many important problems in nonlinear analysis and optimization, such as Nash equilibrium problems, fixed point problems, complementarity problems, vector optimization problems, saddle point problems and problems of game theory, can be formulated as variational inequalities.

For a mapping  $T : C \rightarrow C$ , we denote by  $F(T) = \{x \in C : Tx = x\}$  the set of fixed points of  $T$ . Recall that  $T$  is said to be nonexpansive if  $\|Tx - Ty\| \leq \|x - y\|$  for all  $x, y \in C$ . The problem of finding a common element of the set of fixed points of a nonexpansive mapping and the solution set of a variational inequality has been studied extensively. For instance, Iiduka and Takahashi [10] considered a  $\tau$ -cocoercive mapping  $A$  (see Definition 2.1) and a nonexpansive mapping  $S : C \rightarrow C$ , and proposed the iterative scheme

$$(1.2) \quad x_0 = x \in C, \quad x_{n+1} = \alpha_n x + (1 - \alpha_n)SP_C(x_n - \eta_n Ax_n), \quad n \geq 0,$$

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SCHOOL OF MATHEMATICS AND STATISTICS, LINYI UNIVERSITY, LINYI 276000, SHANDONG, CHINA

Email address: [yuanqing@lyu.edu.cn](mailto:yuanqing@lyu.edu.cn).

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where  $\{\alpha_n\} \subset (0, 1)$  and  $\{\eta_n\} \subset (0, 2\tau)$ . Under suitable conditions on these parameters, they proved that  $\{x_n\}$  converges strongly to an element of  $F(S) \cap VI(C, A)$ .

Motivated by these results, we consider in this paper a relaxed  $(u, v)$ -cocoercive and Lipschitz continuous mapping  $A$  together with a finite family of nonexpansive mappings  $T_1, T_2, \dots, T_N$ . To treat the finite family, we use the  $W$ -mapping, which is defined as follows. For each  $n \geq 1$ , let  $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,N} \in (0, 1)$  and define

$$\begin{aligned}
 (1.3) \quad & U_{n,0} = I, \\
 & U_{n,1} = \lambda_{n,1}T_1U_{n,0} + (1 - \lambda_{n,1})I, \\
 & U_{n,2} = \lambda_{n,2}T_2U_{n,1} + (1 - \lambda_{n,2})I, \\
 & \vdots \\
 & U_{n,N-1} = \lambda_{n,N-1}T_{N-1}U_{n,N-2} + (1 - \lambda_{n,N-1})I, \\
 & W_n = U_{n,N} = \lambda_{n,N}T_NU_{n,N-1} + (1 - \lambda_{n,N})I.
 \end{aligned}$$

The mapping  $W_n$  is called the  $W$ -mapping generated by  $T_1, T_2, \dots, T_N$  and  $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,N}$ . Since each  $T_i$  is nonexpansive, so is each  $U_{n,k}$ ; in particular,  $W_n$  is nonexpansive. Moreover,  $F(W_n) = \bigcap_{i=1}^N F(T_i)$ ; see Lemma 2.5 below.

We introduce a viscosity-type algorithm with two projection steps for finding a common element of  $\bigcap_{i=1}^N F(T_i)$  and  $VI(C, A)$ , and prove a strong convergence theorem under mild conditions on the parameters.

## 2. PRELIMINARIES

Throughout this paper, we write  $x_n \rightarrow x$  and  $x_n \rightharpoonup x$  for the strong and the weak convergence of  $\{x_n\}$  to  $x$ , respectively.

**Definition 2.1.** Let  $A : C \rightarrow H$  be a mapping.

- (i)  $A$  is said to be *monotone* if  $\langle Ax - Ay, x - y \rangle \geq 0$  for all  $x, y \in C$ .
- (ii)  $A$  is said to be  $\mu$ -*Lipschitz continuous* if there exists a constant  $\mu > 0$  such that  $\|Ax - Ay\| \leq \mu\|x - y\|$  for all  $x, y \in C$ .
- (iii)  $A$  is said to be  $\tau$ -*cocoercive* if there exists a constant  $\tau > 0$  such that

$$\langle Ax - Ay, x - y \rangle \geq \tau\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

By the Cauchy–Schwarz inequality, every  $\tau$ -cocoercive mapping is  $1/\tau$ -Lipschitz continuous.

- (iv)  $A$  is said to be *relaxed  $(u, v)$ -cocoercive* if there exist two constants  $u, v > 0$  such that

$$(2.1) \quad \langle Ax - Ay, x - y \rangle \geq (-u)\|Ax - Ay\|^2 + v\|x - y\|^2, \quad \forall x, y \in C.$$

A set-valued mapping  $T : H \rightarrow 2^H$  is said to be monotone if  $\langle x - y, f - g \rangle \geq 0$  whenever  $f \in Tx$  and  $g \in Ty$ . A monotone mapping  $T$  is said to be maximal if its graph  $G(T)$  is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping  $T$  is maximal if and only if, for  $(x, f) \in H \times H$ ,  $\langle x - y, f - g \rangle \geq 0$  for every  $(y, g) \in G(T)$  implies  $f \in Tx$ . Let  $A : C \rightarrow H$  be monotone and continuous, and let

$N_C v = \{w \in H : \langle v - y, w \rangle \geq 0, \forall y \in C\}$  be the normal cone to  $C$  at  $v \in C$ . Define

$$(2.2) \quad T v = \begin{cases} A v + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then  $T$  is maximal monotone, and  $0 \in T v$  if and only if  $v \in VI(C, A)$ ; see [9].

We shall use the following well-known facts on the metric projection. For  $x \in H$  and  $z \in C$ ,

$$(2.3) \quad z = P_C x \iff \langle x - z, y - z \rangle \leq 0, \quad \forall y \in C,$$

and  $P_C$  is firmly nonexpansive, that is,

$$(2.4) \quad \|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y \rangle, \quad \forall x, y \in H.$$

It follows from (2.3) that, for any  $s > 0$ ,  $u \in VI(C, A)$  if and only if  $u = P_C(u - sAu)$ . Moreover, in a Hilbert space, for all  $x_1, x_2, x_3 \in H$  and  $t_1, t_2, t_3 \in [0, 1]$  with  $t_1 + t_2 + t_3 = 1$ ,

$$(2.5) \quad \|t_1 x_1 + t_2 x_2 + t_3 x_3\|^2 \leq t_1 \|x_1\|^2 + t_2 \|x_2\|^2 + t_3 \|x_3\|^2,$$

and, for all  $x, y \in H$ ,

$$(2.6) \quad \|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle.$$

**Lemma 2.2** (Opial [14]). *Let  $\{x_n\}$  be a sequence in  $H$  with  $x_n \rightharpoonup x$ . Then for every  $y \neq x$ ,*

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|.$$

**Lemma 2.3** ([12]). *Let  $\{x_n\}$  and  $\{z_n\}$  be bounded sequences in a Banach space  $X$  and let  $\{\beta_n\}$  be a sequence in  $[0, 1]$  with  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$ . Suppose that  $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$  for all integers  $n \geq 1$  and*

$$\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

*Then  $\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0$ .*

**Lemma 2.4** ([13]). *Let  $\{a_n\}$  be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \theta_n)a_n + \delta_n, \quad n \geq 1,$$

*where  $\{\theta_n\}$  is a sequence in  $(0, 1)$  and  $\{\delta_n\}$  is a sequence of real numbers such that*

- (i)  $\sum_{n=1}^{\infty} \theta_n = \infty$ ;
- (ii)  $\limsup_{n \rightarrow \infty} \delta_n / \theta_n \leq 0$  or  $\sum_{n=1}^{\infty} |\delta_n| < \infty$ .

*Then  $\lim_{n \rightarrow \infty} a_n = 0$ .*

**Lemma 2.5.** *Let  $T_1, T_2, \dots, T_N : C \rightarrow C$  be nonexpansive mappings with  $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ . For  $\lambda = (\lambda_1, \dots, \lambda_N) \in (0, 1)^N$ , let  $U_0 = I$ ,  $U_k = \lambda_k T_k U_{k-1} + (1 - \lambda_k)I$  ( $k = 1, \dots, N$ ), and  $W = U_N$ . Then:*

- (i)  $W$  is nonexpansive and  $F(W) = \bigcap_{i=1}^N F(T_i)$ ;
- (ii) if  $\lambda' = (\lambda'_1, \dots, \lambda'_N) \in (0, 1)^N$  and  $W'$  is the  $W$ -mapping generated by  $T_1, \dots, T_N$  and  $\lambda'$ , then for every  $x \in C$  and every  $p \in \bigcap_{i=1}^N F(T_i)$ ,

$$(2.7) \quad \|W'x - Wx\| \leq \sum_{i=1}^N |\lambda'_i - \lambda_i| \|T_i U_{i-1} x - x\| \leq 2\|x - p\| \sum_{i=1}^N |\lambda'_i - \lambda_i|.$$

*Proof.* (i) The nonexpansivity of  $W$  is clear, and  $F(W) = \bigcap_{i=1}^N F(T_i)$  is [11, Lemma 3.1].

(ii) Let  $U'_k$  be the mappings generated by  $\lambda'$ . For  $k = 1, \dots, N$ ,

$$U'_k x - U_k x = \lambda'_k (T_k U'_{k-1} x - T_k U_{k-1} x) + (\lambda'_k - \lambda_k) (T_k U_{k-1} x - x),$$

so that  $\|U'_k x - U_k x\| \leq \|U'_{k-1} x - U_{k-1} x\| + |\lambda'_k - \lambda_k| \|T_k U_{k-1} x - x\|$ . Since  $U'_0 = U_0 = I$ , induction gives the first inequality in (2.7). For the second one, note that  $p$  is a fixed point of every  $U_k$  and that  $T_i U_{i-1}$  is nonexpansive, so  $\|T_i U_{i-1} x - x\| \leq \|T_i U_{i-1} x - p\| + \|p - x\| \leq 2\|x - p\|$ .  $\square$

**Lemma 2.6.** *Let  $A : C \rightarrow H$  be relaxed  $(u, v)$ -cocoercive and  $\mu$ -Lipschitz continuous with  $v > u\mu^2$ , and put*

$$K = \frac{2(v - u\mu^2)}{\mu^2}.$$

*Then for every  $s \in (0, K]$  and all  $x, y \in C$ ,*

$$(2.8) \quad \|(I - sA)x - (I - sA)y\|^2 \leq \|x - y\|^2 - s(K - s)\|Ax - Ay\|^2.$$

*In particular,  $I - sA$  is nonexpansive on  $C$ . Moreover,  $A$  is monotone.*

*Proof.* By (2.1) and  $\|Ax - Ay\| \leq \mu\|x - y\|$ ,

$$\begin{aligned} \|(I - sA)x - (I - sA)y\|^2 &= \|x - y\|^2 - 2s\langle Ax - Ay, x - y \rangle + s^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 + (2su + s^2)\|Ax - Ay\|^2 - 2sv\|x - y\|^2 \\ &\leq \|x - y\|^2 + \left(2su + s^2 - \frac{2sv}{\mu^2}\right)\|Ax - Ay\|^2 \\ &= \|x - y\|^2 - s(K - s)\|Ax - Ay\|^2. \end{aligned}$$

Finally,  $\langle Ax - Ay, x - y \rangle \geq (-u)\|Ax - Ay\|^2 + v\|x - y\|^2 \geq (v - u\mu^2)\|x - y\|^2 \geq 0$ , so  $A$  is monotone.  $\square$

### 3. MAIN RESULT

**Theorem 3.1.** *Let  $H$  be a real Hilbert space,  $C$  a nonempty closed convex subset of  $H$ , and  $A : C \rightarrow H$  a relaxed  $(u, v)$ -cocoercive and  $\mu$ -Lipschitz continuous mapping with  $v > u\mu^2$ . Let  $f : C \rightarrow C$  be a contraction with coefficient  $\alpha \in (0, 1)$ , and let  $T_1, T_2, \dots, T_N : C \rightarrow C$  be nonexpansive mappings such that*

$$\Omega = \bigcap_{i=1}^N F(T_i) \cap VI(C, A) \neq \emptyset.$$

*Let  $\{x_n\}$  be the sequence generated by*

$$(3.1) \quad \begin{cases} x_1 \in C, \\ y_n = P_C(x_n - s_n Ax_n), \\ \rho_n = P_C(y_n - r_n Ay_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n \rho_n, \quad n \geq 1, \end{cases}$$

*where  $W_n$  is defined by (1.3),  $\{r_n\}$  and  $\{s_n\}$  are positive sequences, and  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\gamma_n\}$  are sequences in  $(0, 1)$ . Assume that the following conditions hold:*

- (C1)  $\alpha_n + \beta_n + \gamma_n = 1$ ;
- (C2)  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ;

- (C3)  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$ ;  
 (C4)  $\lim_{n \rightarrow \infty} |r_{n+1} - r_n| = 0$  and  $\lim_{n \rightarrow \infty} |s_{n+1} - s_n| = 0$ ;  
 (C5)  $\{r_n\}, \{s_n\} \subset [a, b]$  for some constants  $a, b$  with  $0 < a \leq b < \frac{2(v-u\mu^2)}{\mu^2}$ ;  
 (C6) there exist constants  $c, d$  with  $0 < c \leq d < 1$  such that  $\lambda_{n,i} \in [c, d]$  for all  $n \geq 1$  and  $i = 1, \dots, N$ , and  $\lim_{n \rightarrow \infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$  for each  $i = 1, \dots, N$ .

Then  $\{x_n\}$  converges strongly to  $q \in \Omega$ , where  $q = P_\Omega f(q)$ ; equivalently,  $q$  is the unique solution in  $\Omega$  of the variational inequality

$$\langle (I - f)q, p - q \rangle \geq 0, \quad \forall p \in \Omega.$$

*Proof.* Put  $K = 2(v - u\mu^2)/\mu^2$  and  $\kappa = a(K - b) > 0$ . For every  $s \in [a, b]$  we have  $s(K - s) \geq a(K - b) = \kappa$ , so Lemma 2.6 and (C5) give, for all  $n \geq 1$  and all  $x, y \in C$ ,

$$(3.2) \quad \|(I - s_n A)x - (I - s_n A)y\|^2 \leq \|x - y\|^2 - \kappa \|Ax - Ay\|^2,$$

and the same inequality with  $r_n$  in place of  $s_n$ . In particular,  $I - s_n A$  and  $I - r_n A$  are nonexpansive.

The set  $\Omega$  is nonempty, closed and convex. Indeed, each  $F(T_i)$  is closed and convex, and  $VI(C, A) = F(P_C(I - aA))$  is the fixed point set of a nonexpansive mapping. Hence  $P_\Omega$  is well defined,  $P_\Omega f$  is a contraction on  $C$ , and by Banach's contraction principle there is a unique  $q \in \Omega$  with  $q = P_\Omega f(q)$ . By (2.3), this is equivalent to  $\langle f(q) - q, p - q \rangle \leq 0$  for all  $p \in \Omega$ , that is,  $\langle (I - f)q, p - q \rangle \geq 0$  for all  $p \in \Omega$ .

We divide the rest of the proof into several steps.

*Step 1.* The sequences  $\{x_n\}$ ,  $\{y_n\}$  and  $\{\rho_n\}$  are bounded. Fix  $p \in \Omega$ . Since  $p = P_C(p - s_n A p) = P_C(p - r_n A p)$  and  $W_n p = p$ , (3.2) yields

$$(3.3) \quad \|y_n - p\|^2 \leq \|x_n - p\|^2 - \kappa \|Ax_n - Ap\|^2, \quad \|\rho_n - p\|^2 \leq \|y_n - p\|^2 - \kappa \|Ay_n - Ap\|^2.$$

In particular,  $\|W_n \rho_n - p\| \leq \|\rho_n - p\| \leq \|y_n - p\| \leq \|x_n - p\|$ . Therefore

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + \beta_n \|x_n - p\| + \gamma_n \|W_n \rho_n - p\| \\ &\leq [1 - \alpha_n(1 - \alpha)] \|x_n - p\| + \alpha_n(1 - \alpha) \frac{\|f(p) - p\|}{1 - \alpha}, \end{aligned}$$

and by induction  $\|x_n - p\| \leq \max\{\|x_1 - p\|, \|f(p) - p\|/(1 - \alpha)\}$  for all  $n \geq 1$ . Hence  $\{x_n\}$  is bounded, and so are  $\{y_n\}$ ,  $\{\rho_n\}$ ,  $\{W_n \rho_n\}$ ,  $\{f(x_n)\}$ ,  $\{Ax_n\}$  and  $\{Ay_n\}$ . Let  $M > 0$  be a constant bounding the norms of all these sequences, as well as  $\|x_n - y_n\|$ ,  $\|y_n - \rho_n\|$  and  $2\|\rho_n - p\|$ , for all  $n \geq 1$ .

*Step 2.*  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ . Since  $I - s_{n+1} A$  is nonexpansive,

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq \|(I - s_{n+1} A)x_{n+1} - (I - s_{n+1} A)x_n\| + |s_{n+1} - s_n| \|Ax_n\| \\ &\leq \|x_{n+1} - x_n\| + M|s_{n+1} - s_n|, \end{aligned}$$

and similarly

$$(3.4) \quad \begin{aligned} \|\rho_{n+1} - \rho_n\| &\leq \|y_{n+1} - y_n\| + M|r_{n+1} - r_n| \\ &\leq \|x_{n+1} - x_n\| + M(|s_{n+1} - s_n| + |r_{n+1} - r_n|). \end{aligned}$$

By Lemma 2.5(ii) with  $x = \rho_n$ ,

$$(3.5) \quad \|W_{n+1}\rho_n - W_n\rho_n\| \leq M \sum_{i=1}^N |\lambda_{n+1,i} - \lambda_{n,i}|.$$

Define

$$z_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} = \frac{\alpha_n f(x_n) + \gamma_n W_n \rho_n}{1 - \beta_n},$$

so that  $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$ . Since  $\gamma_n/(1 - \beta_n) = 1 - \alpha_n/(1 - \beta_n)$  by (C1), we have

$$z_n = W_n \rho_n + \frac{\alpha_n}{1 - \beta_n} (f(x_n) - W_n \rho_n),$$

and therefore, by (3.4) and (3.5),

$$\begin{aligned} \|z_{n+1} - z_n\| &\leq \|W_{n+1}\rho_{n+1} - W_{n+1}\rho_n\| + \|W_{n+1}\rho_n - W_n\rho_n\| \\ &\quad + 2M \left( \frac{\alpha_{n+1}}{1 - \beta_{n+1}} + \frac{\alpha_n}{1 - \beta_n} \right) \\ &\leq \|x_{n+1} - x_n\| + M \left( |s_{n+1} - s_n| + |r_{n+1} - r_n| + \sum_{i=1}^N |\lambda_{n+1,i} - \lambda_{n,i}| \right) \\ &\quad + 2M \left( \frac{\alpha_{n+1}}{1 - \beta_{n+1}} + \frac{\alpha_n}{1 - \beta_n} \right). \end{aligned}$$

By (C2), (C3), (C4) and (C6), we obtain

$$\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Since  $\{z_n\}$  is bounded, Lemma 2.3 gives  $\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0$ , and consequently

$$(3.6) \quad \lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} (1 - \beta_n) \|z_n - x_n\| = 0.$$

Step 3.  $\lim_{n \rightarrow \infty} \|W_n \rho_n - x_n\| = 0$ . From  $x_{n+1} - x_n = \alpha_n(f(x_n) - x_n) + \gamma_n(W_n \rho_n - x_n)$  we get

$$\gamma_n \|W_n \rho_n - x_n\| \leq \|x_{n+1} - x_n\| + \alpha_n \|f(x_n) - x_n\|.$$

By (C1)–(C3),  $\liminf_{n \rightarrow \infty} \gamma_n = 1 - \limsup_{n \rightarrow \infty} \beta_n > 0$ . Hence, by (3.6) and (C2),

$$(3.7) \quad \lim_{n \rightarrow \infty} \|W_n \rho_n - x_n\| = 0.$$

Step 4.  $\lim_{n \rightarrow \infty} \|Ax_n - Ap\| = \lim_{n \rightarrow \infty} \|Ay_n - Ap\| = 0$  for every  $p \in \Omega$ . By (2.5) and (3.3),

$$\begin{aligned} (3.8) \quad \|x_{n+1} - p\|^2 &\leq \alpha_n \|f(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|\rho_n - p\|^2 \\ &\leq \alpha_n \|f(x_n) - p\|^2 + \|x_n - p\|^2 \\ &\quad - \gamma_n \kappa (\|Ax_n - Ap\|^2 + \|Ay_n - Ap\|^2). \end{aligned}$$

Hence

$$\begin{aligned} &\gamma_n \kappa (\|Ax_n - Ap\|^2 + \|Ay_n - Ap\|^2) \\ &\leq \alpha_n \|f(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\leq \alpha_n \|f(x_n) - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|) \|x_n - x_{n+1}\|. \end{aligned}$$

Since  $\liminf_{n \rightarrow \infty} \gamma_n > 0$ , (3.6) and (C2) imply

$$(3.9) \quad \lim_{n \rightarrow \infty} \|Ax_n - Ap\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|Ay_n - Ap\| = 0.$$

Step 5.  $\lim_{n \rightarrow \infty} \|x_n - y_n\| = \lim_{n \rightarrow \infty} \|y_n - \rho_n\| = \lim_{n \rightarrow \infty} \|W_n x_n - x_n\| = 0$ . Let  $p \in \Omega$ . By (2.4) and the nonexpansivity of  $I - s_n A$ ,

$$\begin{aligned} \|y_n - p\|^2 &\leq \langle (x_n - s_n A x_n) - (p - s_n A p), y_n - p \rangle \\ &= \frac{1}{2} \left\{ \|(I - s_n A)x_n - (I - s_n A)p\|^2 + \|y_n - p\|^2 \right. \\ &\quad \left. - \|(x_n - y_n) - s_n(Ax_n - Ap)\|^2 \right\} \\ &\leq \frac{1}{2} \left\{ \|x_n - p\|^2 + \|y_n - p\|^2 - \|x_n - y_n\|^2 \right. \\ &\quad \left. + 2s_n \langle x_n - y_n, Ax_n - Ap \rangle - s_n^2 \|Ax_n - Ap\|^2 \right\}, \end{aligned}$$

which yields

$$(3.10) \quad \|y_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - y_n\|^2 + 2b\|x_n - y_n\| \|Ax_n - Ap\|.$$

In the same way,

$$(3.11) \quad \|\rho_n - p\|^2 \leq \|y_n - p\|^2 - \|y_n - \rho_n\|^2 + 2b\|y_n - \rho_n\| \|Ay_n - Ap\|.$$

Substituting (3.10) and (3.11) into (3.8), we obtain

$$\begin{aligned} \gamma_n (\|x_n - y_n\|^2 + \|y_n - \rho_n\|^2) &\leq \alpha_n \|f(x_n) - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|) \|x_n - x_{n+1}\| \\ &\quad + 2bM (\|Ax_n - Ap\| + \|Ay_n - Ap\|). \end{aligned}$$

By (3.6), (3.9), (C2) and  $\liminf_{n \rightarrow \infty} \gamma_n > 0$ ,

$$(3.12) \quad \lim_{n \rightarrow \infty} \|x_n - y_n\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|y_n - \rho_n\| = 0.$$

Finally, since  $W_n$  is nonexpansive,

$$\|W_n x_n - x_n\| \leq \|W_n x_n - W_n \rho_n\| + \|W_n \rho_n - x_n\| \leq \|x_n - y_n\| + \|y_n - \rho_n\| + \|W_n \rho_n - x_n\|,$$

and (3.7) and (3.12) give

$$(3.13) \quad \lim_{n \rightarrow \infty} \|W_n x_n - x_n\| = 0.$$

Step 6.  $\limsup_{n \rightarrow \infty} \langle f(q) - q, x_n - q \rangle \leq 0$ . Choose a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that

$$\limsup_{n \rightarrow \infty} \langle f(q) - q, x_n - q \rangle = \lim_{k \rightarrow \infty} \langle f(q) - q, x_{n_k} - q \rangle.$$

Since  $\{x_{n_k}\}$  is bounded and  $\lambda_{n,i} \in [c, d]$ , passing to a further subsequence if necessary, we may assume that  $x_{n_k} \rightharpoonup p$  for some  $p \in C$  (as  $C$  is weakly closed) and that  $\lambda_{n_k,i} \rightarrow \lambda_i \in [c, d]$  for each  $i = 1, \dots, N$ . By (3.12), also  $y_{n_k} \rightharpoonup p$ . We show that  $p \in \Omega$ .

(a)  $p \in \bigcap_{i=1}^N F(T_i)$ . Let  $W$  be the  $W$ -mapping generated by  $T_1, \dots, T_N$  and  $\lambda_1, \dots, \lambda_N$ . By Lemma 2.5(ii), for any fixed  $p_0 \in \Omega$ ,

$$\|W_{n_k} x_{n_k} - W x_{n_k}\| \leq 2\|x_{n_k} - p_0\| \sum_{i=1}^N |\lambda_{n_k,i} - \lambda_i| \rightarrow 0,$$

which together with (3.13) gives  $\|W x_{n_k} - x_{n_k}\| \rightarrow 0$ . If  $Wp \neq p$ , then Lemma 2.2 yields

$$\begin{aligned} \liminf_{k \rightarrow \infty} \|x_{n_k} - p\| &< \liminf_{k \rightarrow \infty} \|x_{n_k} - Wp\| \leq \liminf_{k \rightarrow \infty} (\|x_{n_k} - W x_{n_k}\| + \|W x_{n_k} - Wp\|) \\ &\leq \liminf_{k \rightarrow \infty} \|x_{n_k} - p\|, \end{aligned}$$

which is a contradiction. Hence  $p \in F(W) = \bigcap_{i=1}^N F(T_i)$  by Lemma 2.5(i).

(b)  $p \in VI(C, A)$ . By Lemma 2.6,  $A$  is monotone; being Lipschitz continuous, it is continuous, so the mapping  $T$  in (2.2) is maximal monotone. Let  $(w, g) \in G(T)$ . Then  $g - Aw \in N_C w$ , and since  $y_n \in C$ ,

$$\langle w - y_n, g - Aw \rangle \geq 0.$$

On the other hand,  $y_n = P_C(x_n - s_n Ax_n)$  and (2.3) give  $\langle w - y_n, y_n - x_n + s_n Ax_n \rangle \geq 0$ , that is,

$$\left\langle w - y_n, \frac{y_n - x_n}{s_n} + Ax_n \right\rangle \geq 0.$$

Combining these two inequalities with the monotonicity of  $A$ , we obtain

$$\begin{aligned} \langle w - y_n, g \rangle &\geq \langle w - y_n, Aw \rangle - \left\langle w - y_n, \frac{y_n - x_n}{s_n} + Ax_n \right\rangle \\ &= \langle w - y_n, Aw - Ay_n \rangle + \langle w - y_n, Ay_n - Ax_n \rangle - \left\langle w - y_n, \frac{y_n - x_n}{s_n} \right\rangle \\ &\geq -\|w - y_n\| \left( \mu \|y_n - x_n\| + \frac{\|y_n - x_n\|}{a} \right). \end{aligned}$$

Replacing  $n$  by  $n_k$  and letting  $k \rightarrow \infty$ , we get  $\langle w - p, g \rangle \geq 0$  by (3.12) and  $y_{n_k} \rightarrow p$ . Since  $T$  is maximal monotone,  $0 \in Tp$ , that is,  $p \in VI(C, A)$ .

Therefore  $p \in \Omega$ , and since  $q = P_\Omega f(q)$ ,

$$(3.14) \quad \limsup_{n \rightarrow \infty} \langle f(q) - q, x_n - q \rangle = \lim_{k \rightarrow \infty} \langle f(q) - q, x_{n_k} - q \rangle = \langle f(q) - q, p - q \rangle \leq 0.$$

By (3.6), we also have  $\limsup_{n \rightarrow \infty} \langle f(q) - q, x_{n+1} - q \rangle \leq 0$ .

*Step 7.*  $x_n \rightarrow q$ . Since  $q \in \Omega$ , we have  $\|W_n \rho_n - q\| \leq \|x_n - q\|$  by Step 1. Using (2.6), we obtain

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \|\beta_n(x_n - q) + \gamma_n(W_n \rho_n - q)\|^2 + 2\alpha_n \langle f(x_n) - q, x_{n+1} - q \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \langle f(x_n) - f(q), x_{n+1} - q \rangle \\ &\quad + 2\alpha_n \langle f(q) - q, x_{n+1} - q \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + \alpha_n \alpha_n (\|x_n - q\|^2 + \|x_{n+1} - q\|^2) \\ &\quad + 2\alpha_n \langle f(q) - q, x_{n+1} - q \rangle. \end{aligned}$$

Since  $\alpha_n \rightarrow 0$ , we may assume  $\alpha_n < 1$  and  $\alpha_n \alpha_n < 1$  for all  $n$ . Then

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \frac{1 - (2 - \alpha)\alpha_n + \alpha_n^2}{1 - \alpha\alpha_n} \|x_n - q\|^2 + \frac{2\alpha_n}{1 - \alpha\alpha_n} \langle f(q) - q, x_{n+1} - q \rangle \\ &\leq \left(1 - \frac{2(1 - \alpha)\alpha_n}{1 - \alpha\alpha_n}\right) \|x_n - q\|^2 + \frac{\alpha_n^2 L}{1 - \alpha\alpha_n} + \frac{2\alpha_n}{1 - \alpha\alpha_n} \langle f(q) - q, x_{n+1} - q \rangle \\ &= (1 - \theta_n) \|x_n - q\|^2 + \theta_n t_n, \end{aligned}$$

where  $L = \sup_n \|x_n - q\|^2 < \infty$ ,

$$\theta_n = \frac{2(1 - \alpha)\alpha_n}{1 - \alpha\alpha_n} \quad \text{and} \quad t_n = \frac{\alpha_n L}{2(1 - \alpha)} + \frac{1}{1 - \alpha} \langle f(q) - q, x_{n+1} - q \rangle.$$

Since  $\alpha_n \rightarrow 0$ , we have  $\theta_n \in (0, 1)$  for all sufficiently large  $n$ , and  $\theta_n \geq 2(1 - \alpha)\alpha_n$ , so  $\sum_{n=1}^\infty \theta_n = \infty$  by (C2). Moreover,  $\limsup_{n \rightarrow \infty} t_n \leq 0$  by (C2) and Step 6. Applying Lemma 2.4 with  $a_n = \|x_n - q\|^2$  and  $\delta_n = \theta_n t_n$  (from some index on), we conclude that  $x_n \rightarrow q$ . This completes the proof.  $\square$

**Remark 3.2.** In Theorem 3.1, the two projections in (3.1) are computed with possibly different step sizes  $s_n$  and  $r_n$ , and the only restrictions on them are (C4) and (C5). Condition (C6) is used twice: the bound  $\lambda_{n,i} \in [c, d]$  guarantees  $F(W) = \bigcap_{i=1}^N F(T_i)$  for every limit  $W$  of the mappings  $W_{n_k}$ , and the asymptotic condition on  $|\lambda_{n+1,i} - \lambda_{n,i}|$  controls the difference  $W_{n+1} - W_n$  in Step 2.

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## REFERENCES

- [1] M.R. Bai, Y.Y. Tang, Y.P. Liu, Generalized implicit quasivariational inequalities with relaxed Lipschitz and relaxed monotone mappings, *Appl. Math. Lett.* 12 (1999) 69–76.
- [2] S.S. Chang, Fuzzy quasivariational inclusions in Banach spaces, *Appl. Math. Comput.* 145 (2003) 805–819.
- [3] A.H. Siddiqi, Q.H. Ansari, An algorithm for a class of quasivariational inequalities, *J. Math. Anal. Appl.* 145 (1990) 413–418.
- [4] R.U. Verma, A class of projection-contraction methods applied to monotone variational inequalities, *Appl. Math. Lett.* 13 (2000) 55–62.
- [5] R.U. Verma, A new class of iterative algorithms for approximation-solvability of nonlinear variational inequalities, *Comput. Math. Appl.* 41 (2001) 505–512.
- [6] M.A. Noor, Wiener–Hopf equations and variational inequalities, *J. Optim. Theory Appl.* 79 (1993) 197–206.
- [7] M.A. Noor, Sensitivity analysis for quasi variational inequalities, *J. Optim. Theory Appl.* 95 (1997) 399–407.
- [8] M.A. Noor, Some developments in general variational inequalities, *Appl. Math. Comput.* 152 (2004) 199–277.
- [9] R.T. Rockafellar, On the maximality of sums of nonlinear monotone operators, *Trans. Amer. Math. Soc.* 149 (1970) 75–88.
- [10] H. Iiduka, W. Takahashi, Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings, *Nonlinear Anal.* 61 (2005) 341–350.
- [11] S. Atsushiba, W. Takahashi, Strong convergence theorems for a finite family of nonexpansive mappings and applications, *Indian J. Math.* 41 (1999) 435–453.
- [12] T. Suzuki, Strong convergence of Krasnoselskii and Mann’s type sequences for one-parameter nonexpansive semigroups without Bochner integrals, *J. Math. Anal. Appl.* 305 (2005) 227–239.
- [13] H.K. Xu, Iterative algorithms for nonlinear operators, *J. London Math. Soc.* 66 (2002) 240–256.
- [14] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.* 73 (1967) 591–597.