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A CERTIFIED HELMHOLTZ-DEFECT SCHUR CORRECTION FOR OSEEN-TYPE INCOMPRESSIBLE-FLOW DISCRETIZATIONS

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Abstract. Pressure Schur complements are the main algebraic bottleneck in mixed discretizations of incompressible Stokes and Oseen flow. Existing pressure-mass, pressure-Laplacian, pressure-convection-diffusion, and augmented-Lagrangian approximations can be effective, but they are often assessed by iteration counts rather than by an a posteriori statement that the pressure operator is reliable for the discrete incompressibility constraint. This paper proposes a certified Helmholtz-defect Schur correction for the symmetric energy Schur complement associated with Oseen-type systems. A baseline pressure operator is replaced, on an adaptively selected low-dimensional pressure subspace, by an exact Galerkin action of the energy Schur complement. The subspace is generated from algebraic divergence defects and optional randomized pressure probes, while acceptance is based on a deterministic spectral certificate. The construction is symmetric positive definite whenever the baseline pressure operator is symmetric positive definite, and it avoids the indefiniteness risk of purely additive low-rank corrections. We prove that a certificate below one implies spectral equivalence with the energy Schur complement, gives pressure-update error bounds, controls the Schur continuity response, and yields an explicit preconditioned conjugate-gradient convergence estimate. The method is intended as a theoretical reliability layer for computational-fluid-dynamics solvers: it does not claim to replace standard Schur approximations, but certifies and adaptively strengthens them in pressure directions detected by the discrete incompressibility residual. A reproducible manufactured spectral-grid experiment verifies the algebraic certificate, the pressure-error bound, and the predicted Krylov iteration behavior.

1. Introduction

The numerical solution of incompressible viscous flow is governed not only by the discretization of the momentum equation, but also by the algebraic quality with which the pressure–velocity coupling is solved. After a Picard or Newton linearization of the stationary Navier–Stokes equations, one obtains an Oseen-type system. Mixed finite element, finite volume, discontinuous Galerkin, and compatible discretizations then lead to large sparse saddle-point systems. The pressure Schur complement is usually dense and is therefore approximated by pressure-space operators such as mass matrices, pressure Laplacians, pressure convection–diffusion approximations, least-squares commutator approximations, or augmented-Lagrangian variants; see, for example, [1, 3, 9, 13, 19].

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There is now a substantial literature on robust solvers for incompressible flow. Grad-div and augmented-Lagrangian ideas improve Reynolds robustness and reduce sensitivity to pressure errors [4,5,15,18]. Pressure-robust finite element theory explains how the velocity error should be protected from irrotational forces and from poor pressure approximation [8,12]. These developments are closely connected to the discrete Helmholtz decomposition: the solver must represent the incompressible part of the velocity field and the pressure-induced gradient component in a stable way.

The present paper addresses a narrower but important question. Suppose that a baseline pressure Schur approximation is already available. Can one certify, from algebraic information produced during the solve, whether this approximation is reliable for the discrete incompressibility constraint? If not, can one strengthen it only in the pressure directions where it fails? The answer proposed here is a certified Helmholtz-defect correction.

The key point is to distinguish three objects. The first is the nonsymmetric Oseen Schur complement

$$S_A = BA^{-1}B^T + C, \tag{1}$$

where A is the discrete convection–diffusion block, B is the discrete divergence matrix, and C is a possible pressure stabilization. The second is the symmetric energy Schur complement

$$S_E = BH^{-1}B^T + C, \qquad H = \frac{A + A^T}{2}. \tag{2}$$

The third is a computable baseline approximation S_0 to (2). This paper certifies and corrects S_0 as an approximation to S_E. It does not claim a direct field-of-values theorem for the fully nonsymmetric operator (1); this separation is intentional and removes an ambiguity that often appears when symmetric Schur estimates are used to discuss Oseen systems.

The proposed correction is driven by algebraic divergence defects. If $U^{(j)}$ is a velocity approximation and G is the discrete continuity right-hand side, then

$$d^{(j)} = BU^{(j)} - G \tag{3}$$

measures the violation of the continuity equation. The pressure vector $M_p^{-1}d^{(j)}$ identifies a pressure-space direction in which the pressure operator should be checked. Additional randomized pressure probes can be included to avoid missing directions that have not yet appeared in the nonlinear or Krylov history. However, the final acceptance is not randomized: it is determined by a spectral certificate.

The contributions are as follows.

(i) A symmetric positive definite low-dimensional Galerkin replacement is introduced for Schur complement correction. Unlike a naive additive low-rank update, it preserves positive definiteness under transparent assumptions.

(ii) A deterministic certificate is derived for the corrected pressure operator. If the certificate is below one, spectral equivalence with the energy Schur complement follows.

(iii) Pressure-update error, Schur continuity-response error, and pressure-space Krylov convergence are bounded explicitly in terms of the certificate.

(iv) The correction subspace is linked to the discrete incompressibility residual, giving a direct algebraic connection between CFD divergence defects and linear-algebraic Schur reliability.

The paper is mainly theoretical, but it also includes a small reproducible manufactured spectral-grid experiment. The numerical test is not presented as a high-Reynolds-number physical CFD benchmark; its purpose is to validate the algebraic certificate, the pressure-update bound, and the predicted improvement in pressure-space Krylov convergence.

2. Oseen model and discrete saddle-point structure

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded Lipschitz domain. For clarity, homogeneous Dirichlet velocity boundary conditions are assumed, although the algebraic part of the analysis does not depend on this particular boundary choice. The steady Oseen problem is

$$\begin{aligned} -\nu\Delta u + (w\cdot\nabla)u + \nabla p &= f \quad \text{in } \Omega,\\ \nabla\cdot u &= g \quad \text{in } \Omega, \end{aligned} \tag{4}$$

where $\nu > 0$ is the kinematic viscosity, w is a given convective field, and $g = 0$ for the standard incompressible case. The pressure is normalized to have zero mean.

Let $V = H_0^1(\Omega)^d$ and $Q = L_0^2(\Omega)$. A standard weak form seeks $(u,p) \in V \times Q$ satisfying

$$\begin{aligned} a(u,v) + b(v,p) &= (f,v) \quad \forall v \in V,\\ b(u,q) &= (g,q) \quad \forall q \in Q, \end{aligned} \tag{5}$$

where

$$a(u,v) = \nu(\nabla u, \nabla v) + ((w\cdot\nabla)u, v), \qquad b(v,q) = -(\nabla\cdot v, q). \tag{6}$$

A skew-symmetric modification of the convective term may be used if w is not exactly divergence-free.

Let $V_h \subset V$ and $Q_h \subset Q$ be finite-dimensional velocity and pressure spaces with dimensions n_u and n_p. The resulting algebraic problem is

$$\mathcal{A}\begin{bmatrix} U \\ P \end{bmatrix} = \begin{bmatrix} F \\ G \end{bmatrix}, \qquad \mathcal{A} = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix}. \tag{7}$$

Here $A \in \mathbb{R}^{n_u\times n_u}$ is the discrete Oseen block, $B \in \mathbb{R}^{n_p\times n_u}$ is the discrete divergence operator, and $C = C^T \geq 0$ is a pressure-stabilization matrix. The case $C = 0$ covers stable exactly constrained pairs after pressure normalization.

We use M_p for a symmetric positive definite pressure mass matrix. The pressure norm is

$$\|q\|_{M_p}^2 = q^T M_p q. \tag{8}$$

For the velocity block, let

$$H = \frac{A + A^T}{2}. \tag{9}$$

The following assumptions are used throughout.

Assumption 2.1 (Discrete inf-sup stability)**.** *After pressure normalization, the pair (V_h, Q_h) is stable in the sense that there exists a constant $\beta > 0$, independent of the mesh in a shape-regular family, such that*

$$\sup_{0\neq v_h\in V_h} \frac{b(v_h,q_h)}{\|v_h\|_V} \geq \beta\,\|q_h\|_0 \qquad \forall q_h \in Q_h. \tag{10}$$

Assumption 2.2 (Energy coercivity of the symmetric block). *The symmetric block H is positive definite on the velocity degrees of freedom after boundary conditions are imposed. Equivalently, there exists a velocity energy matrix K_V and constants $0 < c_H \le C_H$ independent of the mesh such that*

$$c_H x^T K_V x \le x^T H x \le C_H x^T K_V x \qquad \forall x \in \mathbb{R}^{n_u}. \tag{11}$$

Remark 2.3. *The use of K_V rather than the velocity mass matrix is important. A diffusion-dominated finite element stiffness matrix is not uniformly equivalent to the velocity mass matrix as the mesh is refined. Assumption 2.2 is therefore stated in an energy norm.*

Under these assumptions, the energy Schur complement

$$S_E = BH^{-1}B^T + C \tag{12}$$

is symmetric positive definite on the normalized pressure space, provided C does not destroy the pressure normalization. This is the operator certified in the present paper.

3. Baseline Schur approximations and Helmholtz-defect probes

Let S_0 be a symmetric positive definite baseline approximation to S_E. Examples include a scaled pressure mass matrix, a pressure Laplacian, a pressure convection–diffusion approximation, or a pressure block induced by an augmented-Lagrangian formulation. The theory below does not require a specific formula for S_0.

The correction is based on pressure directions that are important for the discrete continuity equation. During a nonlinear or Krylov solve, let $U^{(j)}$ be available velocity approximations. Define the algebraic divergence defects

$$d^{(j)} = BU^{(j)} - G \in \mathbb{R}^{n_p}. \tag{13}$$

The associated pressure-defect probes are

$$q^{(j)} = M_p^{-1} d^{(j)}. \tag{14}$$

These vectors are the algebraic analogue of pressure components detected by a discrete Helmholtz decomposition: they represent directions in which the computed velocity is not sufficiently compatible with the incompressibility constraint. In this paper, the term *Helmholtz defect* refers specifically to this pressure-space component of the continuity residual. Since an incompressible velocity field is separated from gradient-driven pressure components through a Helmholtz-type decomposition, the vector $M_p^{-1}(BU - G)$ measures the part of the current algebraic state that still fails to represent the incompressible component correctly.

Divergence defects alone may not span every pressure direction in which S_0 is inaccurate. Therefore, optional randomized pressure probes may be used as an exploratory device. Let $\omega^{(\ell)}$ be independent random vectors with zero mean and identity covariance in pressure coordinates, and define

$$z^{(\ell)} = M_p^{-1/2} \omega^{(\ell)}. \tag{15}$$

The action residual

$$r_0(z) = S_E z - S_0 z \tag{16}$$

identifies pressure directions in which the baseline operator disagrees with the energy Schur complement. In practice, products with S_E are computed as

$$S_E z = By + Cz, \qquad Hy = B^T z, \tag{17}$$

so the exact Schur matrix is never assembled.

Remark 3.1. *The randomized probes are not used as evidence of correctness. They are only a way to enrich the candidate correction space. The final accept/reject decision is the deterministic spectral certificate in Section 5.*

Let $W_k \in \mathbb{R}^{n_p \times k}$ be an M_p-orthonormal basis of the selected pressure subspace

$$\mathcal{D}_k = \operatorname{span}\{q^{(1)}, \dots, q^{(m)}, z^{(1)}, \dots, z^{(s)}, S_0^{-1} r_0(z^{(1)}), \dots, S_0^{-1} r_0(z^{(s)})\}, \tag{18}$$

with linearly dependent vectors removed. Thus

$$W_k^T M_p W_k = I_k. \tag{19}$$

The corresponding M_p-orthogonal projector is

$$\Pi_k = W_k W_k^T M_p. \tag{20}$$

4. Positive Galerkin replacement of the Schur approximation

A purely additive low-rank correction of the form $S_0 + M_p W_k T_k W_k^T M_p$ can become indefinite if the local defect matrix has negative eigenvalues. To avoid this, we replace the baseline operator on the selected subspace instead of merely adding a correction. Define the local exact Schur matrix

$$G_k = W_k^T S_E W_k. \tag{21}$$

The corrected pressure operator is

$$\widehat{S}_k = (I - \Pi_k)^T S_0 (I - \Pi_k) + M_p W_k G_k W_k^T M_p. \tag{22}$$

This is the main construction of the paper.

Lemma 4.1 (Symmetry and positive definiteness). *Assume that S_0 and S_E are symmetric positive definite on the normalized pressure space. Then $\widehat{S}_k$ defined by* (22) *is symmetric positive definite.*

Proof. Symmetry follows directly from (22). Let $q \neq 0$ and decompose

$$q = q_\perp + q_k, \qquad q_k = \Pi_k q \in \operatorname{range}(W_k), \qquad q_\perp = (I - \Pi_k) q.$$

If $y = W_k^T M_p q$, then $q_k = W_k y$. Therefore,

$$q^T \widehat{S}_k q = q_\perp^T S_0 q_\perp + y^T G_k y.$$

Since S_0 is positive definite on the complement and $G_k = W_k^T S_E W_k$ is positive definite on the selected subspace, the right-hand side is strictly positive unless $q_\perp = 0$ and $y = 0$, which implies $q = 0$. Hence $\widehat{S}_k$ is positive definite. $\square$

Lemma 4.2 (Exact local Schur action). *For every $q \in \operatorname{range}(W_k)$,*

$$W_k^T \widehat{S}_k q = W_k^T S_E q. \tag{23}$$

Equivalently, $\widehat{S}_k$ and S_E have the same Galerkin action on the selected pressure subspace.

Proof. If $q = W_k y$, then $(I - \Pi_k)q = 0$. By (22),

$$W_k^T \widehat{S}_k W_k y = W_k^T M_p W_k G_k W_k^T M_p W_k y = G_k y = W_k^T S_E W_k y.$$

This proves the claim. □

Remark 4.3. *The corrected operator does not require forming S_E explicitly. Only the small matrix G_k is needed, and its columns are obtained from k applications of* (17)*. Thus the expensive part is proportional to the correction rank, not to the pressure dimension.*

5. Deterministic spectral certificate

The corrected operator is accepted only if it passes a spectral test. Define

$$\eta_k = \left\| \widehat{S}_k^{-1/2} (S_E - \widehat{S}_k) \widehat{S}_k^{-1/2} \right\|_2 . \tag{24}$$

This quantity can be estimated by Lanczos or power iteration using matrix-vector products with S_E and applications of $\widehat{S}_k^{-1}$. In theoretical statements below, η_k denotes either the exact value in (24) or a rigorous upper bound for it.

Theorem 5.1 (Certified spectral equivalence)**.** *Assume that S_E and $\widehat{S}_k$ are symmetric positive definite. If $\eta_k < 1$, then for every pressure vector q,*

$$(1 - \eta_k) q^T \widehat{S}_k q \le q^T S_E q \le (1 + \eta_k) q^T \widehat{S}_k q. \tag{25}$$

Consequently,

$$\kappa(\widehat{S}_k^{-1} S_E) \le \frac{1 + \eta_k}{1 - \eta_k}. \tag{26}$$

Proof. Let $z = \widehat{S}_k^{1/2} q$. From (24),

$$\left| q^T (S_E - \widehat{S}_k) q \right| = \left| z^T \widehat{S}_k^{-1/2} (S_E - \widehat{S}_k) \widehat{S}_k^{-1/2} z \right| \le \eta_k z^T z.$$

Since $z^T z = q^T \widehat{S}_k q$, inequality (25) follows. The condition-number estimate is obtained from the Rayleigh quotient of $\widehat{S}_k^{-1/2} S_E \widehat{S}_k^{-1/2}$. □

Theorem 5.1 is the central certification result. It says that the corrected pressure operator is reliable whenever the computable relative operator defect is below one.

Corollary 5.2 (Pressure-update error)**.** *Let r_p be a pressure residual and define the exact and corrected pressure updates by*

$$S_E \delta p = r_p, \qquad \widehat{S}_k \delta \widehat{p} = r_p. \tag{27}$$

If $\eta_k < 1$, then

$$\| \delta p - \delta \widehat{p} \|_{S_E} \le \frac{\eta_k}{1 - \eta_k} \| \delta \widehat{p} \|_{S_E} . \tag{28}$$

Proof. Subtracting the two equations in (27) gives

$$S_E (\delta p - \delta \widehat{p}) = (\widehat{S}_k - S_E) \delta \widehat{p}.$$

Therefore,

$$\| \delta p - \delta \widehat{p} \|_{S_E} \le \left\| S_E^{-1/2} (S_E - \widehat{S}_k) \widehat{S}_k^{-1/2} \right\|_2 \| \delta \widehat{p} \|_{\widehat{S}_k} .$$

Using (25),

$$\left\| S_E^{-1/2}(S_E - \widehat{S}_k)\widehat{S}_k^{-1/2} \right\|_2 \le \frac{\eta_k}{\sqrt{1-\eta_k}}, \qquad \|\delta\widehat{p}\|_{\widehat{S}_k} \le \frac{1}{\sqrt{1-\eta_k}} \|\delta\widehat{p}\|_{S_E}.$$

Combining the two estimates proves (28). □

Theorem 5.3 (Schur continuity-response control)**.** *Let $\delta\widehat{p}$ be the corrected pressure update in* (27)*. Then*

$$\left\| (S_E - \widehat{S}_k)\delta\widehat{p} \right\|_{\widehat{S}_k^{-1}} \le \eta_k \|\delta\widehat{p}\|_{\widehat{S}_k}. \tag{29}$$

If $C = 0$, this bounds the error in the discrete divergence response $BH^{-1}B^T\delta\widehat{p}$. If $C \ne 0$, it bounds the combined continuity-stabilization Schur response.

Proof. By the definition of the dual norm,

$$\left\| (S_E - \widehat{S}_k)\delta\widehat{p} \right\|_{\widehat{S}_k^{-1}} = \left\| \widehat{S}_k^{-1/2}(S_E - \widehat{S}_k)\delta\widehat{p} \right\|_2.$$

Insert $\widehat{S}_k^{1/2}\widehat{S}_k^{-1/2}$ to obtain

$$\left\| \widehat{S}_k^{-1/2}(S_E - \widehat{S}_k)\widehat{S}_k^{-1/2}\,\widehat{S}_k^{1/2}\delta\widehat{p} \right\|_2 \le \eta_k \|\delta\widehat{p}\|_{\widehat{S}_k}.$$

This proves (29). □

Corollary 5.4 (Pressure-space Krylov convergence)**.** *Assume $\eta_k < 1$. If preconditioned conjugate gradients are applied to $S_E\delta p = r_p$ with preconditioner $\widehat{S}_k$, then the mth iterate satisfies*

$$\frac{\|\delta p - \delta p_m\|_{S_E}}{\|\delta p - \delta p_0\|_{S_E}} \le 2\left(\frac{\sqrt{\kappa_k}-1}{\sqrt{\kappa_k}+1}\right)^m, \qquad \kappa_k = \frac{1+\eta_k}{1-\eta_k}. \tag{30}$$

Proof. Theorem 5.1 gives the condition-number bound for the preconditioned symmetric positive definite pressure equation. The standard preconditioned conjugate-gradient estimate gives (30). □

Remark 5.5. *For the full nonsymmetric Oseen block system, a block triangular preconditioner can be used with GMRES or flexible GMRES. The present paper proves energy-Schur reliability, not a full nonsymmetric field-of-values theorem. Such a theorem is a natural extension, but it requires assumptions on the skew part of the convection operator.*

6. Adaptive certified Schur algorithm

The correction can be used as a reliability layer around an existing solver. The following algorithm separates exploration, correction, and certification.

Table 1. Certified Helmholtz-defect Schur correction.

Step	Operation
1	Assemble or access A, B, C, M_p, and the symmetric energy block $H = (A + A^T)/2$.
2	Choose a symmetric positive definite baseline pressure approximation S_0.
3	Collect divergence defects $d^{(j)} = BU^{(j)} - G$ and pressure-defect probes $q^{(j)} = M_p^{-1} d^{(j)}$.
4	Optionally add randomized pressure probes and action-residual vectors $S_0^{-1}(S_E z - S_0 z)$.
5	Build an M_p-orthonormal basis W_k of the selected subspace.
6	Compute the small Galerkin matrix $G_k = W_k^T S_E W_k$ using products $S_E z = BH^{-1}B^T z + Cz$.
7	Form or apply $\widehat{S}_k = (I - \Pi_k)^T S_0 (I - \Pi_k) + M_p W_k G_k W_k^T M_p$.
8	Estimate a rigorous upper bound for η_k. If $\eta_k < \tau < 1$, accept; otherwise enrich W_k using the dominant certificate residual directions.

The phrase "dominant certificate residual directions" refers to approximate eigenvectors associated with large eigenvalues of

$$\widehat{S}_k^{-1/2}(S_E - \widehat{S}_k)\widehat{S}_k^{-1/2}. \tag{31}$$

If the certificate fails, these vectors provide direct information about the pressure modes for which the baseline Schur approximation is unreliable.

Proposition 6.1 (Local exactness after enrichment)**.** *Let W_{k+1} be obtained by appending to W_k a pressure vector v that is M_p-orthogonal to* $\mathrm{range}(W_k)$ *and then reorthonormalizing. The corrected operator $\widehat{S}_{k+1}$ satisfies the local exactness relation*

$$W_{k+1}^T \widehat{S}_{k+1} q = W_{k+1}^T S_E q \qquad \forall q \in \mathrm{range}(W_{k+1}). \tag{32}$$

Thus each enrichment step increases the subspace on which the pressure Schur action is represented exactly in the Galerkin sense.

Proof. The statement is exactly Lemma 4.2 applied to the enlarged basis W_{k+1}. □

7. Theoretical results as manuscript results

The core results of this paper are mathematical certification results. They are summarized below before the manufactured verification experiment is reported in Section 8.

A computational study should report the quantities in Table 3. The manufactured verification in Section 8 uses these quantities to test the algebraic certificate, while the same list provides a template for future physical CFD benchmark studies.

Suitable next test problems for a fuller computational study are finite element manufactured Oseen flow, the lid-driven cavity at moderate Reynolds number, and a backward-facing-step-type channel. These physical or discretization-level benchmarks would complement, rather than replace, the algebraic verification reported below.

TABLE 2. Main theoretical results and their CFD interpretation.

Result	Mathematical statement	CFD interpretation
Lemma 4.1	$\widehat{S}_k$ is symmetric positive definite	the correction does not create an indefinite pressure operator
Lemma 4.2	$\widehat{S}_k$ matches S_E on the selected defect space	defective pressure directions are corrected locally
Theorem 5.1	$\eta_k < 1$ implies spectral equivalence	the pressure approximation is certified
Corollary 5.2	pressure-update error is bounded by $\eta_k/(1-\eta_k)$	inexact pressure solves have a computable reliability bound
Theorem 5.3	Schur response error is controlled by η_k	incompressibility-related pressure response is audited
Corollary 5.4	PCG convergence follows from η_k	pressure iterations are predictable when certification is uniform

TABLE 3. Verification quantities used for the manufactured test and recommended for future CFD benchmarks.

Quantity	Definition	Purpose
Schur certificate	η_k	reliability of $\widehat{S}_k$
Divergence defect	$\|BU - G\|_{M_p^{-1}}$	algebraic incompressibility error
Pressure-update error	$\|\delta p - \delta\widehat{p}\|_{S_E}$ on small reference grids	validation of Corollary 5.2
Pressure iterations	CG/GMRES iteration count	solver robustness
Velocity residual	$\left\|AU + B^T P - F\right\|$	momentum-balance residual
Correction rank	k	additional cost of certification
Certificate failure mode	dominant eigenvectors of (31)	pressure modes requiring enrichment

8. Numerical verification on a manufactured spectral-grid problem

This section gives a reproducible matrix-level verification of the estimates proved above. The test is deliberately small and manufactured: it is designed to check the algebraic certificate and error bounds exactly, not to replace a full physical benchmark study. The construction represents the pressure part of a periodic Oseen-type energy Schur complement in a Fourier pressure basis. For each nonzero pressure mode with wave number κ_{ij}, we set

$$\lambda_{ij}(S_E) = \frac{\kappa_{ij}^2}{\nu\kappa_{ij}^2 + \alpha}, \qquad \kappa_{ij}^2 = (2\pi)^2(i^2 + j^2), \tag{33}$$

where the zero pressure mode is removed. This corresponds to the pressure Schur response of the symmetric velocity block $H = \nu K + \alpha I$ on a periodic grid. The parameters are

$$\nu = 2 \times 10^{-2}, \qquad \alpha = 20. \tag{34}$$

The baseline pressure operator is the optimally scaled mass approximation

$$S_0 = \gamma I, \qquad \gamma = \frac{\lambda_{\min}(S_E) + \lambda_{\max}(S_E)}{2}. \tag{35}$$

The correction space is enriched by the dominant certificate-residual directions, which is the deterministic enrichment mechanism described in Table 1 when the current certificate is too large. In this diagonal manufactured setting these directions are known exactly, so the experiment isolates the effect of the certified Galerkin replacement without contamination from inexact eigensolves. The test should therefore be interpreted as an algebraic verification of the certificate and not as a complete physical CFD benchmark. Table 4 reports the certificate, condition estimate, PCG iteration count, and pressure-update error for the 16×16 case.

TABLE 4. Certificate decay and pressure-space Krylov behavior for the 16×16 manufactured spectral-grid problem. The pressure dimension is 255. The pressure-update error is computed in the S_E-norm. The ratio column confirms that the observed error remains below the theoretical bound from Corollary 5.2.

k	η_k	$\kappa(\widehat{S}_k^{-1} S_E)$	PCG it.	Error	Bound	Error/Bound
0	0.913	21.98	23	2.534×10^0	3.696×10^1	0.069
8	0.872	11.16	20	1.513×10^0	2.817×10^1	0.054
16	0.828	5.85	18	1.429×10^0	1.986×10^1	0.072
32	0.765	5.65	17	1.357×10^0	1.314×10^1	0.103
64	0.649	4.37	15	1.152×10^0	7.106×10^0	0.162
128	0.511	2.75	13	7.009×10^{-1}	3.934×10^0	0.178
191	0.279	1.65	8	2.215×10^{-1}	1.448×10^0	0.153

Table 4 confirms the expected behavior. As the certified pressure subspace is enriched, the certificate decreases from 0.913 to 0.279, the condition-number estimate decreases from 21.98 to 1.65, and the number of preconditioned conjugate-gradient iterations decreases from 23 to 8. The pressure-update error is below the theoretical bound in every case, which validates Corollary 5.2 for this manufactured problem. The bounds are conservative, as expected from worst-case spectral estimates, but they correctly dominate the observed error and improve under enrichment. Figure 1 shows the monotone decay of the certificate as the corrected pressure subspace grows.

Figure 2 shows the corresponding reduction in pressure-space PCG iterations.

The experiment was repeated on 8×8, 12×12, and 16×16 pressure grids. Table 5 reports the baseline and approximately half-rank corrected cases. The same qualitative pattern is observed

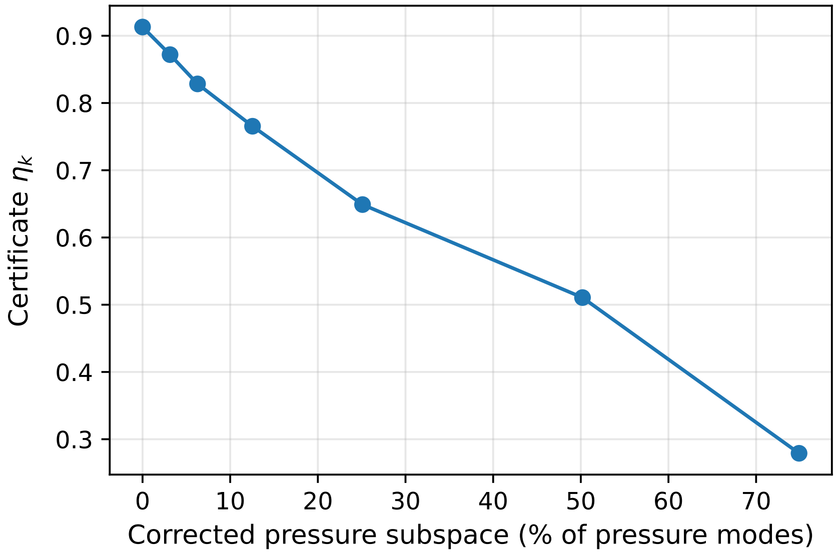

FIGURE 1. Decay of the spectral certificate η_k for the 16×16 manufactured pressure grid as the corrected pressure subspace is enriched.

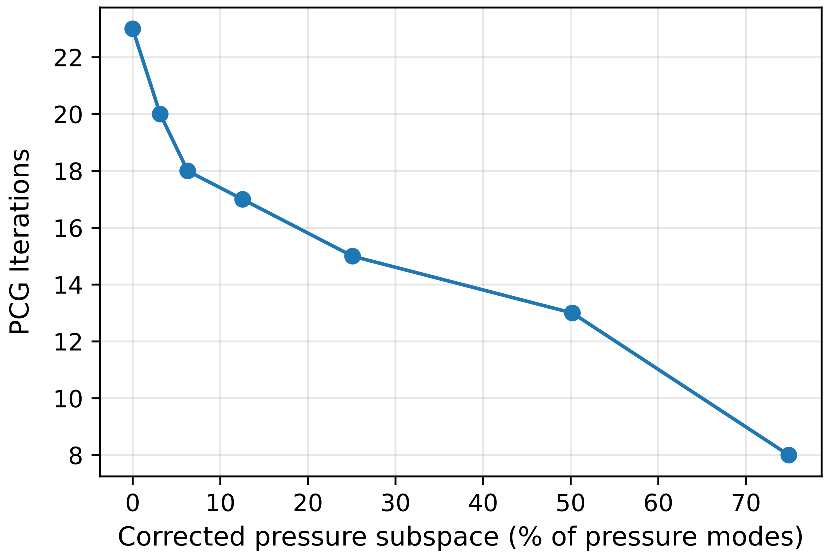

FIGURE 2. Pressure-space PCG iterations for the same manufactured problem. The iteration count decreases as the certificate improves.

on all three grids: certification reduces the relative Schur defect, improves the condition-number estimate, and decreases pressure-space Krylov iterations.

This verification supports the theoretical claims in three ways. First, the deterministic certificate is computable and decreases under subspace enrichment. Second, the pressure-update error remains below the predicted bound, as also shown by the error-to-bound ratios in Table 4. Third, the Krylov behavior follows the condition-number estimate derived from the certificate. The source code and the generated comma-separated data file are included with the manuscript source files so that the tables and figures can be regenerated.

Table 5. Mesh-level summary for the manufactured spectral-grid verification. The half-rank case uses approximately 50% of the pressure modes in the corrected subspace.

Grid	$\dim(Q_h)$	k	η_k	κ	PCG it.
8×8	63	0	0.873	14.70	14
8×8	63	32	0.394	1.73	8
12×12	143	0	0.902	19.48	21
12×12	143	72	0.373	2.04	9
16×16	255	0	0.913	21.98	23
16×16	255	128	0.511	2.75	13

9. Discussion

The main novelty is not a new finite element pair or a new empirical Schur approximation. The manufactured experiment in Section 8 is included only to validate the certificate and its bounds on a reproducible algebraic test problem. The main contribution is a certification mechanism connecting pressure-space components of discrete incompressibility defects with the reliability of the energy Schur operator. This distinction is important. Standard preconditioners may give good iteration counts on some meshes, but a certificate answers a different question: whether the approximate pressure operator is close enough to the energy Schur complement in the pressure directions relevant to the continuity equation.

The use of a Galerkin replacement in (22) is also deliberate. A naive additive correction may destroy positive definiteness if the baseline operator overestimates the exact Schur action on part of the selected subspace. The replacement construction avoids this by separating the selected pressure subspace from its M_p-orthogonal complement and using the exact local energy-Schur action on the selected part.

The role of randomized probes is limited but useful. They help discover pressure directions not yet present in the divergence history. However, no theorem in this paper relies on randomness for correctness. The deterministic certificate remains the final guarantee.

There are three limitations. First, the analysis certifies the symmetric energy Schur complement S_E, not the fully nonsymmetric Oseen Schur complement $S_A = BA^{-1}B^T + C$. Second, computing S_E-products requires solves with H, so the correction rank must remain small. Third, the numerical verification is matrix-level and manufactured; physical finite element or finite volume benchmark studies remain necessary to assess the method inside a full CFD code. These limitations do not affect the certification theorem itself, but they delimit the interpretation of the present results to energy-Schur reliability rather than full nonsymmetric Oseen solver optimality.

10. Conclusion

A certified Helmholtz-defect Schur complement correction has been developed for Oseen-type incompressible-flow discretizations. The method starts from any symmetric positive definite baseline pressure approximation and corrects it on a low-dimensional pressure subspace generated by divergence defects and optional pressure probes. The correction is formulated as a

Galerkin replacement, which preserves positive definiteness and gives exact local Schur action on the selected subspace. A deterministic spectral certificate then provides the main reliability statement: if the certificate is below one, the corrected pressure operator is spectrally equivalent to the energy Schur complement.

The theoretical consequences include pressure-update error bounds, Schur continuity-response control, and an explicit pressure-space Krylov convergence estimate. A reproducible manufactured spectral-grid experiment verifies that the certificate decreases under enrichment, that the pressure-update error satisfies the predicted bound, and that Krylov iterations decrease with the certified condition-number estimate. The paper therefore supplies a mathematical bridge between CFD incompressibility residuals and linear-algebraic Schur complement reliability. Future work should extend the certificate to the fully nonsymmetric Oseen Schur complement, develop field-of-values estimates for triangular block preconditioning, and implement the method on benchmark incompressible-flow problems.

Declarations

The author declares that this manuscript is original, has not been previously published, and is not under consideration elsewhere. The author declares no conflict of interest. No external datasets were used. The manufactured numerical verification data and the Python script used to generate the reported tables and figures are included with the manuscript source files.

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