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INVERSE UNCERTAINTY DISTRIBUTION OF SOLUTIONS FOR UNCERTAIN FRACTIONAL DIFFERENTIAL EQUATIONS

ZEMAN WANG[1], QIUBAO WANG[1,2,*], ZIKUN HAN[2], AND XIUYING GUO[1]

Abstract. Due to the axiomatic differences between uncertainty theory and other stochastic or fuzzy frameworks, developing efficient numerical methods for solving Caputo-type uncertain fractional differential equations remains a challenge. This paper introduces the concept of the α-path for uncertain fractional differential equations (UFDEs) and establishes the existence, uniqueness, and continuity of their solutions. Based on this, the inverse uncertainty distribution of the solution is derived. Finally, a fast numerical algorithm is proposed to efficiently compute the inverse uncertainty distribution of the solution to uncertain fractional differential equations, along with illustrative examples.

1. Introduction

The theory of fractional calculus, which extends beyond the constraints of traditional integer-order calculus to investigate differentiation and integration of arbitrary orders, has garnered significant attention. Its principal advantage lies in characterizing the memory and genetic properties of systems, rendering fractional differential equations (FDEs) a powerful tool for modeling complex systems with historical dependence. Consequently, FDEs have found widespread applications across diverse fields such as physics [1], viscoelasticity [2], chaos theory [3], cybernetics [4], and finance [5] [6].

Unlike the integer-order case, multiple definitions of fractional derivatives exist, including the Grünwald-Letnikov, Riemann-Liouville, Caputo, Hadamard, and Caputo-Hadamard types [7] [8] [9]. Among these, the Caputo-type derivative is particularly advantageous for modeling initial value problems because its initial conditions take the same form as those for integer-order differential equations, making it highly suitable for physical modeling and engineering applications. Current research and applications primarily focus on FDEs involving the Riemann-Liouville and Caputo derivatives.

Uncertainty is ubiquitous in the real world. The frequency of sudden events, such as wars, earthquakes, and financial crises, often lacks sufficient historical data for objective estimation. Similarly, subjective evaluations, like scoring in diving competitions, rely heavily on expert judgment and experience. To quantify the degree of belief associated with such subjective

[1]Department of Mathematics and Physics, Shijiazhuang Tiedao University, Shijiazhuang, 050043, China

[2]Department of Engineering Mechanics, Shijiazhuang Tiedao University, Shijiazhuang, 050043, China

[*]Corresponding author

Email address: wangqiubao12@sina.com.

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Graduated.

information, Professor Baoding Liu established uncertainty theory as an axiomatic mathematical framework, which has been continually refined [10]. To describe dynamic evolution within uncertain environments, the theory further introduced uncertain processes.

As a cornerstone application, Liu [11] defined the Liu process and uncertain differential equations (UDEs), while Chen and Liu [12] proved the existence and uniqueness theorem for their solutions. A pivotal advancement for solving UDEs effectively was the Yao-Chen formula [13], proposed by Yao and Chen, which provides a key formula for computing the inverse uncertainty distribution of the solution. This formula fundamentally transforms the problem of solving the original UDE into solving a family of ordinary differential equations (ODEs). The solutions to this family of ODEs, termed the α-paths, have been proven to constitute the inverse uncertainty distribution of the solution to the original UDE, offering a powerful tool for analyzing the distributional properties of the solution.Further extending this analytical framework, Wang et al. [14] systematically studied second-order and a class of higher-order UDEs, thereby refining the theoretical foundation for more complex uncertain dynamical systems. To model complex systems that simultaneously exhibit uncertainty, memory, and genetic characteristics, Zhu [15] systematically introduced uncertainty into the FDE framework in 2015, pioneering the concept of uncertain fractional differential equations (UFDEs) and providing definitions for both Riemann-Liouville and Caputo types. Subsequent research has explored applications of Caputo-type UFDEs in areas such as interest rate modeling [16], established sufficient conditions for the existence and uniqueness of solutions for both types of UFDEs.

Based on previous studies, Lu [17] investigated the analytical solution of linear UFDEs. Although numerical methods for stochastic [18] and fuzzy fractional differential equations [19] have been developed, these techniques are not directly applicable to UFDEs grounded in uncertainty theory. This incompatibility arises from the fundamental differences in axiomatic systems and core concepts (e.g., the Liu process) between uncertainty theory, probability theory, and fuzzy set theory. Particularly for Caputo-type UFDEs, developing efficient and accurate numerical methods for obtaining approximate solutions remains an open and critical challenge. While existing theoretical analyses, such as the α-path concept, lay the groundwork for understanding the solution's distribution, a systematic and effective algorithm for direct numerical solution is still lacking.

Based on the property that the inverse uncertainty distribution of the solution to a Caputo-type UFDE is given by the solution of a corresponding deterministic Caputo-type FDE, this paper demonstrates that the problem of solving a Caputo-type UFDE with an initial condition can be equivalently transformed into solving a family of deterministic Caputo-type FDEs under the same initial condition. This equivalence establishes a solid theoretical foundation for the numerical solution of UFDEs.

The remainder of this paper is structured as follows. Section 2 reviews fundamental concepts and key theorems, including the Caputo fractional derivative, uncertainty theory (covering Liu processes, uncertain integrals, and uncertain fractional differential equations), and the α-path. Section 3, the core theoretical contribution of this work, presents the inverse uncertainty distribution of the solution for Caputo-type UFDEs. Building upon the theory in Section 3, Section 4 proposes a numerical framework that leverages established numerical solvers for deterministic FDEs to compute the inverse uncertainty distribution of the UFDE solution.

Numerical examples are provided to demonstrate the feasibility and effectiveness of the proposed algorithm. Finally, Section 5 summarizes the findings and outlines potential directions for future research.

2. Preliminary

Definition 1. *(Liu [11]) An uncertain process C_t is said to be a Liu process if*

(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,

(ii) C_t has stationary and independent increments,

(iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2.

Theorem 1. *(Liu [11]) (Measure Inversion Theorem) Let ξ be an uncertain variable with uncertainty distribution Φ. Then for any real number x, we have*

$$\mathcal{M}\{\xi \leq x\} = \Phi(x), \quad \mathcal{M}\{\xi > x\} = 1 - \Phi(x). \tag{1}$$

Theorem 2. *(Zhu [20]) Suppose that C_t is a Liu process, and X_t is an integrable uncertain process on $[a, b]$ with respect to t. Then the inequality*

$$\left| \int_a^b X_t(\gamma)\,\mathrm{d}C_t(\gamma) \right| \leq K_\gamma \int_a^b |X_t(\gamma)|\,\mathrm{d}t \tag{2}$$

holds, where K_γ is the Lipschitz constant of the sample path $X_t(\gamma)$.

Definition 2. *(Liu [16]) Suppose that $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^l$ are two functions. Then,*

$${}^cD^p X_t = f(t, X_t) + g(t, X_t)\frac{\mathrm{d}C_t}{\mathrm{d}t} \tag{3}$$

is called an uncertain fractional differential equation of the Caputo type. And a solution of this equation ($p \in (0, 1]$) with the initial condition x_0 is an uncertain process X_t such that

$$\begin{aligned} X_t = X_0 &+ \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s, X_s)\,\mathrm{d}s \\ &+ \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} g(s, X_s)\,\mathrm{d}C_s \end{aligned} \tag{4}$$

holds almost surely.

Theorem 3. *(Liu [11]) A function $\Phi^{-1} : (0, 1) \to \mathbb{R}$ is the inverse uncertainty distribution of an uncertain variable ξ if and only if it is continuous and*

$$\mathcal{M}\{\xi \leq \Phi^{-1}(\alpha)\} = \alpha. \tag{5}$$

Definition 3. *(Liu [21]) Assume that $\alpha \in (0, 1)$. A Caputo type of uncertain fractional differential equation with initial value conditions*

$$\begin{cases} {}^cD^p X_t = f(t, X_t) + g(t, X_t)\dfrac{\mathrm{d}C_t}{\mathrm{d}t}, \\ X_t^k\Big|_{t=0} = X_k, \quad k = 0, 1, \ldots, n-1. \end{cases} \tag{6}$$

is called to have an α-path X_t^α which is a function of t and solves the associated fractional differential equation

$$ {}^cD^pX_t = f(t,X_t) + |g(t,X_t)|\Phi^{-1}(\alpha) \tag{7}$$

with the same initial value conditions, where $\Phi^{-1}(\alpha)$ is an inverse standard normal uncertainty distribution, that is,

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}. \tag{8}$$

3. Materials and methods

Definition 4. *Suppose f and g are continuous functions. An uncertain fractional differential equation*

$$\begin{cases} {}^cD^pX_t = f(t,X_t) + g(t,X_t)\dfrac{\mathrm{d}C_t}{\mathrm{d}t}, \\ X_t^k\Big|_{t=0} = X_0. \end{cases} \tag{9}$$

is said to satisfy the regular condition if

$$g(t,X_t) > 0, \quad \forall t \geq 0. \tag{10}$$

Theorem 4. *The uncertain fractional differential equation*

$$\begin{cases} {}^cD^pX_t = f(t,X_t) + g(t,X_t)\dfrac{\mathrm{d}C_t}{\mathrm{d}t}, \\ X_t^k\Big|_{t=0} = X_0. \end{cases} \tag{11}$$

has a unique solution if the functions $f(t,x)$ and $g(t,x)$ satisfy the linear growth condition

$$|f(t,x)| + |g(t,x)| \leq L(1+|x|), \quad \forall x \in \mathbb{R},\ t \geq 0 \tag{12}$$

and Lipschitz condition

$$|f(t,x) - f(t,y)| + |g(t,x) - g(t,y)| \leq L(|x-y|), \quad \forall x,y \in \mathbb{R},\ t \geq 0. \tag{13}$$

for some constant L. Moreover, the solution is sample-continuous.

Proof. At first, Theorem 14.1 in [11] says that there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$, such that $C_t(\gamma)$ is Lipschitz continuous with respect to t for each $\gamma \in \Lambda$. Next, we prove the existence and continuity of solution by a successive approximation method. Define $X_t^{(0)}(\gamma) = X_0$, and

$$\begin{aligned} X_t^{(n)}(\gamma) = X_0 &+ \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} f(s, X_s^{(n-1)}(\gamma))\,\mathrm{d}s \\ &+ \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} g(s, X_s^{(n-1)}(\gamma))\,\mathrm{d}C_s(\gamma) \end{aligned} \tag{14}$$

for each integer n. It follows from the linear growth condition that

$$
\begin{aligned}
&|X_t^{(1)}(\gamma) - X_t^{(0)}(\gamma)| \\
&= \left| \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s, X_0)\,\mathrm{d}s + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} g(s, X_0)\,\mathrm{d}C_s(\gamma) \right| \\
&\leq \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} |f(s, X_0)|\,\mathrm{d}s + \frac{K_\gamma}{\Gamma(p)} \int_0^t (t-s)^{p-1} |g(s, X_0)|\,\mathrm{d}s \\
&\leq \frac{1+K_\gamma}{\Gamma(p)} \int_0^t (t-s)^{p-1} |f(s, X_0) + g(s, X_0)|\,\mathrm{d}s \\
&\leq \frac{1+K_\gamma}{\Gamma(p)} \int_0^t (t-s)^{p-1} L(1+|X_0|)\,\mathrm{d}s \\
&= \frac{L(1+|X_0|)(1+K_\gamma)}{\Gamma(p)} \int_0^t (t-s)^{p-1}\,\mathrm{d}s \\
&\leq \frac{L(1+|X_0|)(1+K_\gamma)}{\Gamma(p+1)} t^p.
\end{aligned} \tag{15}
$$

for any $t \geq 0$, where K_γ is the Lipschitz constant to the sample path $C_t(\gamma)$. Assuming

$$
|X_t^{(k)}(\gamma) - X_t^{(k-1)}(\gamma)| \leq \frac{L^k(1+K_\gamma)^k(1+|X_0|)}{(\Gamma(p+1))^k} t^{kp}, \tag{16}
$$

It follows from the Lipschitz condition that

$$
\begin{aligned}
&|X_t^{(k+1)}(\gamma) - X_t^{(k)}(\gamma)| \\
&\leq \left| \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} |f(s, X_s^{(k)}(\gamma)) - f(s, X_s^{(k-1)}(\gamma))|\,\mathrm{d}s \right. \\
&\quad \left. + \frac{K_\gamma}{\Gamma(p)} \int_0^t (t-s)^{p-1} |g(s, X_s^{(k)}(\gamma)) - g(s, X_s^{(k-1)}(\gamma))|\,\mathrm{d}s \right| \\
&\leq \frac{1+K_\gamma}{\Gamma(p)} \int_0^t (t-s)^{p-1} \Big(|f(s, X_s^{(k)}(\gamma)) - f(s, X_s^{(k-1)}(\gamma))| \\
&\quad + |g(s, X_s^{(k)}(\gamma)) - g(s, X_s^{(k-1)}(\gamma))| \Big)\,\mathrm{d}s \\
&\leq \frac{L(1+K_\gamma)}{\Gamma(p)} \int_0^t (t-s)^{p-1} |X_t^{(k)}(\gamma) - X_t^{(k-1)}(\gamma)|\,\mathrm{d}s \\
&\leq \frac{L(1+K_\gamma)}{\Gamma(p)} \cdot \frac{L^k(1+K_\gamma)^k(1+|X_0|)}{(\Gamma(p+1))^k} \int_0^t (t-s)^{p-1}\,\mathrm{d}s \\
&= \frac{L^{k+1}(1+K_\gamma)^{k+1}(1+|X_0|)}{(\Gamma(p+1))^{k+1}} t^{(k+1)p}.
\end{aligned} \tag{17}
$$

The above induction shows that

$$
|X_t^{(n+1)}(\gamma) - X_t^{(n)}(\gamma)| \leq \frac{L^{n+1}(1+K_\gamma)^{n+1}(1+|X_0|)}{(\Gamma(p+1))^{n+1}} t^{(n+1)p} \tag{18}
$$

for each integer n. This means that, for each $\gamma \in \Lambda$, the sequence $X_t^{(n)}(\gamma)$ converges uniformly on any given time interval as $n \to \infty$. Furthermore, since $X_t^{(0)}$ is constant, obviously it is continuous. Suppose $X_t^{(k)}$ is continuous, then $X_t^{(k)}$ is also continuous with respect to t. So, by induction, $X_t^{(n)}$ is continuous for all n. Provided $X_t^{(n)}$ converges uniformly, that is $\lim_{n\to\infty} X_t^{(n)}(\gamma) = X_t(\gamma)$, the limit function X_t is also continuous.Write the limit by $X_t(\gamma)$ that is just a solution of the uncertain fractional differential equation since

$$\begin{aligned} X_t(\gamma) =& X_0 + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} f(s, X_s(\gamma))\,\mathrm{d}s \\ &+ \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} g(s, X_s(\gamma))\,\mathrm{d}C_s(\gamma). \end{aligned} \tag{19}$$

We then prove that the solution is unique. Assume that both $\mathbf{X}_t$ and $\mathbf{X}_t^*$ are solutions of the uncertain fractional differential equation. For each $\gamma \in \Lambda$, it follows from the Lipschitz condition that

$$|X_t(\gamma) - X_t^*(\gamma)| \le \frac{L(1+K_\gamma)}{\Gamma(p)}\int_0^t (t-s)^{p-1}|(X_s(\gamma) - X_s^*(\gamma))|\,\mathrm{d}s. \tag{20}$$

By applying the fractional Gronwall inequality [22], we obtain

$$|X_t(\gamma) - X_t^*(\gamma)| \le 0 \cdot E_p(L(1+K_\gamma)t^p) = 0, \tag{21}$$

where

$$E_p(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(pk+1)} \tag{22}$$

is the Mittag–Leffler function. Hence,

$$|X_t(\gamma) - X_t^*(\gamma)| = 0, \tag{23}$$

which implies

$$X_t(\gamma) = X_t^*(\gamma). \tag{24}$$

Therefore,

$$X_t = X_t^*. \tag{25}$$

The uniqueness is verified. The theorem is proved. □

Theorem 5. *Let X_t^α be the α-path of the uncertain fractional differential equation*

$$\begin{cases} {}^cD^p X_t = f(t, X_t) + g(t, X_t)\dfrac{\mathrm{d}C_t}{\mathrm{d}t}, \\ X_t^k\Big|_{t=0} = X_0. \end{cases} \tag{26}$$

If f and g satisfies the Lipschitz and regular conditions hold, then X_t^α is a continuous function with respect to α at each time $t > 0$.

Proof. Since f and g satisfy the Lipschitz conditions, it follows from Theorem 4 that the uncertain fractional differential equations

$${}^cD^p X_t^\alpha = f(t, X_t^\alpha) + |g(t, X_t^\alpha)|\Phi^{-1}(\alpha) \tag{27}$$

has a unique solution X_t^α, where $\Phi^{-1}(\alpha)$ is the inverse standard normal uncertainty distribution. For any numbers α and β between 0 and 1, it follows from the Lipschitz condition that

$$
\begin{aligned}
|X_t^\alpha - X_t^\beta| \le{} & \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}|f(s,X_s^\alpha)-f(s,X_s^\beta)|\,\mathrm{d}s \\
& + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}|g(s,X_s^\alpha)\Phi^{-1}(\alpha)-g(s,X_s^\beta)\Phi^{-1}(\beta)|\,\mathrm{d}s \\
\le{} & \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}|f(s,X_s^\alpha)-f(s,X_s^\beta)|\,\mathrm{d}s \\
& + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}|g(s,X_s^\alpha)-g(s,X_s^\beta)||\Phi^{-1}(\alpha)|\,\mathrm{d}s \\
& + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}|g(s,X_s^\beta)||\Phi^{-1}(\alpha)-\Phi^{-1}(\beta)|\,\mathrm{d}s \\
\le{} & L(1+\Phi^{-1}(\alpha))\frac{t^p}{\Gamma(p+1)}\int_0^t |X_t^\alpha - X_t^\beta|\,\mathrm{d}s \\
& + \frac{|\Phi^{-1}(\alpha)-\Phi^{-1}(\beta)|}{\Gamma(p)}\int_0^t (t-s)^{p-1}|g(s,X_s^\beta)|\,\mathrm{d}s.
\end{aligned}
\tag{28}
$$

By using Gronwall inequality, we obtain

$$
|X_t^\alpha - X_t^\beta| \le \frac{|\Phi^{-1}(\alpha)-\Phi^{-1}(\beta)|}{\Gamma(p)}\int_0^t (t-s)^{p-1}|g(s,X_s^\beta)|\,\mathrm{d}s\cdot\exp\left\{L(1+\Phi^{-1}(\alpha))\frac{t^p}{\Gamma(p+1)}\right\}. \tag{29}
$$

Thus

$$
\lim_{\alpha\to\beta}|X_t^\alpha - X_t^\beta| = 0 \tag{30}
$$

for each time $t>0$. Hence X_t^α is continuous with respect to α. □

Theorem 6. *Let X_t^α be the α-path of the regular uncertain fractional differential equation*

$$
\begin{cases}
{}^cD^pX_t = f(t,X_t) + g(t,X_t)\dfrac{\mathrm{d}C_t}{\mathrm{d}t}, \\
X_t^k\Big|_{t=0} = X_0.
\end{cases}
\tag{31}
$$

If the linear growth, Lipschitz and regular conditions hold, then X_t^α is a strictly increasing function with respect to α at each time $t>0$.

Proof. Since f and g satisfy the linear growth and Lipschitz conditions, the α-path X_t^α is continuous with respect to t. Let Φ^{-1} be the inverse standard normal uncertainty distribution, and let α and β be numbers with $0<\alpha<\beta<1$. Write

$$
\begin{aligned}
\mu(T,t) &= (T-t)^{p-1}[f(t,X_t^\alpha)+g(t,X_t^\alpha)\Phi^{-1}(\alpha)], \\
\nu(T,t) &= (T-t)^{p-1}[f(t,X_t^\beta)+g(t,X_t^\beta)\Phi^{-1}(\beta)].
\end{aligned}
\tag{32}
$$

We have

$$
\begin{aligned}
\mu(T,0) &= T^{p-1}[f(0,X_0)+g(0,X_0)\Phi^{-1}(\alpha)], \\
\nu(T,0) &= T^{p-1}[f(0,X_0)+g(0,X_0)\Phi^{-1}(\beta)].
\end{aligned}
\tag{33}
$$

Since $g(0,X_0)>0$ (regular condition), we have

$$
\mu(T,0) < \nu(T,0). \tag{34}
$$

Due to the continuous nature of f and g, and the continuity of X_t^α, we can infer that the composite functions μ and ν are also continuous with respect to t. By continuity of μ and ν, there exists a small number $r > 0$ such that

$$\mu(T,t) < \nu(T,t), \quad \forall t \in [0,r]. \tag{35}$$

Thus,

$$\begin{aligned} X_T^\alpha &= X_0 + \frac{1}{\Gamma(p)}\int_0^T (T-t)^{p-1} f(t,X_t^\alpha)\,\mathrm{d}t + \frac{1}{\Gamma(p)}\int_0^T (T-t)^{p-1} g(t,X_t^\alpha)\Phi^{-1}(\alpha)\,\mathrm{d}t \\ &< X_0 + \frac{1}{\Gamma(p)}\int_0^T (T-t)^{p-1} f(t,X_t^\beta)\,\mathrm{d}t + \frac{1}{\Gamma(p)}\int_0^T (T-t)^{p-1} g(t,X_t^\beta)\Phi^{-1}(\beta)\,\mathrm{d}t \\ &= X_T^\beta \end{aligned} \tag{36}$$

for any time $T \in (0,r]$.

If for any $t > r$, $X_T^\alpha < X_T^\beta$, the theorem holds. We will prove that by contradiction.

Suppose there exists a time $b > r$ at which X_T^α and X_T^β first meet, i.e.,

$$X_b^\alpha = X_b^\beta, \quad X_T^\alpha < X_T^\beta,\ \forall T \in (0,b). \tag{37}$$

since $f(t,X_b^\alpha) = f(t,X_b^\beta)$ and $g(t,X_b^\alpha) = g(t,X_b^\beta) > 0$, we have

$$\mu(b,t) < \nu(b,t). \tag{38}$$

By using the continuity, there exists a time $a \in (0,b)$ such that

$$\mu(T,t) < \nu(T,t), \quad \forall t \in [a,b]. \tag{39}$$

Thus

$$X_b^\alpha = X_0 + \frac{1}{\Gamma(p)}\int_a^b \mu(b,t)\,\mathrm{d}t < X_0 + \frac{1}{\Gamma(p)}\int_a^b \nu(b,t)\,\mathrm{d}t = X_b^\beta \tag{40}$$

that is in contradiction with $X_b^\alpha = X_b^\beta$. The theorem is proved. □

Theorem 7. *Let X_t and X_t^α be the solution and α-path of a regular uncertain fractional differential equation*

$$\begin{cases} {}^cD^pX_t = f(t,X_t) + g(t,X_t)\dfrac{\mathrm{d}C_t}{\mathrm{d}t}, \\ X_t^k\Big|_{t=0} = X_0. \end{cases} \tag{41}$$

respectively. If the linear growth, Lipschitz and regular conditions hold, then

$$\mathcal{M}\{X_t \le X_t^\alpha, \forall t\} = \alpha, \tag{42}$$

$$\mathcal{M}\{X_t > X_t^\alpha, \forall t\} = 1-\alpha. \tag{43}$$

Proof. Theorem 14.3 in [11] constructs an event Λ_1 with $\mathcal{M}\{\Lambda_1\} = \alpha$, and shows that for each $\gamma \in \Lambda_1$, there exists a small number $\delta > 0$ such that

$$\frac{C_s(\gamma) - C_t(\gamma)}{s-t} < \Phi^{-1}(\alpha-\delta) \tag{44}$$

for any times s and t with $s > t$, where Φ^{-1} is the inverse standard normal uncertainty distribution. Since f and g satisfy the linear growth and Lipschitz conditions, X_t^α and $X_t(\gamma)$ are continuous with respect to t. Write

$$\lambda(T,t,s) = (T-t)^{p-1}[f(t,X_t(\gamma)) + g(t,X_t(\gamma))\frac{C_s(\gamma)-C_t(\gamma)}{s-t}], \tag{45}$$

$$\begin{aligned} \mu(T,t) &= (T-t)^{p-1}[f(t,X_t(\gamma)) + g(t,X_t(\gamma))\Phi^{-1}(\alpha-\delta)], \\ \nu(T,t) &= (T-t)^{p-1}[f(t,X_t^\alpha) + g(t,X_t^\alpha)\Phi^{-1}(\alpha)]. \end{aligned} \tag{46}$$

Due to the continuous nature of f and g, and the continuity of $X_t(\gamma)$, we can infer that the composite functions λ, μ and ν are also continuous with respect to t. Since $g(0,X_0) > 0$ (regular condition), we have

$$\mu(T,0) < \nu(T,0). \tag{47}$$

By continuity of μ and ν, there exists a small number $r > 0$ such that

$$\mu(T,t) < \nu(T,t), \quad \forall t \in [0,r]. \tag{48}$$

By inequality (44), for any time $t \in [0,r]$ and any time $s \in (t,\infty)$, we have

$$\begin{aligned} \lambda(T,t,s) &= (T-t)^{p-1}[f(t,X_t(\gamma)) + g(t,X_t(\gamma))\frac{C_s(\gamma)-C_t(\gamma)}{s-t}] \\ &< (T-t)^{p-1}[f(t,X_t(\gamma)) + g(t,X_t(\gamma))\Phi^{-1}(\alpha-\delta)] \\ &= \mu(T,t) < \nu(T,t). \end{aligned} \tag{49}$$

Thus

$$\begin{aligned} X_T(\gamma) &= X_0 + \frac{1}{\Gamma(p)}\int_0^T (T-t)^{p-1} f(t,X_t(\gamma))\,\mathrm{d}t + \frac{1}{\Gamma(p)}\int_0^T (T-t)^{p-1} g(t,X_t(\gamma))\,\mathrm{d}C_t(\gamma) \\ &= X_0 + \frac{1}{\Gamma(p)}\lim_{\Delta\to 0}\sum_{i=1}^{k}\lambda(T,t_{i+1},t_i)(t_{i+1}-t_i) \\ &\le X_0 + \frac{1}{\Gamma(p)}\lim_{\Delta\to 0}\sum_{i=1}^{k}\mu(T,t_i)(t_{i+1}-t_i) \\ &= X_0 + \frac{1}{\Gamma(p)}\int_0^T \mu(T,t)\,\mathrm{d}t < X_0 + \frac{1}{\Gamma(p)}\int_0^T \nu(T,t)\,\mathrm{d}t \\ &= X_0 + \frac{1}{\Gamma(p)}\int_0^T (T-t)^{p-1} f(t,X_t^\alpha)\,\mathrm{d}t + \frac{1}{\Gamma(p)}\int_0^T (T-t)^{p-1} g(t,X_t^\alpha)\Phi^{-1}(\alpha)\,\mathrm{d}t \\ &= X_T^\alpha \end{aligned} \tag{50}$$

for any time $T \in (0,r]$.

We will then prove $X_T(\gamma) < X_T^\alpha$ for all $T > r$ by contradiction.

Suppose there exists a time $b > r$ at which $X_T(\gamma)$ and X_T^α first meet, i.e.,

$$X_b(\gamma) = X_b^\alpha, \quad X_T(\gamma) < X_T^\alpha,\ \forall T \in (0,b). \tag{51}$$

since $f(t,X_b(\gamma)) = f(t,X_b^\alpha)$ and $g(t,X_b(\gamma)) = g(t,X_b^\alpha) > 0$, we have

$$\mu(b,t) < \nu(b,t). \tag{52}$$

By using the continuity, there exists a time $a \in (0, b)$ such that

$$\mu(T, t) < \nu(T, t), \quad \forall t \in [a, b]. \tag{53}$$

Thus

$$X_b(\gamma) = X_0 + \frac{1}{\Gamma(p)} \int_a^b \mu(b, t)\,\mathrm{d}t < X_0 + \frac{1}{\Gamma(p)} \int_a^b \nu(b, t)\,\mathrm{d}t = X_b^\alpha \tag{54}$$

that is in contradiction with $X_b(\gamma) = X_b^\alpha$.

$$X_t(\gamma) < X_t^\alpha, \quad \forall t > 0. \tag{55}$$

Since $\mathcal{M}\{\Lambda_1\} = \alpha$, we have

$$\mathcal{M}\{X_t \leq X_t^\alpha, \forall t\} \geq \alpha. \tag{56}$$

Theorem 14.3 in [11] constructs an event Λ_2 with $\mathcal{M}\{\Lambda_2\} = 1 - \alpha$, and shows that for each $\gamma \in \Lambda_2$, there exists a small number $\delta > 0$ such that

$$\frac{C_s(\gamma) - C_t(\gamma)}{s - t} > \Phi^{-1}(\alpha + \delta) \tag{57}$$

for any times s and t with $s > t$. Write

$$\lambda(T, t, s) = (T - t)^{p-1}[f(t, X_t(\gamma)) + g(t, X_t(\gamma))\frac{C_s(\gamma) - C_t(\gamma)}{s - t}], \tag{58}$$

$$\begin{aligned} \mu(T, t) &= (T - t)^{p-1}[f(t, X_t(\gamma)) + g(t, X_t(\gamma))\Phi^{-1}(\alpha + \delta)], \\ \nu(T, t) &= (T - t)^{p-1}[f(t, X_t^\alpha) + g(t, X_t^\alpha)\Phi^{-1}(\alpha)]. \end{aligned} \tag{59}$$

Since $g(0, X_0) > 0$ (regular condition), we have

$$\mu(T, 0) > \nu(T, 0). \tag{60}$$

By continuity of μ and ν, there exists a small number $r > 0$ such that

$$\mu(T, t) > \nu(T, t), \quad \forall t \in [0, r]. \tag{61}$$

By inequality (57), for any time $t \in [0, r]$ and any time $s \in (t, \infty)$, we have

$$\begin{aligned} \lambda(T, t, s) =& (T - t)^{p-1}[f(t, X_t(\gamma)) + g(t, X_t(\gamma))\frac{C_s(\gamma) - C_t(\gamma)}{s - t}] \\ >& (T - t)^{p-1}[f(t, X_t(\gamma)) + g(t, X_t(\gamma))\Phi^{-1}(\alpha + \delta)] \\ =& \mu(T, t) > \nu(T, t). \end{aligned} \tag{62}$$

Thus

$$
\begin{aligned}
X_T(\gamma) &= X_0 + \frac{1}{\Gamma(p)}\int_0^T (T-t)^{p-1} f(t, X_t(\gamma))\,\mathrm{d}t + \frac{1}{\Gamma(p)}\int_0^T (T-t)^{p-1} g(t, X_t(\gamma))\,\mathrm{d}C_t(\gamma) \\
&= X_0 + \frac{1}{\Gamma(p)}\lim_{\Delta\to 0}\sum_{i=1}^{k}\lambda(T, t_{i+1}, t_i)(t_{i+1}-t_i) \\
&\geq X_0 + \frac{1}{\Gamma(p)}\lim_{\Delta\to 0}\sum_{i=1}^{k}\mu(T, t_i)(t_{i+1}-t_i) \\
&= X_0 + \frac{1}{\Gamma(p)}\int_0^T \mu(T,t)\,\mathrm{d}t \\
&> X_0 + \frac{1}{\Gamma(p)}\int_0^T \nu(T,t)\,\mathrm{d}t \\
&= X_0 + \frac{1}{\Gamma(p)}\int_0^T (T-t)^{p-1} f(t, X_t^\alpha)\,\mathrm{d}t + \frac{1}{\Gamma(p)}\int_0^T (T-t)^{p-1} g(t, X_t^\alpha)\Phi^{-1}(\alpha)\,\mathrm{d}t \\
&= X_T^\alpha
\end{aligned} \tag{63}
$$

for any time $T \in (0, r]$.

We will then prove $X_T(\gamma) > X_T^\alpha$ for all $T > r$ by contradiction.

Suppose there exists a time $b > r$ at which $X_T(\gamma)$ and X_T^α first meet, i.e.,

$$X_b(\gamma) = X_b^\alpha, \quad X_T(\gamma) > X_T^\alpha,\ \forall T \in (0, b). \tag{64}$$

since

$$f(t, X_b(\gamma)) = f(t, X_b^\alpha), \quad g(t, X_b(\gamma)) = g(t, X_b^\alpha) > 0, \tag{65}$$

we have

$$\mu(b,t) > \nu(b,t). \tag{66}$$

By using the continuity, there exists a time $a \in (0, b)$ such that

$$\mu(T,t) > \nu(T,t), \quad \forall T \in [a, b]. \tag{67}$$

Thus

$$X_b(\gamma) = X_0 + \frac{1}{\Gamma(p)}\int_a^b \mu(b,t)\,\mathrm{d}t > X_0 + \frac{1}{\Gamma(p)}\int_a^b \nu(b,t)\,\mathrm{d}t = X_b^\alpha \tag{68}$$

that is in contradiction with $X_b(\gamma) = X_b^\alpha$.

$$X_t(\gamma) > X_t^\alpha, \quad \forall t > 0. \tag{69}$$

Since $\mathcal{M}\{\Lambda_2\} = 1 - \alpha$, we have

$$\mathcal{M}\{X_t > X_t^\alpha, \forall t\} \geq 1 - \alpha. \tag{70}$$

It follows from (56), (70) and

$$\mathcal{M}\{X_t \leq X_t^\alpha, \forall t\} + \mathcal{M}\{X_t > X_t^\alpha, \forall t\} \leq 1, \tag{71}$$

that

$$\mathcal{M}\{X_t \leq X_t^\alpha, \forall t\} = \alpha, \tag{72}$$

$$\mathcal{M}\{X_t > X_t^\alpha, \forall t\} = 1 - \alpha. \tag{73}$$

The proof is now complete. □

Theorem 8. *Let X_t and X_t^α be the solution and α-path of the uncertain fractional differential equation*

$$\begin{cases} {}^cD^pX_t = f(t, X_t) + g(t, X_t)\dfrac{\mathrm{d}C_t}{\mathrm{d}t}, \\ X_t^k\Big|_{t=0} = X_0. \end{cases} \tag{74}$$

respectively. If the linear growth, Lipschitz and regular conditions hold, then at any time $t > 0$, we have

$$\mathcal{M}\{X_t \le X_t^\alpha\} = \alpha, \tag{75}$$

$$\mathcal{M}\{X_t > X_t^\alpha\} = 1 - \alpha. \tag{76}$$

Proof. Note that $\{X_t \le X_t^\alpha\} \supset \{X_s \le X_s^\alpha, \forall s > 0\}$ holds for any time $t > 0$. By using the Theorems 6 and 7, we obtain

$$\mathcal{M}\{X_t \le X_t^\alpha\} \ge \mathcal{M}\{X_s \le X_s^\alpha, \forall s > 0\} = \alpha. \tag{77}$$

Similarly, we also have

$$\mathcal{M}\{X_t > X_t^\alpha\} \ge \mathcal{M}\{X_s > X_s^\alpha, \forall s > 0\} = 1 - \alpha. \tag{78}$$

Since $\{X_t \le X_t^\alpha\}$ and $\{X_t > X_t^\alpha\}$ are opposite events for each $t > 0$, the duality axiom makes

$$\mathcal{M}\{X_t \le X_t^\alpha\} + \mathcal{M}\{X_t > X_t^\alpha\} = 1. \tag{79}$$

the theorem is proved. □

Theorem 9. *Let X_t and X_t^α be the solution and α-path of the uncertain fractional differential equation*

$$\begin{cases} {}^cD^pX_t = f(t, X_t) + g(t, X_t)\dfrac{\mathrm{d}C_t}{\mathrm{d}t}, \\ X_t^k\Big|_{t=0} = X_0. \end{cases} \tag{80}$$

If the linear growth, Lipschitz and regular conditions hold, then X_t has an inverse uncertainty distribution

$$\Psi_t^{-1}(\alpha) = X_t^\alpha. \tag{81}$$

Proof. Theorem 5 says that X_t^α is a continuous function with respect to α at each time $t > 0$. Theorem 7 says that

$$\mathcal{M}\{X_t \le X_t^\alpha\} = \alpha \tag{82}$$

for any $\alpha \in (0, 1)$. It follows from Theorem 3 that the inverse uncertainty distribution of X_t is $\Psi_t^{-1}(\alpha) = X_t^\alpha$. □

4. Fast Algorithm for the Inverse Uncertainty Distribution of Solutions

The Caputo fractional derivative, due to its ability to capture memory effects and long-range correlations, is widely used in modeling nonlocal processes in fields such as physics, biology, and finance. However, its numerical computation faces significant challenges. Traditional methods (e.g., the L_1 algorithm) require storing function values at all historical time steps, leading to high storage and computational complexity. To overcome this bottleneck, it is imperative to develop fast algorithms that reduce both computational and memory overhead. In this chapter, focusing on the case $0 < p < 1$, we decompose the convolution integral of the Caputo derivative on the time grid into the sum of a local part and a history part, thereby providing a feasible scheme for numerical simulation. Algorithm 1 presents a fast algorithm for computing the uncertain Caputo fractional derivative.

Algorithm 1: Fast algorithm for inverse uncertainty distribution

Input: p, Δt, T, ϵ
Output: inverse uncertainty distribution
Find an SOE approximation for t^{-p}: determine weights w_i and exponents s_i such that error $< \epsilon$, record N_{exp};
for $i = 1$ *to* N_{exp} **do**
  $U_i \leftarrow 0$// historical state variable for recursive computation;
end
for $n = 1$ *to* N **do**
  $C_l(t_n) \leftarrow \dfrac{f(t_n) - f(t_{n-1})}{\Delta t^p \cdot \Gamma(2-p)}$;
  for $i = 1$ *to* N_{exp} **do**
    $U_i(t_n) \leftarrow U_i(t_{n-1}) \cdot \exp(-s_i \Delta t)$;
    local integral term $\leftarrow$ approximate $\displaystyle\int_{t_{n-2}}^{t_{n-1}} f(\tau) e^{-s_i(t_n-\tau)} d\tau$ using linear interpolation;
    $U_i(t_n) \leftarrow U_i(t_n) +$ local integral term;
  end
  boundary term $\leftarrow$ combination of $f(t_{n-1})$ and $f(t_0)$;
  weighted sum $\leftarrow 0$;
  for $i = 1$ *to* N_{exp} **do**
    weighted sum $\leftarrow$ weighted sum $+ w_i \cdot U_i(t_n)$;
  end
  $C_h(t_n) \leftarrow \dfrac{\text{boundary term} - p \cdot \text{weighted sum}}{\Gamma(1-p)}$;
  Caputo derivative $\leftarrow C_l(t_n) + C_h(t_n)$;
  inverse uncertainty distribution $\leftarrow$ compute inverse uncertainty distribution Caputo derivative other relevant parameters;
end

Below, we apply the fast algorithm for the inverse uncertainty distribution of solutions to uncertain fractional differential equations to the following problem.

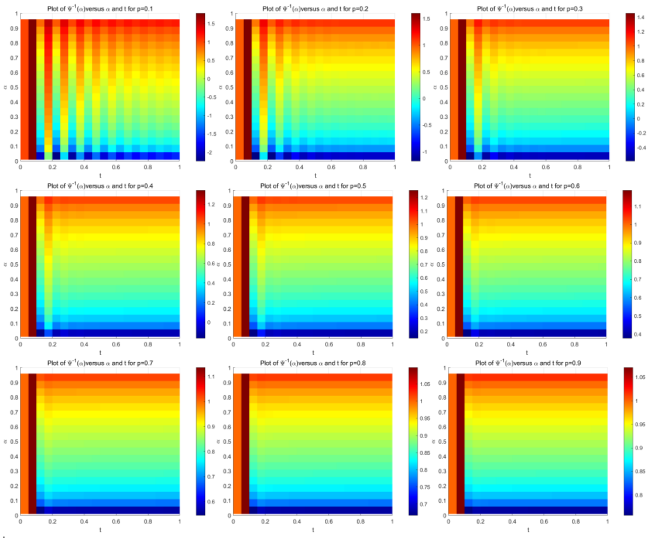

FIGURE 1. Variation of the Inverse Uncertainty Distribution $\Psi_t^{-1}(\alpha)$ with α and t for Different Fractional Orders.

Example Metabolism and clearance of a drug in human tissues with complex fractal structures (such as bones and tumors). The decline rate of drug concentration X_t may not follow the traditional exponential model but instead exhibit anomalous kinetic behavior with a "heavy-tailed" pattern. Therefore, an uncertain fractional differential equation model is established

$$\begin{cases} {}^{c}D^{p}X_t = -\lambda X_t + g(t, X_t)\dfrac{\mathrm{d}C_t}{\mathrm{d}t}, \\ X_t^k\Big|_{t=0} = X_0. \end{cases} \tag{83}$$

where p is the fractional order, $g(t, X_t)$ represents uncertain infusion, and λ denotes the drug clearance rate.

For convenience, let $\lambda = 1$ and $g(t, X_t) = 1$. Using the fast algorithm for uncertain fractional differential equations, the inverse uncertainty distribution of $\Psi_t^{-1}(\alpha)$ under different fractional orders is plotted against α and t, as shown in Figure 1.

From Figure 1, it can be observed that when the fractional order p is relatively small, the inverse uncertainty distribution of the equation is non-uniform, the memory effect is strong, and the tail is pronounced; the uncertain process does not reach stability within a short time. When the fractional order is large, the memory effect is weak, the tail is not obvious, and the uncertain process quickly attains stability, exhibiting the characteristics of an inverse uncertainty distribution. Figure 2 more clearly compares the variation of the inverse uncertainty distribution with α at the same time for different fractional orders. For all values of p, $\Psi_t^{-1}(\alpha)$ increases monotonically with α, which complies with the fundamental property of uncertainty distributions. When p is small, the curve as a whole exhibits an asymmetric shape with a narrow lower tail and a wide upper tail. This indicates that the strong memory effect associated with a low fractional order results in a more pronounced right-tail feature in the drug concentration distribution, implying that the clearance process of the drug in fractal tissue is strongly influenced by its historical concentration, leading to slow elimination and large fluctuations. When p is large, the curve is nearly linear and its symmetry is enhanced. At this point, the memory effect of the system is weak, the uncertainty mainly comes from the current excitation, and the system behaves more closely to a traditional integer-order model, with more uniform uncertainty propagation.

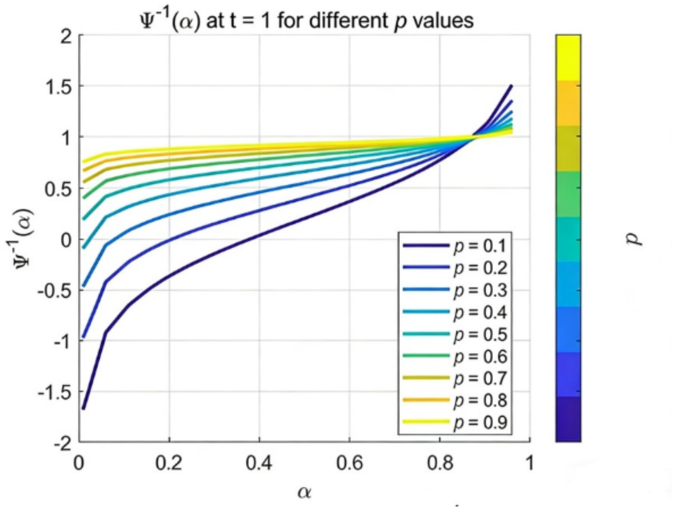

FIGURE 2. $\Psi_t^{-1}(\alpha)$ versus α at $t = 1$ for different fractional orders p

5. Conclusions

This paper makes significant contributions to the theory and computation of Caputo-type uncertain fractional differential equations. The main theoretical achievement is the proof that, under linear growth, Lipschitz, and regularity conditions, the inverse uncertainty distribution of the solution to a UFDE is equivalent to the solution of a family of fractional differential equations, referred to as the α-path. This fundamental connection bridges uncertain dynamics with fractional calculus and provides a clear pathway for analyzing the dynamics of uncertain fractional differential equations.

Building on this foundation, we have developed a fast numerical algorithm employing exponential sum approximation to efficiently compute the fractional integral kernel. This method significantly reduces the computational cost associated with the long memory of fractional operators. Consequently, it enables the computation of the inverse uncertainty distribution for solutions to uncertain fractional differential equations, even for problems exhibiting strong memory effects and heavy-tailed behavior, as demonstrated in pharmacokinetic models of drug clearance in fractal tissues.

Despite these advances, it must be acknowledged that the required conditions impose certain limitations. Future research may explore the statistical properties of solutions to UNDEs and other applications, with the aim of relaxing these constraints.

Informed by our findings, our research group is currently focusing on the inverse uncertainty distribution of solutions to time-delay uncertain differential equations. Further exploration of deeper questions, such as uncertain dynamics, uncertain bifurcations, and uncertain chaos, represents a promising direction for future work.

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