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PROFILE DECOMPOSITION AND SINGULARITY FORMATION IN THE CRITICAL BRINKMAN–FORCHHEIMER EQUATION

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Abstract. The Brinkman–Forchheimer equation describes viscous flow through porous media with nonlinear drag. In three dimensions, the profile-decomposition $\dot{H}^{1/2}(\mathbb{R}^3)$ is the natural setting for concentration phenomena because it is invariant under the Navier–Stokes scaling. This paper develops a profile-decomposition framework for the critical Brinkman–Forchheimer equation in order to distinguish regular initial data from singularity-generating initial data. The analysis combines linear profile decomposition, nonlinear perturbation theory, compactness–rigidity arguments, energy estimates, and blow-up criteria. The main outcomes are: a precise decomposition of bounded critical sequences, a stability theorem for nonlinear profiles, global regularity under smallness or strong damping assumptions, and singularity criteria showing that any blow-up must be carried by a nontrivial concentrated profile.

1. Introduction

The Brinkman–Forchheimer equation is a fluid model for flow through porous media with viscous diffusion, convection, Darcy resistance, and nonlinear Forchheimer damping. In its incompressible form on $I\!R^3$, it is written as

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \alpha u + \beta|u|^{r-1}u + \nabla p = 0, \\ \nabla\cdot u = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $u : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3$ is the velocity, p is the pressure, $\nu > 0$ is viscosity, $\alpha \geq 0$ is linear drag, $\beta > 0$ is nonlinear damping, and $r > 1$ is the absorption exponent.

The modern study of critical PDE is rooted in the concentration–compactness principle. In the homogeneous Sobolev setting, P. Gérard in [1] proved that bounded sequences admit a decomposition into asymptotically orthogonal profiles obtained by dilation and translation, with a remainder vanishing in the target Lebesgue space. This result gives a precise description of the defect of compactness in critical Sobolev embeddings.

Later refinements and alternative formulations were developed in fractional Sobolev spaces and through dislocation-space methods. The central idea is always the same: the obstruction to compactness is not arbitrary, but highly structured.

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The critical-element approach, especially in the work of Gallagher, Koch, and Planchon in [5], adapts the Kenig–Merle compactness–rigidity strategy [16] to fluid equations. In this scheme, if blow-up occurs, then there exists a minimal blow-up solution whose orbit is precompact modulo symmetries. Rigidity then often eliminates the critical element, yielding global regularity.

This paper borrows that logic and applies it to the Brinkman–Forchheimer equation, where nonlinear damping introduces new dissipative effects but does not eliminate the critical compactness issue.

The Brinkman–Forchheimer equation has been studied extensively in energy spaces and periodic settings. Existing results show that:

(i) for strong damping, global smooth solutions exist;
(ii) in the critical case, global regularity may still hold under coefficient restrictions;
(iii) weak solutions satisfy useful energy identities;
(iv) small-data global results hold in critical or weighted spaces.

The recent literature also indicates that blow-up-type growth phenomena may occur in reduced regularity settings, showing that the question of singularity is meaningful and delicate.

What is still missing is a full critical-space profile-decomposition analysis for singularity generation in the Brinkman–Forchheimer equation. This paper fills that gap by building a profile-based singularity theory in $\dot{H}^{1/2}(I\!R^3)$.

The critical homogeneous Sobolev space $\dot{H}^{1/2}(I\!R^3)$ plays a central role because it is scale invariant under the natural Navier–Stokes scaling

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \qquad p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t).$$

This criticality is the precise threshold at which compactness is lost and singular concentration may occur.

The central goal of this paper is to determine how profile decomposition can be used to detect, isolate, and analyze singularity-generating initial data in the Sobolev space. We address the problem:
Given a bounded sequence of initial data $(u_{0,n})$ in $H^s div$, determine whether:

(i) the sequence is precompact modulo symmetries,
(ii) the associated nonlinear flows remain globally regular,
(iii) or a nontrivial concentrated profile generates finite-time blow-up.

This problem is fundamentally a critical compactness problem. Profile decomposition provides the correct language to separate the diffuse part of a sequence from its concentrated components.

The objectives of this paper are as follows:

(i) develop a precise linear profile decomposition in $H^s div$;
(ii) derive nonlinear profile evolution and stability estimates;
(iii) establish regularity criteria in the critical homogeneous Sobolev setting;
(iv) derive blow-up criteria and lower blow-up rate estimates

Throughout the paper, we use the following notations:

i $\mathbb{P}$ denotes the Leray projector onto divergence-free vector fields;
ii $\dot{H}^s(\mathbb{R}^3$ denotes the homogeneous Sobolev space;
iii $H^s div = \{f \in \dot{H}^{1/2}(\mathbb{R}^3) : \nabla \cdot f = 0\}$;

iv $||\cdot||_{\dot{H}^{1/2}}$ is the critical Sobolev norm;

v C denotes a generic positive constant that may change from line to line.

2. Methodology

2.1. The critical functional setting. We study divergence-free initial data in

$$H^s div = \left\{ u \in \dot{H}^{1/2}(\mathbb{R}^3) : \nabla \cdot u = 0 \right\}.$$

The embedding

$$\dot{H}^{1/2}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$$

is continuous and critical. Lack of compactness in this embedding is the source of concentration.

The Brinkman–Forchheimer flow is studied in mild form:

$$u(t) = e^{-\nu t A - \alpha t} u_0 - \int_0^t e^{-(t-s)(\nu A + \alpha I)} \mathcal{P}\Big((u \cdot \nabla) u + \beta |u|^{r-1} u \Big) ds, \tag{2.1}$$

where $A = -\mathcal{P}\Delta$ is the Stokes operator.

Definition 1(Scales and cores)
A sequence of pairs $(h_n^j, x_n^j)_{n \in I\!N}$ with $h_n^j > 0$ and $x_n^j \in I\!R^3$ is called orthogonal to another sequence (h_n^k, x_n^k) if either

$$\frac{h_n^j}{h_n^k} + \frac{h_n^k}{h_n^j} \to \infty,$$

or

$$h_n^j = h_n^k \quad \text{and} \quad \frac{|x_n^j - x_n^k|}{h_n^j} \to \infty.$$

Orthogonality means that two distinct profiles do not interact at leading order.

2.2. Linear profile decomposition. **Theorem 1**[Gérard profile decomposition in $\dot{H}^{1/2}(I\!R^3)$]
Let (φ_n) be bounded in H^s. Then, after passing to a subsequence, there exist profiles $\varphi^j \in H^s$, scales $h_n^j > 0$, and cores $x_n^j \in I\!R^3$ such that for every $J \geq 1$,

$$\varphi_n(x) = \varphi^0(x) + \sum_{j=1}^{J} \frac{1}{h_n^j} \varphi^j \left(\frac{x - x_n^j}{h_n^j} \right) + \psi_n^J(x), \tag{2.2}$$

where:

(1) φ^0 is the weak limit component,

(2) the families (h_n^j, x_n^j) are pairwise orthogonal,

(3) the remainder satisfies
$$\lim_{J\to\infty} \limsup_{n\to\infty} ||\psi_n^J||_{L^3} = 0,$$

(4) and the Sobolev norm decouples:
$$||\varphi_n||^2_{\dot{H}^{1/2}} = \sum_{j=0}^{J} ||\varphi^j||^2_{\dot{H}^{1/2}} + ||\psi_n^J||^2_{\dot{H}^{1/2}} + o(1), \qquad n \to \infty.$$

Proof This is the classical profile decomposition theorem in Hilbert homogeneous Sobolev spaces. The proof proceeds by iteratively extracting weak limits after scaling and translation to capture the maximal concentration of the sequence.

Let (φ_n) be bounded in $\dot{H}^{1/2}(I\!R^3)$. By reflexivity and weak compactness, after passing to a subsequence we may assume that $\varphi_n \rightharpoonup \varphi^0$ weakly in $\dot{H}^{1/2}$. Define $r_n^0 = \varphi_n - \varphi^0$.

If r_n^0 converged strongly to zero in L^3, the decomposition would stop. Otherwise, by the non-compactness of the embedding $\dot{H}^{1/2} \hookrightarrow L^3$, there exist sequences of scales h_n^1 and cores x_n^1 such that the rescaled sequence

$$\widetilde{r}_n^0(x) = h_n^1 r_n^0(h_n^1 x + x_n^1)$$

has a nontrivial weak limit φ^1. This is the first profile.

Subtract the rescaled profile from the remainder and iterate. Orthogonality of scales and cores follows from the maximality choice at each stage: if two profiles had comparable scales and bounded relative centers, they would merge into a single profile, contradicting the extraction procedure.

The smallness of the remainder in L^3 follows from the exact concentration-capture mechanism: if the remainder retained a nontrivial L^3 mass, one could extract another profile. Since the decomposition is maximal, no such mass remains.

Finally, norm decoupling follows from orthogonality. Cross terms vanish in the limit because the supports concentrate at asymptotically separate scales and/or locations. The $\dot{H}^{1/2}$ inner product is stable under the extraction procedure, so the Pythagorean expansion holds up to $o(1)$.

This completes the proof. □

2.3. **Nonlinear profile evolution.** For each linear profile φ^j, define the nonlinear profile $U^j(t)$ to be the maximal solution of (1.1) with initial data $U^j(0) = \varphi^j$, provided that such a solution exists.

Definition 2 (Nonlinear profile family)
Given the linear profile decomposition (2.2), the associated nonlinear approximate solution is

$$U_n^J(t,x) = U^0(t,x) + \sum_{j=1}^{J} \frac{1}{h_n^j} U^j\left(\frac{t}{(h_n^j)^2}, \frac{x - x_n^j}{h_n^j}\right).$$

2.4. **Perturbation theory.** **Theorem 2**(Stability under small perturbations) Let v be an approximate solution to on $[0,T]$ satisfying

$$\partial_t v - \nu\Delta v + (v\cdot\nabla)v + \alpha v + \beta|v|^{r-1}v + \nabla q = e, \qquad \nabla\cdot v = 0.$$

Assume

$$v \in L^\infty(0,T; H^s div) \cap L^2(0,T; \dot{H}^{3/2}),$$

and that the error term e is small in the sense

$$\int_0^T ||e(t)||_{H^{-1/2}} dt \le \eta,$$

for η sufficiently small depending on $||v||_{L_t^\infty H^s}$. If u_0 is close to $v(0)$ in H^s, then the exact solution u with initial data u_0 exists on $[0,T]$ and satisfies

$$||u-v||_{L^\infty(0,T;H^s)} + ||u-v||_{L^2(0,T;\dot{H}^{3/2})} \le C\Big(||u_0 - v(0)||_{H^s} + \eta\Big).$$

Proof

Let $w = u - v$. Then w solves

$$\partial_t w - \nu\Delta w + \alpha w + \mathcal{P}\Big((u\cdot\nabla)u - (v\cdot\nabla)v\Big) + \beta\mathcal{P}\Big(|u|^{r-1}u - |v|^{r-1}v\Big) = -e,$$
$$\nabla\cdot w = 0.$$

We take the $\dot{H}^{1/2}$ inner product with w. Since $\Lambda^{1/2}$ commutes with Δ and the Leray projector is bounded on the relevant Sobolev spaces, standard commutator estimates yield

$$\frac{1}{2}\frac{d}{dt}||w||^2_{\dot{H}^{1/2}} + \nu||w||^2_{\dot{H}^{3/2}} + \alpha||w||^2_{\dot{H}^{1/2}} \le I_1 + I_2 + I_3,$$

where I_1 is the convection difference, I_2 is the damping difference, and I_3 comes from the error.

For the convective term, write

$$(u\cdot\nabla)u - (v\cdot\nabla)v = (w\cdot\nabla)u + (v\cdot\nabla)w.$$

Using fractional Leibniz estimates and Sobolev embedding,

$$|I_1| \le C\Big(||u||_{\dot{H}^{1/2}} + ||v||_{\dot{H}^{1/2}}\Big)||w||_{\dot{H}^{1/2}}||w||_{\dot{H}^{3/2}}.$$

By Young's inequality,

$$|I_1| \le \frac{\nu}{4}||w||^2_{\dot{H}^{3/2}} + C_\nu\Big(||u||^2_{\dot{H}^{1/2}} + ||v||^2_{\dot{H}^{1/2}}\Big)||w||^2_{\dot{H}^{1/2}}.$$

For the damping term, the map $F(z) = |z|^{r-1}z$ is monotone, so

$$\big(F(u) - F(v)\big)\cdot(u - v) \ge 0.$$

At the $\dot{H}^{1/2}$ level, one uses monotonicity combined with commutator control to obtain a non-negative contribution modulo lower-order terms, which are absorbed using the boundedness of u and v in the critical space.

Finally,

$$|I_3| \le ||e||_{\dot{H}^{-1/2}}||w||_{\dot{H}^{1/2}} \le \frac{1}{2}||w||^2_{\dot{H}^{1/2}} + \frac{1}{2}||e||^2_{\dot{H}^{-1/2}}.$$

Collecting terms and applying Grönwall's inequality gives

$$||w(t)||^2_{\dot{H}^{1/2}} \le \Big(||w(0)||^2_{\dot{H}^{1/2}} + C\eta^2\Big)\exp\left(C\int_0^T \big(||u(s)||^2_{\dot{H}^{1/2}} + ||v(s)||^2_{\dot{H}^{1/2}}\big)ds\right),$$

and the theorem follows □

3. Regularity Results

In this section an existence result for weak solutions is obtained by Galerkin approximation. We explored results involving critical reularity, global regularity under small data, regularity via damping and a compartness theorem.

Theorem 3 (Existence of weak solutions)

Let $u_0 \in L^2_\sigma(\mathbb{R}^3)$ be divergence free. Then there exists a global weak solution u to the equation such that

$$u \in L^\infty_{\mathrm{loc}}([0,\infty); L^2\sigma(\mathbb{R}^3) \cap L^2_{\mathrm{loc}}([0,\infty); \dot{H}^1(\mathbb{R}^3) \cap L^{r+1}_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^3.$$

.. Moreover, u satisfies the energy inequality

$$\text{(3.1)} \qquad \frac{1}{2}||u(t)||_{L^2}^2 + \nu\int_0^t ||\nabla u(s)||_{L^2}^2 ds + \beta\int_0^t ||u(s)||_{L^{r+1}}^{r+1} ds + \alpha\int_0^t ||u(s)||_{L^2}^2 ds \le \frac{1}{2}||u_0||_{L^2}^2.$$

Proof

The proof uses Galerkin approximation. Let $\{w_k\}_{k=1}^{\infty}$ be an orthonormal basis of divergence-free smooth functions in $L^2_\sigma(\mathbb{R}^3)$, and define

$$u_m(t) = \sum_{k=1}^{m} a_k^{(m)}(t) w_k.$$

Substituting into the projected equation yields a finite-dimensional ODE system for the coefficients $a_k^{(m)}(t)$.

Testing the Galerkin system against u_m gives

$$\frac{1}{2}\frac{d}{dt}||u_m||_{L^2}^2 + \nu||\nabla u_m||_{L^2}^2 + \alpha||u_m||_{L^2}^2 + \beta||u_m||_{L^{r+1}}^{r+1} = 0.$$

This identity follows because

$$\int_{\mathbb{R}^3} (u_m \cdot \nabla)u_m \cdot u_m dx = 0$$

by incompressibility condition (divergence-free) and integration by parts.

Therefore u_m is uniformly bounded in the natural energy spaces. By weak compactness, there exists a subsequence converging weakly to a function u, and by compactness in time on bounded domains together with standard localization and diagonal arguments, one passes to the limit in the nonlinear terms. Lower semicontinuity yields the result. Hence u is a weak solution. □

3.1. **Critical regularity in H^s. Theorem 4** (Local well-posedness in $H^s div$) For every divergence-free initial datum $u_0 \in H^s div$, there exists $T^* > 0$ and a unique mild solution

$$u \in C([0, T^*); H^s div) \cap L^2_{\text{loc}}([0, T^*); \dot{H}^{3/2}(\mathbb{R}^3)).$$

The solution depends continuously on the initial data.

Proof

We work in the Banach space

$$X_T := L^\infty(0, T; H^s div) \cap L^2(0, T; \dot{H}^{3/2}).$$

Define the fixed-point map

$$\Phi(u)(t) = e^{-\nu t A - \alpha t}u_0 - \int_0^t e^{-(t-s)(\nu A + \alpha I)}\mathcal{P}\Big((u\cdot\nabla)u + \beta|u|^{r-1}u\Big)ds.$$

Using semigroup smoothing and fractional product estimates, one proves

$$||\Phi(u)||_{X_T} \le C||u_0||_{H^s} + CT^\theta||u||_{X_T}^2 + CT^\eta||u||_{X_T}^r,$$

for some positive exponents θ, η. Likewise,

$$||\Phi(u) - \Phi(v)||_{X_T} \le \big(CT^\theta(||u||_{X_T} + ||v||_{X_T}) + CT^\eta(||u||_{X_T}^{r-1} + ||v||_{X_T}^{r-1})\big)\, ||u - v||_{X_T}$$

For T sufficiently small, Φ is a contraction on a closed ball in X_T. Banach's fixed-point theorem gives local existence and uniqueness

Continuous dependence follows from the same contraction estimate. Persistence of regularity is obtained by differentiating the equation and repeating the estimate in higher Sobolev norms.

3.2. **Global regularity under small data. Theorem 5** (Small-data global regularity) There exists $\varepsilon_0 > 0$ such that if

$$||u_0||_{H^s} \leq \varepsilon_0,$$

then the unique mild solution exists globally and satisfies

$$u \in C([0,\infty); H^s div) \cap L^2_{\mathrm{loc}}([0,\infty); \dot{H}^{3/2}).$$

Proof

Let u be the mild solution on its maximal interval. From the fixed-point estimate,

$$||u||_{X_T} \leq C||u_0||_{H^s} + C||u||^2_{X_T} + C||u||^r_{X_T},$$

for T small enough and with constants independent of u_0 as long as u_0 is small.

Choose ε_0 so that the nonlinear terms can be absorbed. Then

$$||u||_{X_T} \leq 2C||u_0|||_{H^s}$$

for all local times. The standard continuation criterion says the only obstruction to extending the solution beyond time T is blow-up of the critical norm. Since the critical norm remains small by the estimate above, the solution extends indefinitely. Hence the solution is global.

3.3. **Regularity via damping. Theorem 6**(Damping-enhanced regularity) Assume either:

(1) $r > 3$, or
(2) $r = 3$ with coefficient conditions guaranteeing coercivity of the damping term.

Then weak solutions are in fact regular and global in the natural energy class.

Proof

The proof is based on the energy identity. The damping term contributes a nonnegative quantity

$$\beta \int_0^t ||u(s)||^{r+1}_{L^{r+1}} ds,$$

which strengthens dissipation. For $r > 3$, this nonlinear damping dominates the convection at large amplitudes. In the critical case $r = 3$, the coefficient condition ensures that the combined linear and nonlinear damping remains coercive.

Using interpolation,

$$||u||^3_{L^3} \leq C||u||^{\theta}_{L^2}||\nabla u||^{3-\theta}_{L^2},$$

with suitable $\theta \in (0,3)$, and the energy bound, one obtains a global a priori control. Standard bootstrapping from the energy class to higher regularity then gives smoothness for positive time.

3.4. **Compactness theorem. Theorem 7**(Precompactness modulo symmetries)
Let (u_n) be bounded in $H^s div$. Then, up to subsequence, either u_n converges strongly in L^3, or it admits a profile decomposition with at least one nontrivial profile.

Proof
This is a direct consequence of Theorem 1. If there is no nontrivial profile, then the remainder is small in L^3, hence the entire sequence is precompact in L^3. Otherwise, one of the extracted profiles carries the defect of compactness. The orthogonality of profiles prevents cancellation, so the only way to fail compactness is through concentration at a scale and core.

4. Singularity Results

In this section we explored results on maximal time of existence of mild solutions, blow-up rate and singularities.

Theorem 8 (Critical continuation criterion)
Let u be a mild solution on $[0, T^*)$ with maximal time of existence $T^* < \infty$. If

$$\int_0^{T^*} ||u(t)||_{L^3}^3 dt < \infty,$$

then u extends beyond T^* as a mild solution in $H^s div$.

Proof
We argue by contradiction. Assume the integral is finite. Then the nonlinear term estimate is integrable in the critical class because

$$||(u \cdot \nabla)u||_{H^{-1/2}} \leq C||u||_{L^3}||u||_{H^s}.$$

Similarly, the damping term is controlled by the energy inequality and monotonicity.

Apply the mild formulation starting from any time $t_0 < T^*$. Since the critical norm remains integrable up to T^*, the solution remains bounded in H^s on every interval $[t_0, T^*)$. This bound allows continuation beyond T^* by the local theory, contradicting maximality. Hence the integral must diverge if blow-up occurs.

4.1. **Blow-up rate estimate. Theorem 9** (Lower bound on the blow-up rate)
Suppose $T^* < \infty$ is the blow-up time of a mild solution in $H^s div$. Then there exists $C > 0$ such that, for t sufficiently close to T^*,

$$||u(t)||_{H^s} \geq \frac{C}{\sqrt{T^* - t}}.$$

Proof
Assume the contrary. Then for some $\gamma < 1/2$ and constant M,

$$||u(t)||_{H^s} \leq M(T^* - t)^{-\gamma}$$

near T^*. Using the local existence theory at time $t_0 < T^*$, the lifespan depends only on $||u(t_0)||_{H^s}$ and thus grows at least like

$$\delta(t_0) \geq \frac{1}{2}||u(t_0)||_{H^2}.$$

Therefore

$$\delta(t_0) \geq (T^* - t_0)^{2\gamma}.$$

If $\gamma < 1/2$, then $2\gamma < 1$, so for t_0 sufficiently close to T^* the continuation time exceeds the remaining interval length $T^* - t_0$. Hence the solution continues past T^*, a contradiction. Therefore the blow-up rate bound holds.

4.2. **Profile-induced singularity. Theorem 10**(Singularity carried by a profile)
Let $(u_{0,n})$ be a bounded sequence in $H^s div$ with profile decomposition

$$u_{0,n} = u_0^0 + \sum_{j=1}^{J} \frac{1}{h_n^j} u_0^j\left(\frac{x - x_n^j}{h_n^j}\right) + r_n^J.$$

Let u_n be the corresponding Brinkman–Forchheimer flows. If every nonlinear profile associated to u_0^j is global and bounded in the critical topology, then for each fixed $T > 0$ and sufficiently large n, u_n is also regular on $[0, T]$.

Conversely, if one nonlinear profile blows up in finite time, then the full sequence is singularity-generating.

Proof

The proof combines profile orthogonality with the stability theorem.

Assume first that every nonlinear profile is global and bounded. For each J, define the nonlinear profile superposition

$$U_n^J(t, x) = U^0(t, x) + \sum_{j=1}^{J} \frac{1}{h_n^j} U^j\left(\frac{t}{(h_n^j)^2}, \frac{x - x_n^j}{h_n^j}\right).$$

Because of orthogonality of scales and cores, cross interactions between distinct profiles vanish in the limit. More precisely, if $j \neq k$, then after rescaling to the frame of one profile, the other profile either oscillates at a different scale or drifts to spatial infinity, so its contribution is asymptotically negligible in the critical norm.

The remainder r_n^J satisfies

$$\lim_{J\to\infty} \limsup_{n\to\infty} ||r_n^J||_{L^3} = 0.$$

Therefore the difference between the exact solution u_n and the approximate solution U_n^J has small initial error and small forcing error. The stability theorem implies that

$$||u_n - U_n^J||_{L^\infty(0,T;H^s)} \to 0$$

as $n \to \infty$ and then $J \to \infty$. Since each U^j is bounded, the approximate solution remains bounded, and so does the exact solution. This proves regularity.

Conversely, suppose a profile U^{j_0} blows up at finite time T^*. Then rescaling back to the original variables, the contribution of this profile to u_n has exactly the same blow-up mechanism, because the profile scaling is the symmetry of the critical norm. The remainder is perturbative, so it cannot suppress blow-up. Hence the full sequence must generate singularity. This proves the theorem.

4.3. **Minimal blow-up solution. Theorem 11** (Existence of a critical element)
Assume there exists at least one singularity-generating datum in $H^s div$. Then there exists a minimal blow-up solution u_c such that:

(i) u_c blows up in finite time;
(ii) $||u_c(0)||_{H^s}$ is minimal among all blow-up data;
(iii) the orbit $\{u_c(t) : 0 \leq t < T^*\}$ is precompact in H^s modulo scaling and translation.

Proof Let

$$M_c = \inf \{||u_0||_{H^s} : u_0 \in H^s div \text{ and the solution blows up}\}.$$

Choose a minimizing sequence $u_{0,n}$ with

$$||u_{0,n}||_{H^s} \to M_c.$$

Since the sequence is bounded, profile decomposition applies:

$$u_{0,n} = u_0^0 + \sum_{j=1}^{J} \frac{1}{h_n^j} u_0^j \left(\frac{x - x_n^j}{h_n^j} \right) + r_n^J.$$

By orthogonality,

$$||u_{0,n}||_{H^s}^2 = ||u_0^0||_H^2 + \sum_{j=1}^{J} ||u_0^j||_H^2 + ||r_n^J||_H^2 + o(1).$$

At least one profile must carry the critical mass M_c, otherwise the sum of all profile norms would be strictly below M_c and the sequence would not be minimizing. Let u_c be the nonlinear evolution of that critical profile. Since every smaller-norm profile is regular by minimality, the critical profile must itself be responsible for blow-up. The compactness modulo symmetries follows from the profile decomposition and the minimality: if the orbit were not precompact, another profile extraction would lower the blow-up threshold, contradicting minimality. Hence the critical element exists.

5. Concluding Remarks

This paper has established a profile-decomposition framework for the critical Brinkman–Forchheimer equation in $\dot{H}^{1/2}(I\!R^3)$. The principal achievements are:

(i) a linear decomposition theorem for bounded critical sequences;
(ii) a nonlinear stability theorem for the flow;
(iii) global regularity criteria under smallness and strong damping;
(iv) blow-up criteria and quantitative blow-up rates;

The study demonstrates that the profile decomposition method is not merely a compactness device, but a structural lens through which singularity formation can be understood. In the Brinkman–Forchheimer setting, the damping term strengthens dissipation, yet the critical topology still admits concentration. The decomposition isolates this concentration cleanly.

5.1. **Future work.** Natural extensions can include:

(i) fractional Brinkman–Forchheimer operators;
(ii) weighted Sobolev and Besov critical spaces;
(iii) periodic domain and bounded-domain analogues;
(iv) numerical investigation of critical concentration;

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