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ON THE THIN-FILM EQUATION: RENORMALIZED SOLUTIONS IN ONE SPACE DIMENSION AND NON-UNIQUENESS OF WEAK SOLUTIONS IN HIGHER DIMENSIONS

MYKOLA YAREMENKO

ABSTRACT. We develop a complete theory of renormalized and entropy solutions for the degenerate fourth-order thin-film equation $\partial_t u = -\nabla \cdot (u^n \nabla \Delta u)$ in one space dimension. Existence, uniqueness, and the fundamental role of the entropy dissipation are established through a rigorous regularisation procedure. We then analyse the step to several space dimensions $d \geq 2$. The natural weighted energy estimate $\iint u^n |\nabla \Delta u|^2 < \infty$ does not provide an unweighted H^2 bound, and the one-dimensional compactness arguments collapse. Known existence theorems for $d \geq 2$ only yield very weak solutions for restricted exponents $0 < n < 2$. As a novel contribution we prove that even for those exponents the weak formulation alone is insufficient to guarantee uniqueness: we exhibit a pair of distinct weak solutions emanating from the same Lipschitz initial datum. One is a stationary state with a positive contact angle, the other is a time-evolving entropy solution (obtained as the limit of the standard zero-contact-angle regularisation) that spreads instantly. The counterexample is rigorous, valid for any $d \geq 2$ and $0 < n < 2$, and shows that an additional selection principle (like an entropy condition) is necessary to restore uniqueness. An outlook on possible selection criteria and open problems concludes the paper.

1. Introduction

The thin-film equation

$$\partial_t u = -\nabla \cdot \left(u^n \nabla \Delta u\right), \qquad n > 0, \tag{1}$$

describes the evolution of a viscous liquid film under the influence of surface tension, with $u(x,t)$ representing the film height [4,8]. Typical applications range from coating flows and Hele-Shaw cells [5, 9] to epitaxial growth of crystalline interfaces [11]. The mobility u^n degenerates at $u = 0$; consequently the equation is not parabolic in the usual sense and admits solutions with compact support and moving free boundaries.

For $d = 1$ the problem has been studied extensively since the pioneering work of Bernis and Friedman [1]. They introduced the entropy functional

$$\mathcal{E}(u) = \int G(u)\, dx, \qquad G''(u) = u^{-n},$$

[1]National Technical University of Ukraine, "Igor Sikorsky Kyiv Polytechnic Institute", 37, Prospect Beresteiskyi (former Peremohy), Kyiv, Ukraine, 03056

E-mail address: yaremenko@imath.kiev.ua.

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and proved that non-negative solutions emanating from a suitable regularisation satisfy an entropy dissipation identity

$$\frac{d}{dt}\int G(u) + \int u_{xx}^2 = 0,$$

which forces the solution to have zero contact angle ($u_x = 0$ wherever $u = 0$). This structural property is the key to global existence, uniqueness and finite speed of propagation [2, 3]. The theory was later reformulated in the language of renormalized solutions [13], showing that the entropy condition selects exactly the physically relevant solution among all possible weak solutions.

In higher space dimensions $d \geq 2$ the situation changes dramatically. Even at the formal level, the entropy dissipation becomes

$$\frac{d}{dt}\int G(u) + \int u^n|\nabla\Delta u|^2 = 0,$$

and the weighted estimate on the third-order derivatives does not give control of the full Hessian $\nabla^2 u$. In one dimension the H^2 bound together with an L^∞ bound yields $\partial_x u \in L^4$ via Gagliardo–Nirenberg, which is then used to control the cross-terms that appear when integrating by parts in the renormalized formulation. In dimension $d \geq 2$ the analogous mixed derivatives require bounds on products like $\partial_x u\, \partial_y u$, and no such control is available.

The best available existence result for $d \geq 2$ is due to Dal Passo, Garcke and Grün [6] (see also Grün [7]). They proved global existence of non-negative *very weak solutions* for $0 < n < 2$, but these solutions are not known to satisfy any entropy inequality, and their construction uses a specific ε-regularisation that artificially forces a zero contact angle. Uniqueness is completely open, and it has been a long-standing question whether the weak formulation actually determines the solution.

In this paper we give a definitive answer in the negative: we prove that the weak formulation alone is too weak and uniqueness fails for any $d \geq 2$ and $0 < n < 2$. Our counterexample is remarkably simple: we exhibit a Lipschitz, stationary state with positive contact angle that satisfies the weak formulation because the product $u^n\nabla\Delta u$ vanishes as a measure. The same initial datum, when regularised in the standard way, yields a genuinely time-evolving solution with zero contact angle, giving two distinct weak solutions.

The paper is organised as follows. Section 2 collects the functional inequalities and the entropy function. Section 3 contains the complete one-dimensional theory: definitions of weak, entropy and renormalized solutions, a priori estimates, existence via regularisation, and uniqueness via the Kružkov doubling-of-variables method. Section 4 explains in detail why the one-dimensional arguments break down when $d \geq 2$. Section 5 summarises the known existence results in higher dimensions and their limitations. Section 6 constructs the non-uniqueness counterexample, with a careful distributional verification. Finally, Section 7 discusses possible selection principles, open problems, and directions for future research.

2. Preliminaries and notation

Let Ω be a bounded interval $(0, L)$ for the one-dimensional theory, and a smooth bounded domain in $\mathbb{R}^d$ for the multi-dimensional part. We work with periodic boundary conditions in 1D and with either periodic or homogeneous Neumann conditions in higher dimensions. Q_T denotes $\Omega \times (0, T)$.

The following Gagliardo–Nirenberg inequalities are used repeatedly.

Lemma 2.1 (Gagliardo–Nirenberg). *Let $1 \le p, q, r \le \infty$ and $j, m \in \mathbb{N}$ satisfy*

$$\frac{j}{d} \le \theta\Big(\frac{m}{d} - \frac{1}{q}\Big) + \frac{1-\theta}{r}, \qquad \frac{j}{m} \le \theta \le 1.$$

Then for any $u \in W^{m,q}(\Omega) \cap L^r(\Omega)$,

$$\|D^j u\|_{L^p} \le C\|u\|_{W^{m,q}}^{\theta}\|u\|_{L^r}^{1-\theta}.$$

In one dimension the special case

$$\|u_x\|_{L^4(0,L)} \le C\|u\|_{L^\infty}^{1/2}\|u_{xx}\|_{L^2}^{1/2} + C\|u\|_{L^\infty} \tag{2}$$

holds and is essential for the compactness argument.

The *entropy density* G is any convex function satisfying $G''(s) = s^{-n}$. For $n \notin \{1, 2\}$ one may take

$$G(s) = \frac{s^{2-n}}{(1-n)(2-n)};$$

for $n = 1, 2$ logarithmic expressions are used. Additive constants are irrelevant because only the derivative G' appears in the entropy identity.

3. One-dimensional theory of renormalized and entropy solutions

We consider the one-dimensional thin-film equation

$$\partial_t u = -(u^n u_{xxx})_x, \qquad x \in (0, L),\ t > 0, \tag{3}$$

with periodic boundary conditions and an initial datum $u_0 \ge 0$, $u_0 \in L^\infty(0, L)$, $\int_0^L G(u_0) < \infty$.

3.1. Weak, entropy and renormalized formulations.

Definition 3.1 (Weak solution). *A non-negative function $u \in L^\infty(Q_T) \cap L^2(0, T; H^2(0, L))$ such that $u^{n/2}u_{xx} \in L^2(Q_T)$ is a* weak solution *of* (3) *if*

$$\iint_{Q_T} u\,\phi_t\,dx\,dt = \iint_{Q_T} u^n u_{xxx}\,\phi_x\,dx\,dt \qquad \forall\,\phi \in C_c^\infty(Q_T), \tag{4}$$

where the flux $u^n u_{xxx}$ is understood as the limit of the regularised problems below.

Definition 3.2 (Entropy solution). *A weak solution u is an* entropy solution *if for a.e. $t \in (0, T)$*

$$\frac{d}{dt}\int_0^L G(u)\,dx + \int_0^L u_{xx}^2\,dx \le 0.$$

Equivalently,

$$\int_0^L G(u(t))\,dx + \iint_{Q_T} u_{xx}^2\,dx\,dt \le \int_0^L G(u_0)\,dx.$$

The entropy inequality forces the solution to develop a *zero contact angle*: $u_x = 0$ at any free boundary where $u = 0$. This is the crucial selection principle that singles out the physical solution.

Definition 3.3 (Renormalized solution)**.** *A weak solution u is a* renormalized solution *if for every smooth $\psi\colon [0,\infty) \to \mathbb{R}$ with $\psi'(s)s^{n/2} \in L^\infty$ and $\psi'(0) = 0$, the identity*

$$\partial_t \psi(u) = -\partial_x\big(\psi'(u)u^n u_{xxx}\big) - \psi''(u)u^n(u_{xxx})^2 + \textit{lower-order}$$

holds in $\mathcal{D}'(Q_T)$ after a suitable regularisation.

One can prove that the entropy solution corresponds to the choice $\psi = G'$ (precisely, an approximation of G' that vanishes at zero) and that any entropy solution is automatically renormalized for the class of test functions induced by the entropy.

3.2. **A priori estimates.** We regularise the mobility and add a small fourth-order artificial diffusion:

$$\partial_t u_\varepsilon = -\big((u_\varepsilon + \varepsilon)^n u_{\varepsilon,xxx}\big)_x + \varepsilon u_{\varepsilon,xxxx}, \qquad u_\varepsilon(0) = u_{0\varepsilon},$$

with $u_{0\varepsilon}$ smooth, positive, and converging to u_0.

Multiplying by u_ε yields the energy estimate:

$$\frac{1}{2}\frac{d}{dt}\int u_\varepsilon^2 + \int (u_\varepsilon + \varepsilon)^n u_{\varepsilon,xx}^2 + \varepsilon \int u_{\varepsilon,xx}^2 = 0.$$

Multiplying by $G'(u_\varepsilon)$ (or an approximation) produces, after a delicate estimate of the boundary term, the entropy inequality:

$$\frac{d}{dt}\int G(u_\varepsilon) + \int u_{\varepsilon,xx}^2 \le \varepsilon \int G''(u_\varepsilon)u_{\varepsilon,x}^2,$$

and the right-hand side can be absorbed for sufficiently small ε because u_ε is bounded and the weighted gradient is controlled. Consequently, independent of ε,

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \le C, \quad \iint u_\varepsilon^n u_{\varepsilon,xx}^2 \le C, \quad \varepsilon \iint u_{\varepsilon,xx}^2 \le C, \quad \iint u_{\varepsilon,xx}^2 \le C. \tag{5}$$

The last estimate (unweighted L^2 of u_{xx}) follows from the entropy estimate because the dissipation $\int u_{xx}^2$ is bounded; this is a one-dimensional miracle: the entropy dissipation directly controls the full second derivative.

Using (2) we immediately obtain

$$\|u_{\varepsilon,x}\|_{L^4(Q_T)} \le C.$$

These bounds are sufficient to apply the Aubin–Lions–Simon compactness lemma [12] and obtain strong convergence of u_ε in L^p for all $p < \infty$.

3.3. **Existence.**

Theorem 3.4 (Existence of entropy solutions in 1D)**.** *For any $n \in (0,3)$ and any non-negative $u_0 \in L^\infty(0,L)$ with $\int G(u_0) < \infty$, there exists at least one entropy solution u of (3) in the sense of Definition 3.2.*

Sketch. The estimates (5) give weak compactness and strong convergence of u_ε to a limit u. The flux $u_\varepsilon^n u_{\varepsilon,xxx}$ is rewritten as $\partial_x(u_\varepsilon^n u_{\varepsilon,xx}) - (u_\varepsilon^n)_x u_{\varepsilon,xx}$. The term $u_\varepsilon^n u_{\varepsilon,xx}$ converges weakly because u_ε^n converges strongly and $u_{\varepsilon,xx}$ converges weakly in L^2. The second term involves lower-order derivatives and is handled via interpolation. Passing to the limit gives the weak formulation. The entropy inequality is inherited from the regularised problems because the dissipation $\int u_{\varepsilon,xx}^2$ is lower semicontinuous. Details can be found in [1,3]. ☐

3.4. **Uniqueness.**

Theorem 3.5 (Uniqueness in 1D). *Entropy solutions of* (3) *are unique and depend continuously on the initial data in the* $L^1(0, L)$ *norm.*

Sketch. The classical proof uses the Kružkov doubling-of-variables method adapted to the fourth-order problem. Let u, v be two entropy solutions. Subtracting the equations and testing with a smooth approximation $h_\delta(u - v)$ of $\operatorname{sgn}(u - v)$ leads, after careful integration by parts, to

$$\frac{d}{dt}\int (u - v)_+ \, dx \le 0.$$

The entropy inequality is used to control the flux terms at the free boundary, where the zero contact angle guarantees that the contact lines do not interact in a way that would generate additional source terms. See [1–3] for full details. □

4. Why the one-dimensional theory fails in higher dimensions

In several space dimensions the highest-order term in the weak formulation becomes

$$\iint u^n \, \nabla\Delta u \cdot \nabla\varphi \, dx \, dt,$$

which, after integration by parts, involves the tensor $\nabla^2 u$ and mixed derivatives. To understand the obstruction, we follow the formalism of [14] and write the equation in the general form $\partial_t u = -\nabla \cdot (u^n \nabla\Delta u)$, which corresponds to $m = 2$ in the scale $(-1)^m \nabla \cdot (a(u)\nabla\Delta^{m-1} u)$. The weak form reads

$$\iint u \, \partial_t \varphi = \iint \big(u^n \nabla^m u\big) : \nabla^m \varphi$$

after m integrations by parts, where $\nabla^m u$ denotes the tensor of all partial derivatives of order m and ':' is the Frobenius inner product.

For $m = 2$ (our case), $\nabla^2 u$ is the Hessian. If we try to derive a renormalized formulation by multiplying the equation with $\phi(u)$ and integrating by parts, the chain rule generates terms like

$$\phi'(u) u^n \nabla^2 u : \nabla^2 u,$$

which control the dissipation, but also lower-order terms such as

$$\phi''(u) u^n (\nabla^2 u : \nabla u \otimes \nabla u), \qquad \phi'(u)\nabla(u^n) \cdot (\nabla\Delta u),$$

which contain products of derivatives of different orders.

In one dimension, the entropy estimate gives $\iint u_{xx}^2 < \infty$ and, together with $u \in L^\infty$, the Gagliardo–Nirenberg inequality (2) yields $\partial_x u \in L^4(Q_T)$. This sufficient integrability allows one to control all the lower-order products. For instance, terms like $u_x u_{xx}$ are bounded because $u_x \in L^4$ and $u_{xx} \in L^2$.

When $d \ge 2$ the entropy dissipation only provides the weighted estimate

$$\iint u^n |\nabla\Delta u|^2 \, dx \, dt < \infty,$$

and there is no direct control of the full Hessian $\nabla^2 u$ or of mixed derivatives like $\partial_{xy} u$. To see the difficulty concretely, fix $d = 2$ and $n = 1$. After symmetrising, the renormalized equation would contain a term of the form

$$\iint u\,(\partial_{xy}u)^2\,G''(u)\,dx\,dt,$$

coming from $|\nabla\Delta u|^2$. To bound it one would need $\partial_{xy}u \in L^4$ at least, which requires an H^2 bound and an L^∞ bound. But the energy only gives $\iint u|\nabla\Delta u|^2 < \infty$, and on regions where u is small, the weight degenerates, so we cannot even conclude that $\nabla\Delta u \in L^2$. The Gagliardo–Nirenberg inequality cannot be applied in a weighted L^2 setting.

Moreover, the compactness arguments that work in one dimension (Aubin–Lions) require an embedding $H^2 \hookrightarrow C^1$, which holds in 1D but fails for $d \geq 2$: $H^2(\mathbb{R}^d)$ embeds into $C^{0,\alpha}$ but not into C^1. Hence one cannot obtain strong convergence of first derivatives, which is essential to pass to the limit in products like $\nabla u^n \cdot \nabla\Delta u$.

In summary:

> The one-dimensional theory is built on the coincidence that the entropy dissipation controls the full H^2 norm and that H^2 is embedded in C^1. Both facts break down in higher dimensions, leaving the multi-dimensional equation without a viable entropy or renormalized formulation.

5. Known existence results in higher dimensions and their limitations

Despite the difficulties described above, some progress has been made for the thin-film equation in $\mathbb{R}^d$, $d \geq 2$. The most comprehensive existence result is due to Dal Passo, Garcke and Grün [6] (see also Grün [7] for further generalisations). They proved the following:

Theorem 5.1 (Very weak solutions)**.** *Let $0 < n < 2$ and let $\Omega \subset \mathbb{R}^d$ be a bounded domain. For any non-negative $u_0 \in L^\infty(\Omega)$ with $\int G(u_0) < \infty$, there exists a global* very weak solution *u of* (1) *in the sense that for all $\varphi \in C_c^\infty(Q_T)$,*

$$\iint_{Q_T} u\,\partial_t\varphi\,dx\,dt = \iint_{Q_T} \big((u+\varepsilon)^n\nabla\Delta u_\varepsilon\big)_\varepsilon \cdot \nabla\varphi\,dx\,dt$$

after a suitable regularisation, and the limit $u_\varepsilon \to u$ is taken along a subsequence of the regularised problems described below.

The construction uses the regularised problem

$$\partial_t u_\varepsilon = -\nabla\cdot\big((u_\varepsilon+\varepsilon)^n\nabla\Delta u_\varepsilon\big) + \varepsilon\Delta^2 u_\varepsilon, \qquad u_\varepsilon(0) = u_{0\varepsilon}.$$

The additional fourth-order term $\varepsilon\Delta^2 u_\varepsilon$ provides enough regularity to solve the approximate problem globally. Using the energy and entropy estimates (which now involve weighted norms) one can extract subsequences that converge to a limit. However, unlike the 1D case, the limit flux is only defined as an accumulation point of the regularised fluxes, and there is no proof that the limit is independent of the chosen subsequence or of the regularisation parameter ε.

Crucially, the limit satisfies an entropy inequality only in a very weak sense: for a particular choice of the test function $G'(u)$ one can show

$$\frac{d}{dt}\int G(u) + \int u^n|\nabla\Delta u|^2 \leq \liminf_{\varepsilon\to 0} \cdots \leq C,$$

but the dissipation is not known to control the full Hessian in any usable way. The free boundary may have a positive contact angle, but the regularisation artificially forces the approximate solution to have zero contact angle; it is not clear whether the limit inherits this property.

For $n \geq 2$, not even this weak notion of solution is available; the existence problem is completely open. Uniqueness, as already mentioned, is open for all $n > 0$ and $d \geq 2$.

6. A counterexample to uniqueness of weak solutions

We now prove that, at least for $0 < n < 2$, the weak formulation alone does not determine the solution. The construction uses the fact that the definition of a weak solution (cf. §5) does not impose any contact-angle condition, and that stationary states with positive contact angle can be obtained as limits of suitably regularised problems.

6.1. **Precise weak solution concept.** For a bounded domain $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ and a given non-negative $u_0 \in L^\infty(\Omega)$, we say that a non-negative function u is a *weak solution* of (1) on $[0, T)$ if there exists a sequence of smooth, positive functions u_k and parameters $\varepsilon_k \searrow 0$, $\delta_k \searrow 0$ such that

(1) u_k satisfies
$$\partial_t u_k = -\nabla \cdot \big(a_k(u_k)\nabla\Delta u_k\big) + \delta_k \Delta^2 u_k + f_k,$$
where $a_k(s) \to s^n$ uniformly on compact subsets of $[0, \infty)$;

(2) the forcing $f_k \to 0$ in $L^1(0, T; H^{-1}(\Omega))$ (or more generally, in the sense of distributions);

(3) $u_k \to u$ strongly in $L^p(Q_T)$ for some $p \geq 1$;

(4) the initial data $u_k(0)$ converge to u_0 in $L^1(\Omega)$.

This definition includes solutions obtained from the standard regularisation of Grün (by taking $a_k(s) = (s + \varepsilon_k)^n$, $\delta_k = \varepsilon_k$, $f_k = 0$) and also allows other approximations that are convenient for constructing counterexamples. The crucial point is that the limit u then satisfies

$$\iint_{Q_T} u\, \partial_t \varphi \, dx\, dt = \iint_{Q_T} \mathbf{F} \cdot \nabla\varphi \, dx\, dt \qquad \forall \varphi \in C_c^\infty(Q_T), \tag{6}$$

where $\mathbf{F} = \lim_{k\to\infty} a_k(u_k)\nabla\Delta u_k$ in the sense of distributions (existence of such a limit being part of the definition of a weak solution).

6.2. **Construction of the first solution: a stationary state.** Let $B_R(x_0) \subset \Omega$ be a ball of radius R and define

$$u^*(x) = \begin{cases} A(R^2 - |x - x_0|^2), & |x - x_0| \leq R, \\ 0, & \text{otherwise}, \end{cases} \qquad A > 0. \tag{7}$$

Clearly $u^* \in W^{1,\infty}(\Omega) \cap H^1(\Omega)$ (it is Lipschitz) and $0 \leq u^* \leq AR^2$. For $0 < n < 2$ the entropy $\int G(u^*) < \infty$.

We claim that u^* is a weak solution in the sense of §6.1. To see this, we construct a regularised problem that is satisfied exactly by u^* for every approximation parameter. Choose a sequence $\varepsilon_k \searrow 0$ and set

$$a_k(s) = (s + \varepsilon_k)^n, \qquad \delta_k = \varepsilon_k.$$

Define the forcing term

$$f_k = \nabla \cdot \big(a_k(u^*)\nabla\Delta u^*\big) - \varepsilon_k \Delta^2 u^*.$$

Because u^* is time-independent, the function $u_k(x, t) = u^*(x)$ satisfies

$$\partial_t u_k = -\nabla \cdot \big(a_k(u_k)\nabla\Delta u_k\big) + \varepsilon_k \Delta^2 u_k + f_k.$$

Now we examine the limit of f_k. As computed in the Introduction, $\nabla\Delta u^* = 2Ad\,\delta_{\partial B_R}\mathbf{n}$ and $\Delta^2 u^*$ consists of derivatives of that Dirac mass. However, the product $a_k(u^*)\nabla\Delta u^*$ involves $(u^* + \varepsilon_k)^n$ evaluated on ∂B_R. Since $u^*|_{\partial B_R} = 0$, we have $a_k(u^*) = \varepsilon_k^n$, so

$$a_k(u^*)\nabla\Delta u^* = \varepsilon_k^n \cdot 2Ad\,\delta_{\partial B_R}\mathbf{n}.$$

Thus the first part of f_k is

$$\nabla \cdot \big(\varepsilon_k^n\, 2Ad\,\delta_{\partial B_R}\mathbf{n}\big) + \text{smooth terms inside } B_R,$$

which is a distribution of order at most one whose norm in H^{-1} tends to zero as $\varepsilon_k \to 0$. The term $\varepsilon_k\Delta^2 u^*$ is also of order ε_k times derivatives of the Dirac mass and vanishes in H^{-1}. Consequently, $f_k \to 0$ in $L^\infty(0,T;H^{-s}(\Omega))$ for any sufficiently large s, and the required strong convergence $u_k \to u^*$ is trivial (they are all equal). Hence u^* is a weak solution with initial datum $u_0 = u^*$.

Remark 6.1. *The above regularisation shows that u^* fits the definition of a weak solution. One may also directly verify that u^* satisfies the distributional equation $\partial_t u^* = -\nabla \cdot ((u^*)^n\nabla\Delta u^*)$ because both sides vanish: the right-hand side is zero as a distribution as shown in the Introduction. Thus u^* is a stationary weak solution.*

6.3. **Construction of the second solution: the standard regularisation limit.** Now take the same initial datum $u_0 = u^*$ and consider the standard regularised problem of Grün:

$$\partial_t u_\varepsilon = -\nabla \cdot \big((u_\varepsilon + \varepsilon)^n\nabla\Delta u_\varepsilon\big) + \varepsilon\Delta^2 u_\varepsilon, \qquad u_\varepsilon(0) = u_0 * \rho_\varepsilon,$$

where ρ_ε is a smooth mollifier. The mollified initial data are smooth, positive in the interior of their support, and have zero contact angle at the boundary of the support. By Theorem 5.1 (Dal Passo–Garcke–Grün [6]), there exists a subsequence $\varepsilon_k \to 0$ and a limit u such that u is a weak solution (in the sense of §6.1) and satisfies the entropy inequality

$$\frac{d}{dt}\int G(u) + \int u^n|\nabla\Delta u|^2 \le 0.$$

A fundamental consequence of this entropy inequality is the *zero-contact-angle* property (see [7], Theorem 3.3 and [1]): for all $t > 0$,

$$|\nabla u| = 0 \quad \text{on the free boundary } \partial\{x : u(x,t) > 0\},$$

in the sense of a suitable capacity. In particular, the gradient vanishes wherever u touches zero for positive times. The initial datum u^*, on the other hand, has non-zero gradient on its free boundary. Therefore $u(t) \neq u^*$ for almost every $t > 0$ (indeed, u immediately spreads and its contact line moves, whereas u^* remains stationary). Hence u is a genuinely time-evolving weak solution.

6.4. **Non-uniqueness.** We have exhibited two distinct weak solutions (in the precise sense of §6.1) with the same initial datum $u_0 = u^*$:

(1) $u^{(1)}(x,t) = u^*(x)$ (stationary, positive contact angle);
(2) $u^{(2)}(x,t)$ obtained from the standard regularisation (time-evolving, zero contact angle).

This proves the following theorem.

Theorem 6.2 (Non-uniqueness in higher dimensions)**.** *Let $d \geq 2$ and $0 < n < 2$. Then there exists a non-negative initial datum $u_0 \in W^{1,\infty}(\Omega)$ with finite entropy, such that the thin-film equation (1) admits at least two different weak solutions.*

Remark 6.3. *The counterexample is robust; the ball B_R can be arbitrarily placed, and even multiple plateaux would generate a wealth of distinct solutions. The mechanism illustrates that without an entropy condition, the free boundary can choose any contact angle, producing infinitely many weak solutions.*

7. Open problems and outlook

The results of this paper clarify the status of the thin-film equation:

(1) In one space dimension, the entropy formulation provides a complete and rigorous theory of existence and uniqueness, with finite speed of propagation and a well-behaved free boundary.
(2) In several space dimensions, even the weakest reasonable formulation leads to non-uniqueness, as shown by the counterexample above.

This negative result is not merely a technicality; it underscores the fundamental structural gap created by the loss of the unweighted H^2 estimate.

The following open questions now appear as the central challenges:

(1) **Selection principle.** What additional condition (a contact-angle law, an entropy defect inequality, a maximal monotonicity property) must be added to the weak formulation to restore uniqueness? The physical intuition suggests that the correct condition is $\nabla u = 0$ at the free boundary, but can this be formulated intrinsically without reference to a specific regularisation?
(2) **Entropy solutions in multi-D.** Is it possible to define a class of functions for which the entropy dissipation
$$\iint u^n |\nabla \Delta u|^2$$
controls enough derivatives to close the estimates? Candidate routes include the use of weighted Sobolev spaces [15] or Muckenhoupt weights, and the development of a parabolic Lipschitz truncation method for higher-order equations [16].
(3) **Multi-dimensional compactness.** The Aubin–Lions–Simon lemma requires an embedding that fails. Can one prove a compensated compactness result for the nonlinear terms using the structure of the equation, similar to the Div-Curl lemma?
(4) **Waiting times and free boundary regularity.** In 1D the free boundary is smooth and waiting times can be computed analytically [8]. Are there analogues in radially symmetric higher dimensions, and what insights do they provide for the general case?
(5) **Non-uniqueness for larger n.** Our counterexample works for $0 < n < 2$ because we rely on the existence theorem of Grün. Extending the counterexample to $n \geq 2$ (where even existence of very weak solutions is open) would require a different construction, perhaps using self-similar solutions with different contact angles.
(6) **Connection to gradient flow formulations.** The thin-film equation can be written as a gradient flow in the Wasserstein space [9]. In one dimension this yields a unique

curve of maximal slope. In higher dimensions the geometry changes; can one use this framework to characterise the unique physical solution?

Progress on any of these problems would significantly advance the theory.

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