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ON FUZZY g-GROUPS

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Abstract. We develop the notion of fuzzy structures on g-groups, with emphasis on fuzzy g-subgroups, fuzzy g-ideals, and fuzzy g-filters, and give their characterizations.

1. Introduction

The notion of a g-group is an element-wise form of group-like structure in which associativity is retained, but identities and inverses are attached to individual elements rather than imposed globally [1, 4]. In this setting, many familiar group-theoretic arguments must be reformulated carefully, because a single global identity need not exist and the identities or inverses attached to a fixed element need not be unique [1, 4]. This makes g-groups a natural setting for fuzzy algebra, where level-subset methods often provide the most reliable bridge between crisp and fuzzy structure.

Fuzzy algebra begins with Zadeh's theory of fuzzy sets [2] and the early notion of a fuzzy subgroup [3]. Since then, fuzzy versions of ideals, filters, congruences, and subalgebras have been studied in a wide range of algebraic systems. Related constructions for several nonclassical algebraic settings are already available in the literature, including work on BCC-algebras, pseudo MV-algebras, BL-algebras, pseudo BE-algebras, pseudo-BL-algebras, JJ-algebras, Menger algebras, and lattices; see, for example, [5–12]. The aim of the present paper is to provide a similarly rigorous entry point for fuzzy structures on g-groups.

In an ordinary group, every element has the same identity and a unique inverse; hence, classical fuzzy subgroup conditions can be expressed directly in terms of products and inverses. In a g-group, however, an element may possess several identities and several inverses. Consequently, if one wishes the nonempty level subsets of a fuzzy set to recover the intended crisp objects exactly, then the inverse condition must be formulated with care.

The paper records practical crisp characterizations of g-subgroups and introduces corresponding notions of left, right, and two-sided g-ideals and g-filters. It then defines fuzzy g-subgroups, fuzzy g-ideals, and fuzzy g-filters by means of ordinary upper level subsets, and strong fuzzy versions by means of strong upper level subsets. Exact pointwise characterizations are established for all of these notions, together with characteristic-function, support, and monotone-transformation theorems.

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2. Preliminaries

Throughout the paper, $(G,\cdot)$ denotes a nonempty set equipped with a binary operation, and we write xy in place of $x \cdot y$.

Definition 2.1. A g-group is a nonempty set G with a binary operation $\cdot$ such that the following conditions hold [1,4]:

- (g1) $(xy)z = x(yz)$ for all $x, y, z \in G$;
- (g2) for each $x \in G$, there exists an element $e_x \in G$, called an identity element of x, such that $xe_x = e_x x = x$;
- (g3) for each $x \in G$, there exists an element $x^{-1} \in G$, called an inverse of x, such that $xx^{-1} = x^{-1}x = e_x$ for at least one identity element e_x of x.

If $xy = yx$ for all $x, y \in G$, then G is called Abelian.

Since identities and inverses attached to an element need not be unique [1,4], we use the set-valued notation

$$E_G(x) = \{e \in G : xe = x = ex\} \tag{1}$$

and

$$\mathrm{Inv}_G(x) = \{y \in G : xy = yx = e \text{ for some } e \in E_G(x)\}. \tag{2}$$

Thus $E_G(x)$ is the set of all identities of x, while $\mathrm{Inv}_G(x)$ is the set of all inverses of x. Every ordinary group is a g-group, in which case $E_G(x) = \{e\}$ and $\mathrm{Inv}_G(x) = \{x^{-1}\}$ for all $x \in G$.

Definition 2.2. Let G be a g-group. A nonempty subset $H \subseteq G$ is called a g-subgroup of G if H, with the operation inherited from G, is itself a g-group [4].

A fuzzy subset of G is a function $\mu : G \to [0,1]$. For $t \in (0,1]$ we define the upper level subset

$$U(\mu; t) = \{x \in G : \mu(x) \geq t\}, \tag{3}$$

and for $t \in [0,1)$ we define the strong upper level subset

$$U^+(\mu; t) = \{x \in G : \mu(x) > t\}. \tag{4}$$

3. Crisp auxiliary notions

The first criterion will be used repeatedly.

Proposition 3.1. *Let H be a nonempty subset of a g-group G. Then H is a g-subgroup of G if and only if the following conditions hold:*

- *(a) $xy \in H$ for all $x, y \in H$;*
- *(b) for every $x \in H$, there exists $y \in H \cap \mathrm{Inv}_G(x)$.*

Proof. Assume first that H is a g-subgroup. Then H is closed under the inherited operation, so (a) holds. Let $x \in H$. Since H is a g-group, there exist $y, e \in H$ such that $xy = e = yx$ and e is an identity of x in H. Because the operation on H is inherited from G, the same equalities hold in G, so $y \in H \cap \mathrm{Inv}_G(x)$. Thus (b) holds.

Conversely, assume (a) and (b). Associativity on H is inherited from G. Let $x \in H$. By (b), choose $y \in H \cap \mathrm{Inv}_G(x)$. Then there exists $e \in E_G(x)$ such that $xy = e = yx$. Since $x, y \in H$, condition (a) implies that $e = xy \in H$. Hence e is an identity of x lying in H, and y is an inverse of x lying in H. Therefore every element of H has an identity and an inverse in H, and H satisfies the axioms of a g-group. Thus H is a g-subgroup of G. □

Remark 3.2. In Proposition 3.1, no separate assumption on identities in H is needed. Once $x \in H$ admits an inverse $y \in H \cap \mathrm{Inv}_G(x)$, the corresponding identity $e = xy = yx$ automatically belongs to H by closure.

Definition 3.3. Let G be a g-group and let $I \subseteq G$ be nonempty.

(i) I is a left g-ideal if I is a g-subgroup of G and $GI \subseteq I$.
(ii) I is a right g-ideal if I is a g-subgroup of G and $IG \subseteq I$.
(iii) I is a two-sided g-ideal, or simply a g-ideal, if it is both a left and a right g-ideal.

Proposition 3.4. *A nonempty subset I of a g-group G is a left g-ideal if and only if the following conditions hold:*

(a) $xy \in I$ for all $x, y \in I$;
(b) for every $x \in I$, there exists $y \in I \cap \mathrm{Inv}_G(x)$;
(c) $ax \in I$ for all $a \in G$ and $x \in I$.

The analogous statement holds for right g-ideals.

Proof. This is immediate from Proposition 3.1 and the definition of a left g-ideal. □

Proposition 3.5. *In an Abelian g-group, the notions of left g-ideal and right g-ideal coincide.*

Proof. If G is Abelian and I is a left g-ideal, then for every $x \in I$ and $a \in G$ we have $xa = ax \in I$, so I is also a right g-ideal. The converse is identical. □

Definition 3.6. Let G be a g-group and let $F \subseteq G$ be nonempty.

(i) F is a left g-filter if F is a g-subgroup of G and, for all $a, x \in G$, the implication $ax \in F \Rightarrow x \in F$ holds.
(ii) F is a right g-filter if F is a g-subgroup of G and, for all $a, x \in G$, the implication $xa \in F \Rightarrow x \in F$ holds.
(iii) F is a two-sided g-filter, or simply a g-filter, if it is both a left and a right g-filter.

Proposition 3.7. *A nonempty subset F of a g-group G is a left g-filter if and only if the following conditions hold:*

(a) $xy \in F$ for all $x, y \in F$;
(b) for every $x \in F$, there exists $y \in F \cap \mathrm{Inv}_G(x)$;
(c) whenever $a \in G$ and $x \in G$ satisfy $ax \in F$, one has $x \in F$.

The analogous statement holds for right g-filters.

Proof. This is immediate from Proposition 3.1 and the definition of a left g-filter. □

Proposition 3.8. *In an Abelian g-group, the notions of left g-filter and right g-filter coincide.*

Proof. If G is Abelian and F is a left g-filter, then $xa = ax$ for all $a, x \in G$. Hence $xa \in F$ implies $ax \in F$, so $x \in F$. Thus F is also a right g-filter. The converse is identical. □

4. Fuzzy g-subgroups

Definition 4.1. Let G be a g-group and let $\mu : G \to [0,1]$ be a fuzzy subset. We say that μ is a fuzzy g-subgroup of G if, for every $t \in (0,1]$, the set $U(\mu;t)$ is either empty or a g-subgroup of G.

Definition 4.2. Let G be a g-group and let $\mu : G \to [0,1]$ be a fuzzy subset. We say that μ is a strong fuzzy g-subgroup of G if, for every $t \in [0,1)$, the set $U^+(\mu;t)$ is either empty or a g-subgroup of G.

Theorem 4.3. *Let G be a g-group and let $\mu : G \to [0,1]$ be a fuzzy subset. Then μ is a fuzzy g-subgroup of G if and only if the following conditions hold:*

(i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in G$;
(ii) for every $x \in G$, there exists $y \in \mathrm{Inv}_G(x)$ such that $\mu(y) \geq \mu(x)$.

Proof. Assume first that μ is a fuzzy g-subgroup. To prove (i), let $x, y \in G$ and put $t = \min\{\mu(x), \mu(y)\}$. If $t = 0$, there is nothing to prove. If $t > 0$, then $x, y \in U(\mu;t)$, and by definition $U(\mu;t)$ is a g-subgroup. Hence $xy \in U(\mu;t)$, so $\mu(xy) \geq t = \min\{\mu(x), \mu(y)\}$.

To prove (ii), let $x \in G$. If $\mu(x) = 0$, choose any element of $\mathrm{Inv}_G(x)$, which exists by Definition 2.1. Suppose that $\mu(x) > 0$ and put $t = \mu(x)$. Then $x \in U(\mu;t)$, so $U(\mu;t)$ is a g-subgroup. By Proposition 3.1, there exists $y \in U(\mu;t) \cap \mathrm{Inv}_G(x)$. Therefore $\mu(y) \geq t = \mu(x)$.

Conversely, assume (i) and (ii). Fix $t \in (0,1]$ and suppose that $U(\mu;t)$ is nonempty. Let $x, z \in U(\mu;t)$. Then

$$\mu(xz) \geq \min\{\mu(x), \mu(z)\} \geq t, \tag{5}$$

so $xz \in U(\mu;t)$. Next, let $x \in U(\mu;t)$. By (ii), there exists $y \in \mathrm{Inv}_G(x)$ such that $\mu(y) \geq \mu(x) \geq t$, whence $y \in U(\mu;t)$. Proposition 3.1 now shows that $U(\mu;t)$ is a g-subgroup. Therefore μ is a fuzzy g-subgroup. □

Theorem 4.4. *Let G be a g-group and let $\mu : G \to [0,1]$ be a fuzzy subset. Then μ is a strong fuzzy g-subgroup of G if and only if the following conditions hold:*

(i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in G$;
(ii) $\sup\{\mu(y) : y \in \mathrm{Inv}_G(x)\} \geq \mu(x)$ for all $x \in G$.

Proof. Assume first that μ is a strong fuzzy g-subgroup. To prove (i), let $x, y \in G$ and put $m = \min\{\mu(x), \mu(y)\}$. If $m = 0$, there is nothing to prove. If $0 < s < m$, then $x, y \in U^+(\mu;s)$, so $xy \in U^+(\mu;s)$. Hence $\mu(xy) > s$ for every $s < m$, and therefore $\mu(xy) \geq m$.

To prove (ii), let $x \in G$. If $\mu(x) = 0$, the conclusion is automatic. Suppose that $\mu(x) > 0$ and let $s < \mu(x)$. Then $x \in U^+(\mu;s)$, so $U^+(\mu;s)$ is a g-subgroup. By Proposition 3.1, there exists $y_s \in U^+(\mu;s) \cap \mathrm{Inv}_G(x)$. Hence $\mu(y_s) > s$. Since this happens for every $s < \mu(x)$, it follows that

$$\sup\{\mu(y) : y \in \mathrm{Inv}_G(x)\} \geq \mu(x). \tag{6}$$

Conversely, assume (i) and (ii). Fix $t \in [0,1)$ and suppose that $U^+(\mu;t)$ is nonempty. Let $x, z \in U^+(\mu;t)$. Then

$$\mu(xz) \geq \min\{\mu(x), \mu(z)\} > t, \tag{7}$$

so $xz \in U^+(\mu;t)$. Next, let $x \in U^+(\mu;t)$. Then $\mu(x) > t$, and by (ii) we have

$$\sup\{\mu(y) : y \in \mathrm{Inv}_G(x)\} \geq \mu(x) > t. \tag{8}$$

Therefore there exists $y \in \mathrm{Inv}_G(x)$ such that $\mu(y) > t$, that is, $y \in U^+(\mu;t)$. Proposition 3.1 now implies that $U^+(\mu;t)$ is a g-subgroup. Hence μ is a strong fuzzy g-subgroup. □

Remark 4.5. The inverse condition in Theorem 4.3 is existential, whereas the strong-cut characterization naturally involves a supremum. Consequently, every fuzzy g-subgroup is a strong fuzzy g-subgroup, but the converse requires an attainment hypothesis on inverse sets. In particular, if for every $x \in G$ the supremum of $\{\mu(y) : y \in \mathrm{Inv}_G(x)\}$ is attained, then the ordinary and strong notions coincide.

Corollary 4.6. *Every fuzzy g-subgroup of a g-group G is a strong fuzzy g-subgroup of G.*

Proof. If μ is a fuzzy g-subgroup, then by Theorem 4.3 for each $x \in G$ there exists $y \in \mathrm{Inv}_G(x)$ with $\mu(y) \geq \mu(x)$. Hence

$$\sup\{\mu(z) : z \in \mathrm{Inv}_G(x)\} \geq \mu(x), \tag{9}$$

and Theorem 4.4 applies. □

Corollary 4.7. *Let μ be a strong fuzzy g-subgroup of a g-group G. If for every $x \in G$ the supremum of $\{\mu(y) : y \in \mathrm{Inv}_G(x)\}$ is attained, then μ is a fuzzy g-subgroup.*

Proof. By Theorem 4.4, for each $x \in G$ we have

$$\sup\{\mu(y) : y \in \mathrm{Inv}_G(x)\} \geq \mu(x). \tag{10}$$

By hypothesis there exists $y_x \in \mathrm{Inv}_G(x)$ attaining this supremum. Then $\mu(y_x) \geq \mu(x)$, so Theorem 4.3 yields that μ is a fuzzy g-subgroup. □

Corollary 4.8. *If μ is a fuzzy g-subgroup of a g-group G and $x \in G$, then there exist $y \in \mathrm{Inv}_G(x)$ and $e \in E_G(x)$ such that*

$$\mu(y) \geq \mu(x) \quad \textit{and} \quad \mu(e) \geq \mu(x). \tag{11}$$

Proof. By Theorem 4.3, there exists $y \in \mathrm{Inv}_G(x)$ such that $\mu(y) \geq \mu(x)$. Choose $e \in E_G(x)$ with $xy = e = yx$. Then

$$\mu(e) = \mu(xy) \geq \min\{\mu(x), \mu(y)\} = \mu(x) \tag{12}$$

by Theorem 4.3(i). □

Theorem 4.9. *Let H be a nonempty subset of a g-group G, and let χ_H be its characteristic function. Then the following are equivalent:*

(i) H is a g-subgroup of G;
(ii) χ_H is a fuzzy g-subgroup of G;
(iii) χ_H is a strong fuzzy g-subgroup of G.

Proof. The equivalence of (i) and (ii) is proved exactly as in the standard characteristic-function argument. Assume (i). Then, by the proof of (i) ⇒ (ii), the function χ_H satisfies Theorem 4.3; hence Corollary 4.6 shows that χ_H is also a strong fuzzy g-subgroup. Thus (i) implies (iii).

Finally, assume (iii). Since the values of χ_H are only 0 and 1, the supremum condition in Theorem 4.4 implies that whenever $x \in H$, there exists $y \in \mathrm{Inv}_G(x)$ with $\chi_H(y) = 1$, that is, $y \in H$. The closure condition gives $xy \in H$ for all $x, y \in H$. Proposition 3.1 now yields that H is a g-subgroup. Therefore (iii) implies (i). □

Proposition 4.10. *If μ is a fuzzy g-subgroup of a g-group G and* $\mathrm{Supp}(\mu) \neq \emptyset$*, then* $\mathrm{Supp}(\mu)$ *is a g-subgroup of G.*

Proof. By Corollary 4.6, the fuzzy subset μ is a strong fuzzy g-subgroup. Since

$$\mathrm{Supp}(\mu) = \{x \in G : \mu(x) > 0\} = U^+(\mu; 0), \tag{13}$$

Definition 4.2 implies that $\mathrm{Supp}(\mu)$ is a g-subgroup whenever it is nonempty. □

Proposition 4.11. *Let μ be a fuzzy g-subgroup of a g-group G, and let $\varphi : [0, 1] \to [0, 1]$ be nondecreasing. Then $\varphi \circ \mu$ is a fuzzy g-subgroup of G.*

Proof. For all $x, y \in G$,

$$(\varphi \circ \mu)(xy) = \varphi(\mu(xy)) \geq \varphi\big(\min\{\mu(x), \mu(y)\}\big) = \min\{\varphi(\mu(x)), \varphi(\mu(y))\}, \tag{14}$$

where the last equality follows because a nondecreasing map preserves the lesser of two comparable real numbers. Next, let $x \in G$. By Theorem 4.3, there exists $y \in \mathrm{Inv}_G(x)$ with $\mu(y) \geq \mu(x)$. Applying φ gives

$$(\varphi \circ \mu)(y) = \varphi(\mu(y)) \geq \varphi(\mu(x)) = (\varphi \circ \mu)(x). \tag{15}$$

Hence Theorem 4.3 applies to $\varphi \circ \mu$. □

Proposition 4.12. *If G is an ordinary group, then the fuzzy g-subgroups of G are exactly the classical fuzzy subgroups of G [3].*

Proof. Assume first that μ is a fuzzy g-subgroup of the group G. Since inverses are unique in an ordinary group, Theorem 4.3 yields

$$\mu(xy) \geq \min\{\mu(x), \mu(y)\} \quad \text{and} \quad \mu(x^{-1}) \geq \mu(x) \qquad (x, y \in G). \tag{16}$$

Applying the second inequality to x^{-1} gives $\mu(x) \geq \mu(x^{-1})$, so $\mu(x^{-1}) = \mu(x)$ for all $x \in G$. Therefore,

$$\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y^{-1})\} = \min\{\mu(x), \mu(y)\}, \tag{17}$$

which is the classical subgroup condition [3].

Conversely, assume that μ is a classical fuzzy subgroup in the sense of [3], so that

$$\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\} \qquad (x, y \in G). \tag{18}$$

Taking $y = x$ gives $\mu(e) \geq \mu(x)$ for all $x \in G$. Taking $x = e$ gives

$$\mu(x^{-1}) = \mu(ex^{-1}) \geq \min\{\mu(e), \mu(x)\} = \mu(x). \tag{19}$$

Applying the same argument to x^{-1} yields $\mu(x) \geq \mu(x^{-1})$, hence $\mu(x^{-1}) = \mu(x)$. Finally,

$$\mu(xy) = \mu\big(x(y^{-1})^{-1}\big) \geq \min\{\mu(x), \mu(y^{-1})\} = \min\{\mu(x), \mu(y)\}. \tag{20}$$

Thus μ satisfies Theorem 4.3 and is therefore a fuzzy g-subgroup. □

5. Fuzzy g-ideals

Definition 5.1. Let G be a g-group and let $\mu : G \to [0,1]$ be a fuzzy subset.

(i) μ is a fuzzy left g-ideal if, for every $t \in (0,1]$, the set $U(\mu;t)$ is either empty or a left g-ideal of G.

(ii) μ is a fuzzy right g-ideal if, for every $t \in (0,1]$, the set $U(\mu;t)$ is either empty or a right g-ideal of G.

(iii) μ is a fuzzy g-ideal if it is both a fuzzy left g-ideal and a fuzzy right g-ideal.

Definition 5.2. Let G be a g-group and let $\mu : G \to [0,1]$ be a fuzzy subset.

(i) μ is a strong fuzzy left g-ideal if, for every $t \in [0,1)$, the set $U^+(\mu;t)$ is either empty or a left g-ideal of G.

(ii) μ is a strong fuzzy right g-ideal if, for every $t \in [0,1)$, the set $U^+(\mu;t)$ is either empty or a right g-ideal of G.

(iii) μ is a strong fuzzy g-ideal if it is both a strong fuzzy left g-ideal and a strong fuzzy right g-ideal.

Theorem 5.3. *Let G be a g-group and let $\mu : G \to [0,1]$ be a fuzzy subset. Then μ is a fuzzy left g-ideal if and only if the following conditions hold:*

(i) μ is a fuzzy g-subgroup of G;

(ii) $\mu(ax) \geq \mu(x)$ for all $a, x \in G$.

The analogous statement for fuzzy right g-ideals is obtained by replacing (ii) with the condition $\mu(xa) \geq \mu(x)$ for all $a, x \in G$.

Proof. Assume first that μ is a fuzzy left g-ideal. Since every left g-ideal is, in particular, a g-subgroup, each nonempty upper level subset $U(\mu;t)$ is a g-subgroup. Hence μ is a fuzzy g-subgroup.

Now let $a, x \in G$ and set $t = \mu(x)$. If $t = 0$, the desired inequality is immediate. If $t > 0$, then $x \in U(\mu;t)$, and by hypothesis $U(\mu;t)$ is a left g-ideal. Therefore $ax \in U(\mu;t)$, so $\mu(ax) \geq t = \mu(x)$.

Conversely, assume (i) and (ii). Fix $t \in (0,1]$ and suppose that $U(\mu;t)$ is nonempty. Since μ is a fuzzy g-subgroup, $U(\mu;t)$ is a g-subgroup. Let $a \in G$ and $x \in U(\mu;t)$. Then

$$\mu(ax) \geq \mu(x) \geq t, \tag{21}$$

so $ax \in U(\mu;t)$. Thus $U(\mu;t)$ is a left g-ideal. The right-sided statement is proved in the same manner. □

Theorem 5.4. *Let G be a g-group and let $\mu : G \to [0,1]$ be a fuzzy subset. Then μ is a strong fuzzy left g-ideal if and only if the following conditions hold:*

(i) μ is a strong fuzzy g-subgroup of G;

(ii) $\mu(ax) \geq \mu(x)$ for all $a, x \in G$.

The analogous statement for strong fuzzy right g-ideals is obtained by replacing (ii) with the condition $\mu(xa) \geq \mu(x)$ for all $a, x \in G$.

Proof. Assume first that μ is a strong fuzzy left g-ideal. Since every left g-ideal is a g-subgroup, each nonempty strong upper level subset $U^+(\mu;t)$ is a g-subgroup. Hence μ is a strong fuzzy g-subgroup.

Now let $a, x \in G$. If $\mu(x) = 0$, the inequality is immediate. Let $s < \mu(x)$. Then $x \in U^+(\mu; s)$, so $ax \in U^+(\mu; s)$. Thus $\mu(ax) > s$ for every $s < \mu(x)$, and therefore $\mu(ax) \geq \mu(x)$.

Conversely, assume (i) and (ii). Fix $t \in [0, 1)$ and suppose that $U^+(\mu; t)$ is nonempty. Since μ is a strong fuzzy g-subgroup, $U^+(\mu; t)$ is a g-subgroup. Let $a \in G$ and $x \in U^+(\mu; t)$. Then $\mu(x) > t$, and (ii) yields

$$\mu(ax) \geq \mu(x) > t. \tag{22}$$

Hence $ax \in U^+(\mu; t)$. Thus $U^+(\mu; t)$ is a left g-ideal. The right-sided argument is identical. □

Corollary 5.5. *In an Abelian g-group, a fuzzy left g-ideal is the same thing as a fuzzy right g-ideal, and a strong fuzzy left g-ideal is the same thing as a strong fuzzy right g-ideal.*

Proof. The crisp part follows from Proposition 3.5. The fuzzy statements follow by combining Proposition 3.5 with Theorems 5.3 and 5.4. □

Theorem 5.6. *Let I be a nonempty subset of a g-group G, and let χ_I be its characteristic function. Then the following are equivalent:*

(i) I is a left g-ideal of G;
(ii) χ_I is a fuzzy left g-ideal of G;
(iii) χ_I is a strong fuzzy left g-ideal of G.

The analogous statement holds for right g-ideals and two-sided g-ideals.

Proof. The equivalence of (i) and (ii) is the usual characteristic-function argument based on Theorem 5.3. Since every fuzzy left g-ideal is a strong fuzzy left g-ideal by Corollary 4.6 and Theorem 5.4, (ii) implies (iii).

Assume (iii). Then χ_I is a strong fuzzy g-subgroup, so Theorem 4.9 yields that I is a g-subgroup. Moreover, if $a \in G$ and $x \in I$, then $\chi_I(x) = 1$, so the inequality in Theorem 5.4 gives $\chi_I(ax) \geq 1$. Hence $ax \in I$. Therefore I is a left g-ideal. □

Proposition 5.7. *If μ is a fuzzy left g-ideal of a g-group G and* $\mathrm{Supp}(\mu) \neq \emptyset$, *then* $\mathrm{Supp}(\mu)$ *is a left g-ideal of G. The analogous statement holds for fuzzy right g-ideals and fuzzy two-sided g-ideals.*

Proof. Since every fuzzy left g-ideal is a strong fuzzy left g-ideal by Theorem 5.4 and Corollary 4.6, and since $\mathrm{Supp}(\mu) = U^+(\mu; 0)$, the claim follows immediately from Definition 5.2. □

Proposition 5.8. *Let μ be a fuzzy left g-ideal of a g-group G, and let $\varphi : [0, 1] \to [0, 1]$ be nondecreasing. Then $\varphi \circ \mu$ is a fuzzy left g-ideal of G. The analogous statement holds for fuzzy right g-ideals and fuzzy two-sided g-ideals.*

Proof. By Proposition 4.11, $\varphi \circ \mu$ is a fuzzy g-subgroup. For all $a, x \in G$,

$$(\varphi \circ \mu)(ax) = \varphi(\mu(ax)) \geq \varphi(\mu(x)) = (\varphi \circ \mu)(x), \tag{23}$$

since $\mu(ax) \geq \mu(x)$ and φ is nondecreasing. Theorem 5.3 completes the proof. □

Theorem 5.9. *If G is an ordinary group and μ is a fuzzy left g-ideal or a fuzzy right g-ideal of G, then μ is constant on G.*

Proof. Assume that μ is a fuzzy left g-ideal. Let e be the identity of G. For any $g \in G$, Theorem 5.3 gives

$$\mu(g) = \mu(ge) \geq \mu(e). \tag{24}$$

On the other hand, Proposition 4.12 shows that μ is a classical fuzzy subgroup [3], so $\mu(g^{-1}) = \mu(g)$. Hence

$$\mu(e) = \mu(gg^{-1}) \geq \min\{\mu(g), \mu(g^{-1})\} = \mu(g). \tag{25}$$

Therefore $\mu(g) = \mu(e)$ for all $g \in G$. The proof for fuzzy right g-ideals is identical, using $\mu(g) = \mu(eg) \geq \mu(e)$. □

Corollary 5.10. *If G is an ordinary group, then the only nonempty left, right, or two-sided g-ideal of G is G itself.*

Proof. Let I be a nonempty left g-ideal of G. By Theorem 5.6, χ_I is a fuzzy left g-ideal. Theorem 5.9 implies that χ_I is constant. Since I is nonempty, this constant must be 1, so $I = G$. The right-sided and two-sided cases are identical. □

6. Fuzzy g-filters

Definition 6.1. Let G be a g-group and let $\mu : G \to [0, 1]$ be a fuzzy subset.

(i) μ is a fuzzy left g-filter if, for every $t \in (0, 1]$, the set $U(\mu; t)$ is either empty or a left g-filter of G.
(ii) μ is a fuzzy right g-filter if, for every $t \in (0, 1]$, the set $U(\mu; t)$ is either empty or a right g-filter of G.
(iii) μ is a fuzzy g-filter if it is both a fuzzy left g-filter and a fuzzy right g-filter.

Definition 6.2. Let G be a g-group and let $\mu : G \to [0, 1]$ be a fuzzy subset.

(i) μ is a strong fuzzy left g-filter if, for every $t \in [0, 1)$, the set $U^+(\mu; t)$ is either empty or a left g-filter of G.
(ii) μ is a strong fuzzy right g-filter if, for every $t \in [0, 1)$, the set $U^+(\mu; t)$ is either empty or a right g-filter of G.
(iii) μ is a strong fuzzy g-filter if it is both a strong fuzzy left g-filter and a strong fuzzy right g-filter.

Theorem 6.3. *Let G be a g-group and let $\mu : G \to [0, 1]$ be a fuzzy subset. Then μ is a fuzzy left g-filter if and only if the following conditions hold:*

(i) μ is a fuzzy g-subgroup of G;
(ii) $\mu(x) \geq \mu(ax)$ for all $a, x \in G$.

The analogous statement for fuzzy right g-filters is obtained by replacing (ii) with the condition $\mu(x) \geq \mu(xa)$ for all $a, x \in G$.

Proof. Assume first that μ is a fuzzy left g-filter. Since every left g-filter is, in particular, a g-subgroup, each nonempty upper level subset $U(\mu; t)$ is a g-subgroup. Hence μ is a fuzzy g-subgroup.

Now let $a, x \in G$ and put $t = \mu(ax)$. If $t = 0$, the inequality is immediate. If $t > 0$, then $ax \in U(\mu; t)$. Because $U(\mu; t)$ is a left g-filter, the implication defining a left g-filter yields $x \in U(\mu; t)$. Thus $\mu(x) \geq t = \mu(ax)$.

Conversely, assume (i) and (ii). Fix $t \in (0,1]$ and suppose that $U(\mu;t)$ is nonempty. Since μ is a fuzzy g-subgroup, $U(\mu;t)$ is a g-subgroup. Let $a, x \in G$ and assume $ax \in U(\mu;t)$. Then $\mu(ax) \geq t$, so (ii) gives $\mu(x) \geq \mu(ax) \geq t$. Hence $x \in U(\mu;t)$. Thus $U(\mu;t)$ is a left g-filter. The right-sided statement is analogous. □

Theorem 6.4. *Let G be a g-group and let $\mu : G \to [0,1]$ be a fuzzy subset. Then μ is a strong fuzzy left g-filter if and only if the following conditions hold:*

(i) μ is a strong fuzzy g-subgroup of G;

(ii) $\mu(x) \geq \mu(ax)$ for all $a, x \in G$.

The analogous statement for strong fuzzy right g-filters is obtained by replacing (ii) with the condition $\mu(x) \geq \mu(xa)$ for all $a, x \in G$.

Proof. Assume first that μ is a strong fuzzy left g-filter. Since every left g-filter is a g-subgroup, each nonempty strong upper level subset $U^+(\mu;t)$ is a g-subgroup. Hence μ is a strong fuzzy g-subgroup.

Now let $a, x \in G$. If $\mu(ax) = 0$, the inequality is immediate. Let $s < \mu(ax)$. Then $ax \in U^+(\mu;s)$. Because $U^+(\mu;s)$ is a left g-filter, one has $x \in U^+(\mu;s)$. Thus $\mu(x) > s$ for every $s < \mu(ax)$, and therefore $\mu(x) \geq \mu(ax)$.

Conversely, assume (i) and (ii). Fix $t \in [0,1)$ and suppose that $U^+(\mu;t)$ is nonempty. Since μ is a strong fuzzy g-subgroup, $U^+(\mu;t)$ is a g-subgroup. Let $a, x \in G$ and assume $ax \in U^+(\mu;t)$. Then $\mu(ax) > t$, so (ii) implies $\mu(x) \geq \mu(ax) > t$. Hence $x \in U^+(\mu;t)$. Therefore $U^+(\mu;t)$ is a left g-filter. The right-sided argument is identical. □

Corollary 6.5. *In an Abelian g-group, a fuzzy left g-filter is the same thing as a fuzzy right g-filter, and a strong fuzzy left g-filter is the same thing as a strong fuzzy right g-filter.*

Proof. Combine Proposition 3.8 with Theorems 6.3 and 6.4. □

Theorem 6.6. *Let F be a nonempty subset of a g-group G, and let χ_F be its characteristic function. Then the following are equivalent:*

(i) F is a left g-filter of G;

(ii) χ_F is a fuzzy left g-filter of G;

(iii) χ_F is a strong fuzzy left g-filter of G.

The analogous statement holds for right g-filters and two-sided g-filters.

Proof. The equivalence of (i) and (ii) follows from Theorem 6.3. Also, every fuzzy left g-filter is a strong fuzzy left g-filter by Corollary 4.6 and Theorem 6.4, so (ii) implies (iii).

Assume (iii). Then χ_F is a strong fuzzy g-subgroup, so Theorem 4.9 yields that F is a g-subgroup. If $a, x \in G$ and $ax \in F$, then $\chi_F(ax) = 1$, and Theorem 6.4 gives $\chi_F(x) \geq \chi_F(ax) = 1$. Hence $x \in F$. Thus F is a left g-filter. □

Proposition 6.7. *If μ is a fuzzy left g-filter of a g-group G and $\mathrm{Supp}(\mu) \neq \emptyset$, then $\mathrm{Supp}(\mu)$ is a left g-filter of G. The analogous statement holds for fuzzy right g-filters and fuzzy two-sided g-filters.*

Proof. Since every fuzzy left g-filter is a strong fuzzy left g-filter and $\mathrm{Supp}(\mu) = U^+(\mu;0)$, the conclusion follows from Definition 6.2. □

Proposition 6.8. *Let μ be a fuzzy left g-filter of a g-group G, and let $\varphi : [0,1] \to [0,1]$ be nondecreasing. Then $\varphi \circ \mu$ is a fuzzy left g-filter of G. The analogous statement holds for fuzzy right g-filters and fuzzy two-sided g-filters.*

Proof. By Proposition 4.11, $\varphi \circ \mu$ is a fuzzy g-subgroup. For all $a, x \in G$,

$$(\varphi \circ \mu)(x) = \varphi(\mu(x)) \geq \varphi(\mu(ax)) = (\varphi \circ \mu)(ax), \tag{26}$$

since $\mu(x) \geq \mu(ax)$ and φ is nondecreasing. Theorem 6.3 completes the proof. □

Theorem 6.9. *If G is an ordinary group and μ is a fuzzy left g-filter or a fuzzy right g-filter of G, then μ is constant on G.*

Proof. Assume that μ is a fuzzy left g-filter. Let e be the identity of G. For any $g \in G$, Theorem 6.3 gives

$$\mu(e) = \mu(g^{-1}g) \leq \mu(g) \tag{27}$$

and also

$$\mu(g) = \mu(ge) \leq \mu(e). \tag{28}$$

Hence $\mu(g) = \mu(e)$ for all $g \in G$. The proof for fuzzy right g-filters is identical. □

Corollary 6.10. *If G is an ordinary group, then the only nonempty left, right, or two-sided g-filters of G is G itself.*

Proof. Let F be a nonempty left g-filter of G. By Theorem 6.6, χ_F is a fuzzy left g-filter. Theorem 6.9 implies that χ_F is constant. Since F is nonempty, this constant must be 1, so $F = G$. The right-sided and two-sided cases are identical. □

7. Examples

The standard finite example in the g-group literature is the set

$$S_6 = \{0, 1, 2, 3, 4, 5\} \tag{29}$$

with multiplication modulo 6. It is known that $(S_6, \times_6)$ is an Abelian g-group [4].

Example 7.1. Let

$$H = \{1, 5\}, \qquad I = \{0, 2, 3, 4\}. \tag{30}$$

Then H is a g-subgroup of S_6. Indeed, H is closed under multiplication modulo 6, and each of its elements has an inverse in H: for 1 we may take the inverse 1 with identity 1, while for 5 we may take the inverse 5 with identity 1 because $5 \cdot 5 \equiv 1 \pmod 6$ and $5 \cdot 1 \equiv 5 \equiv 1 \cdot 5 \pmod 6$.

The set I is a two-sided g-ideal of S_6. First, I is closed under multiplication modulo 6. Next, each element of I has an inverse in I: 0 is self-inverse with identity 0, 3 is self-inverse with identity 3, 4 is self-inverse with identity 4, and 2 is self-inverse relative to the identity 4 since $2 \cdot 2 \equiv 4 \pmod 6$ and $2 \cdot 4 \equiv 2 \equiv 4 \cdot 2 \pmod 6$. Thus I is a g-subgroup by Proposition 3.1. Finally, for every $a \in S_6$ and every $x \in I$, one has $ax \in I$; since S_6 is Abelian, Proposition 3.5 yields that I is a two-sided g-ideal.

The set H is also a two-sided g-filter of S_6. Indeed, H is already a g-subgroup. If $a, x \in S_6$ and $ax \in H$, then ax is a unit modulo 6, so x must also be a unit modulo 6, whence $x \in H$. Since S_6 is Abelian, Proposition 3.8 implies that H is a two-sided g-filter.

Example 7.2. Define the characteristic function $\mu_H : S_6 \to [0,1]$ by

$$\mu_H(x) = \begin{cases} 1, & x \in H, \\ 0, & x \notin H. \end{cases} \tag{31}$$

By Theorems 4.9 and 6.6, μ_H is both a fuzzy g-subgroup and a fuzzy g-filter of S_6.

Example 7.3. Define $\nu : S_6 \to [0,1]$ by

$$\nu(0) = 1, \qquad \nu(2) = \nu(3) = \nu(4) = 0.7, \qquad \nu(1) = \nu(5) = 0. \tag{32}$$

Then ν is a fuzzy g-ideal of S_6. Indeed, for $0 < t \leq 0.7$ we have

$$U(\nu; t) = \{0, 2, 3, 4\} = I, \tag{33}$$

which is a g-ideal by Example 7.1. For $0.7 < t \leq 1$ we have

$$U(\nu; t) = \{0\}. \tag{34}$$

The set $\{0\}$ is a two-sided g-ideal of S_6 because it is closed under multiplication, 0 is self-inverse with identity 0, and $a \cdot 0 = 0 = 0 \cdot a$ for all $a \in S_6$. Thus every nonempty upper level subset of ν is a g-ideal, so ν is a fuzzy g-ideal.

Example 7.4. Define $\lambda : S_6 \to [0,1]$ by

$$\lambda(1) = \lambda(5) = 1, \qquad \lambda(0) = \lambda(2) = \lambda(3) = \lambda(4) = 0.4. \tag{35}$$

Then λ is a fuzzy g-filter of S_6. Indeed, if $0 < t \leq 0.4$, then $U(\lambda; t) = S_6$, which is trivially a two-sided g-filter of itself. If $0.4 < t \leq 1$, then

$$U(\lambda; t) = \{1, 5\} = H, \tag{36}$$

which is a two-sided g-filter by Example 7.1. Hence every nonempty upper level subset of λ is a g-filter.

Example 7.5. Let G be any ordinary group. By Theorems 5.9 and 6.9, every fuzzy g-ideal and every fuzzy g-filter on G is constant. Thus the theories of fuzzy g-ideals and fuzzy g-filters become genuinely nontrivial only in proper g-groups, where identities and inverses depend on the element.

8. Conclusion

We have developed a level-subset approach to fuzzy structures on g-groups through the study of fuzzy g-subgroups, fuzzy g-ideals, and fuzzy g-filters. Because identities and inverses in a g-group are not unique, the level-subset method is not merely convenient but structurally appropriate.

Three features deserve particular emphasis. First, the subgroup theory is compatible with the classical case: when the underlying g-group is an ordinary group, the resulting notion coincides with the classical fuzzy subgroup notion [3]. Second, both the ideal theory and the filter theory are genuinely new: in ordinary groups, fuzzy g-ideals and fuzzy g-filters degenerate to constant fuzzy subsets, whereas nonconstant examples arise naturally in proper g-groups such as $(S_6, \times_6)$. Third, the strong upper level viewpoint makes clear where inverse nonuniqueness forces a supremum-based formulation.

Natural directions for further work include normality-type conditions, homomorphism theorems, quotient constructions, and intuitionistic or interval-valued fuzzy analogues. Any such extension will likely require additional hypotheses controlling the interaction of element-wise identities and inverses.

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