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LOCAL EXISTENCE AND REGULARITY OF SOLUTIONS IN α-NORM FOR SOME INTEGRODIFFERENTIAL EQUATIONS WITH INFINITE DELAY IN BANACH SPACES

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Abstract. In this work, we prove the existence and regularity of solutions in α-norm for some partial functional integrodifferential equations in Banach spaces under the resolvent operator's theory developed by Grimmer.

1. Introduction

In this work, we study the existence and regularity in α-norm of solutions for the following partial functional integrodifferential equation

$$\begin{cases} u'(t) = -Au(t) + \displaystyle\int_0^t B(t-s)u(s)ds + f(u_t) \text{ for } t \geq 0 \\ \\ u_0 = \varphi \in \mathcal{B}, \end{cases} \tag{1.1}$$

where $-A$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators, $\mathcal{B}_\alpha$, is a subset of $\mathcal{B}$, where $\mathcal{B}$ is a Banach space of functions mapping from $]-\infty, 0]$ into X and satisfying some axioms that will be introduced later, and $0 < \alpha < 1$, the operator is the fractional α-power of A. This operator $(A^\alpha, D(A^\alpha))$ will be describe later. For $t \geq 0$, $B(t)$ is closed linear operator with domain $D(B) \supset D(A)$, f is a continuous functions from $\mathbb{R}^+ \times \mathcal{B}_\alpha$ with values respectively in X where $\mathcal{B}_\alpha$ is defined by

$$\mathcal{B}_\alpha = \{\varphi \in \mathcal{B};\ \varphi(\theta) \in D(A^\alpha) \text{ for } \theta < 0 \text{ and } A^\alpha\varphi \in \mathcal{B}\}$$

with the norm

$$\|\varphi\|_{\mathcal{B}_\alpha} = \|A^\alpha\varphi\|_{\mathcal{B}}.$$

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For every $t \geq 0$, the history function $x_t \in \mathcal{B}_\alpha$ is defined by

$$x_t(\theta) = x(t+\theta) \text{ for } \theta \in]-\infty, 0],$$

In [6], the author investigated the existence and regularity of solutions of the following integro-differential equation

$$\left\{ \begin{array}{l} u'(t) = -Au(t) + \displaystyle\int_0^t B(t-s)u(s)ds + g(t) \text{ for } t \geq 0 \\ \\ u(0) = u_0 \in X, \end{array} \right. \tag{1.2}$$

the existence, uniqueness, representation of solutions via variation of constants formula and other properties of a resolvent operator have been studied (see e.g. [4, 5, 10] and the bibliography therein). The resolvent operator play an important role to solve equation (1.2) in weak and strict sens, it replaces the role of c_0-semigroup. In this work, we use the basic theory developed for equation (1.2) in [6] to build basic results on the existence of so-called mild and strict solutions of equation (1.1). Firstly, we use the strict contraction principle to show the local existence of mild solutions which may blow up in finite time. Then, we reformulate some sufficient conditions to get the existence of strict solutions. The results obtained in this work generalizes the well known results developed in [11] when $B = 0$.

The organization of this work is as follows, in section 2, we recall some preliminary results about analytic semigroups and fractional power associated to its generator will be used throughout this work and some useful results on the analytic resolvent operator. In section 3, we study the existence of local mild solutions of equation (1.1). We prove that in the case of local existence, the solutions blows up, we also show the dependence continuous with the initial data. In section 4, we will show the existence of strict solutions for equation (1.1) by using regularity theorem due to Grimmer [6]. For illustration, we propose to study the existence of solutions for some partial functional integrodifferential equations with diffusion.

2. Preliminary results

Let $(X, \|.\|)$ be a Banach space and α be a constant such that $0 < \alpha < 1$ and $-A$ be the infinitesimal generator of a bounded analytic semigroup of linear operator $(T(t))_{t\geq 0}$ on X. We assume without loss of generality that $0 \in \rho(A)$. Note that if the assumption $0 \in \rho(A)$ is not satisfied, one can substitute the operator A by the operator $(A - \sigma)$ with σ large enough such that $0 \in \rho(A - \sigma)$. This allows us to define the fractional power A^α for $0 < \alpha < 1$, as a closed linear invertible operator with domain $D(A^\alpha)$ dense in X. The closeness of A^α implies that $D(A^\alpha)$, endowed with the graph norm of A^α, $|x| = \|x\| + \|A^\alpha x\|$, is a Banach space. Since A^α is invertible, its graph norm $|.|$ is equivalent to the norm $|x|_\alpha = \|A^\alpha x\|$. Thus, $D(A^\alpha)$ equipped with the norm $|.|_\alpha$, is a Banach space, which we denote by X_α. For $0 < \beta \leq \alpha < 1$, the imbedding $X_\alpha \hookrightarrow X_\beta$ is compact if the resolvent operator of A is compact. Also, the following properties are well known.

Proposition 2.1. *[9] Let $0 < \alpha < 1$. Assume that the operator $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on the Banach space X satisfying $0 \in \rho(A)$. Then we have*

i) $T(t) : X \to D(A^\alpha)$ *for every* $t > 0$.
ii) $T(t)A^\alpha x = A^\alpha T(t)x$ *for every* $x \in D(A^\alpha)$ *and* $t \geq 0$.
iii) for every $t > 0$, $A^\alpha T(t)$ *is bounded on* X *and there exist* $M_\alpha > 0$ *and* $\omega > 0$ *such that*

$$\|A^\alpha T(t)\| \leq M_\alpha e^{-\omega t} t^{-\alpha} \textit{ for } t > 0.$$

iv) If $0 < \alpha \leq \beta < 1$, $D(A^\beta) \hookrightarrow D(A^\alpha)$.
v) There exists $N_\alpha > 0$ *such that*

$$\|(T(t) - I)A^{-\alpha}\| \leq N_\alpha t^\alpha \textit{ for } t > 0.$$

Remark 2.2. Recall that $A^{-\alpha}$ is given by the following formula

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} T(t) dt,$$

where the integral converges in the uniform operator topology for every $\alpha > 0$.
Consequently, if $T(t)$ is compact for each $t > 0$, then $A^{-\alpha}$ is compact.

We also collect some basic results about resolvent operators. Consider the following linear homogeneous equation

$$\begin{cases} v'(t) = -Av(t) + \displaystyle\int_0^t B(t-s)v(s)ds \text{ for } t \geq 0 \\ \\ v(0) = v_0 \in X, \end{cases} \tag{2.1}$$

where A and $B(t)$ are closed linear operators on a Banach space X.

Definition 2.3. [6] A resolvent operator for equation (2.1) is a bounded operator valued function $R(t) \in \mathcal{B}(X)$ for $t \geq 0$ such that
(a) $R(0) = I$ and $|R(t)| \leq Ne^{-\beta t}$ for some positives constants N and β.
(b) For all $x \in X$, $R(t)x$ is strongly continuous for $t \geq 0$.
(c) $R(t) \in \mathcal{B}(D(A))$ for $t \geq 0$. For $x \in Y = (D(A)$, $R(.)x \in C^1([0, +\infty[; X) \cap C([0, +\infty[; D(A))$ and

$$\begin{aligned} R'(t)x &= -AR(t)x + \int_0^t B(t-s)R(s)xds \\ &= -R(t)Ax + \int_0^t R(t-s)B(s)xds \text{ for } t \geq 0. \end{aligned}$$

The resolvent operator play an interesting role to investigate the existence of solution of equation (2.1). The existence of an analytic resolvent operator has been discussed in [6] under the following assumptions. The notation f^* denotes the Laplace transform of f.

($\mathbf{V}_1$) $-A$ generates an analytic semigroup $(T(t))_{t\geq 0}$ on X. $(B(t))_{t\geq 0}$ is a closed operator on X with domain at least $D(A)$ a.e $t \geq 0$ with $B(t)x$ strongly measurable for each $x \in D(A)$ and $\|B(t)x\| \leq b(t)\|x\|_1$ for $b \in L^1_{loc}(0, \infty)$ with $b^*(\lambda)$ absolutely convergent for $Re\lambda > 0$.

($\mathbf{V}_2$) $\rho(\lambda) = (\lambda I + A - B^*(\lambda))^{-1}$ exists as a bounded operator on X which is analytic for $\lambda \in \Lambda = \{\lambda \in \mathbf{C} : \ |arg(\lambda)| < \frac{\pi}{2} + \delta\}$, where $0 < \delta < \frac{\pi}{2}$. In Λ if $|\lambda| \geq \varepsilon > 0$ there exists $M = M(\varepsilon) > 0$ so that $\|\rho(\lambda)\| \leq \frac{M}{|\lambda|}$.

($\mathbf{V}_3$) $A\rho(\lambda) \in \mathcal{L}(X)$ for $\lambda \in \Lambda$ and is analytic from Λ to $\mathcal{L}(X)$. $B^*(\lambda) \in \mathcal{L}(Y, X)$ and $B^*(\lambda)\rho(\lambda) \in \mathcal{L}(Y, X)$ for $\lambda \in \Lambda$. Given $\varepsilon > 0$, there exists a positive constant $M = M(\varepsilon)$ so that for $x \in Y$ and $\lambda \in \Lambda$ with $|\lambda| \geq \varepsilon$ $\|A\rho(\lambda)x\| + \|B^*(\lambda)\rho(\lambda)x\| < \frac{M}{|\lambda|}\|x\|$ and $\|B^*(\lambda)\| \to 0$ as $|\lambda| \to +\infty$ in Λ. In addition, $\|A\rho(\lambda)x\| \leq M|\lambda|^n$ for some $n > 0$, $\lambda \in \Lambda$ with , $|\lambda| \geq \varepsilon$. Further, there exists $D \subset D(A^2)$ which is dense in Y such that $A(D)$ and $B^*(\lambda)(D)$ are contained in Y and $\|B^*(\lambda)x\|$ is bounded for each $x \in D$ and $\lambda \in \Lambda$ with $|\lambda| \geq \varepsilon$.

($\mathbf{V}_4$) For any $y \in Y$, the map $t \to B(t)x$ belongs to $W^{1,1}_{loc}(\mathbb{R}^+, X)$ and $\|B'(t)x\| \leq b(t)\|x\|_1$ for $b \in L^1_{loc}(0, \infty)$

From [6], there exists a resolvent operator $(R(t))_{t\geq 0}$ for linear equation (2.1) which is analytic. Moreover for $a > 0$, there exist $K_\alpha > 0$ such that

$$\|A^\alpha R(t)\| \leq K_\alpha t^{-\alpha} \text{ for } t \in (0, a] \text{ and } \ 0 < \alpha < 1.$$

Definition 2.4. [7]. We say that a continuous function $u : [0, +\infty[\to X$ is a strict solution of equation (1.2) if the following conditions hold
(i) $u \in C^1([0, +\infty[; X) \cap C([0, +\infty[; D(A))$.
(ii) u satisfies equation (1.2) for $t \geq 0$.

The following theorem makes a relationship between resolvent operator and the expression of strict solution.

Theorem 2.5. *[6] Assume that (*$\boldsymbol{V}_1$*), (*$\boldsymbol{V}_2$*) and (*$\boldsymbol{V}_3$*) hold. If u is a strict solution of equation (1.2), then*

$$u(t) = R(t)u_0 + \int_0^t R(t-s)g(s)ds \text{ for } t \geq 0. \tag{2.2}$$

The basic theory for the existence of resolvent operator is given in [2, 6, 7].

Theorem 2.6. *[2] Assume that (*$\boldsymbol{V}_1$*), (*$\boldsymbol{V}_2$*) and (*$\boldsymbol{V}_3$*) hold. Let $(T(t))_{t\geq 0}$ be a compact semi-group for $t > 0$. Then the corresponding resolvent operator $(R(t))_{t\geq 0}$ of equation (2.1) is also compact for $t > 0$.*

Now consider the following system:

$$\begin{cases} x'(t) = Ax(t) + \displaystyle\int_0^t [B_1(t-s) - B_2(t-s)]x(s)ds \text{ for } t \geq 0 \\ \\ x(0) = x_0 \in X. \end{cases} \tag{2.3}$$

where $B_1(t)$ and $B_2(t)$ are closed linear operators in X. Then we have the following Lemma and corollary coming from [2].

Lemma 2.7. *[2] (Perturbation result) .Assume that (*$\boldsymbol{V}_1$*), (*$\boldsymbol{V}_2$*), (*$\boldsymbol{V}_3$*) and (*$\boldsymbol{V}_4$*) hold and Let $(R_{B_1}(t))_{t\geq 0}$ be a resolvent operator of equation (2.1) and $(R_{B_1+B_2}(t))_{t\geq 0}$ be a resolvent operator* *of equation (2.3). Then*

$$R_{B_1+B_2}(t)x - R_{B_1}(t)x = \int_0^t R_{B_1}(t-s)Q(s)xds,$$

where the operator Q is defined by

$$Q(t)x = \int_0^t B_2'(t-s)\Big(\int_0^s R_{B_1+B_2}(\tau)xd\tau\Big)ds + B_2(0)\int_0^t R_{B_1+B_2}(s)xds,$$

Q is uniformly bounded on bounded intervals, and for each $x \in X$, $Q(.)x$ belongs to $C([0,\infty);X)$.

Corollary 2.8. *[2] Let A be a closed, densely defined linear operator in X, $B(t) = 0$ for all $t \geq 0$, and $(R(t))_{t\geq 0}$ be a resolvent operator for equation (2.1). Then $(R(t))_{t\geq 0}$ is a C_0-semigroup with infinitesimal generator A.*

The following theorem gives sufficient conditions ensuring the existence of strict solutions which generalizes the well-known results on the c_0-semigroup theory.

Theorem 2.9. *[7]. Let $g \in C^1([0,+\infty[;X)$ and v be defined by*

$$v(t) = R(t)v_0 + \int_0^t R(t-s)g(s)ds \;\; \text{for} \;\; t \geq 0.$$

If $v_0 \in D(A)$, then v is a strict solution of equation (1.2).

In the whole of this work, we suppose that $(\mathbf{V}_1)$, $(\mathbf{V}_2)$, $(\mathbf{V}_3)$ and $(\mathbf{V}_4)$ hold which imply the existence of an analytic resolvent operator $(R(t))_{t\geq 0}$.

Lemma 2.10. *[9] Let $v : [0,a] \to [0,+\infty[$ be continuous. If there are positive constants E, F, $0 < \alpha < 1$ such that*

$$v(t) \leq E + F\int_0^t \frac{v(s)}{(t-s)^\alpha}ds \;\; \text{for} \;\; t \in [0,a].$$

Then, there is a constant K such that

$$v(t) \leq K.$$

3. Existence, global continuation and blowing up of solutions

Definition 3.1. We say that a continuous function $u :]-\infty,+\infty[\to X_\alpha$ is a strict solution of equation (1.1) if the following conditions hold
(i) $u \in C^1([0,+\infty[;X_\alpha)$.
(ii) u satisfies equation (1.1) on $[0,+\infty[$.
(iii) $u(\theta) = \varphi(\theta)$ for $-\infty < \theta \leq 0$.

Lemma 3.2. *[3] If u is a solution of equation (1.1) then*

$$u(t) = R(t)\varphi(0) + \int_0^t R(t-s)f(u_s)ds \; \text{for} \; t \geq 0. \tag{3.1}$$

Remark 3.3. The converse is not true. In fact if u satisfies equation (3.1), u may be not differentiable, that is why we distinguish between mild and strict solutions.

Definition 3.4. We say that a continuous function $u :]-\infty, +\infty[\to X_\alpha$ is a mild solution of equation (1.1) if u satisfies the following equation

$$\begin{cases} u(t) = R(t)\varphi(0) + \displaystyle\int_0^t R(t-s)f(u_s)ds \text{ for } t \geq 0 \\ \\ u_0 = \varphi \in \mathcal{B}_\alpha. \end{cases}$$

In all this work, we assume that the state space $(\mathcal{B}, |.|_\mathcal{B})$ is a normed linear space of functions mapping $]-\infty, 0]$ into X and satisfying the following fundamental axioms due to Hale and Kato in [8].

$(\mathbf{A}_1)$ There exist a positive constant H and functions $K(.)$, $\mathcal{M}(.) : \mathbb{R}^+ \to \mathbb{R}^+$, with K continuous and $\mathcal{M}$ locally bounded, such that for any $\sigma \in \mathbb{R}$ and $a > 0$, if $u :]-\infty, a] \to X$, $u_\sigma \in \mathcal{B}$, and $u(.)$ is continuous on $[\sigma, \sigma + a]$, then for every $t \in [\sigma, \sigma + a]$ the following conditions hold

(i) $u_t \in \mathcal{B}$.

(ii) $|u(t)| \leq H|u_t|_\mathcal{B}$, which is equivalent to $|\varphi(0)| \leq H|\varphi|_\mathcal{B}$ for every $\varphi \in \mathcal{B}$.

(iii) $|u_t|_\mathcal{B} \leq K(t-\sigma) \sup\limits_{\sigma \leq s \leq t} |u(s)| + \mathcal{M}(t-\sigma)|u_\sigma|_\mathcal{B}$.

$(\mathbf{A}_2)$ For the function $u(.)$ in $(\mathbf{A}_1)$, $t \mapsto u_t$ is a $\mathcal{B}$-valued continuous function for $t \in [\sigma, \sigma + a]$.

$(\mathbf{B})$ The space $\mathcal{B}$ is complete.

In the following, we prove the existence of mild solutions of equation (3.4). For this, we make the following assumption.

$(\mathbf{H}_1)$The function $f : \mathcal{B}_\alpha \to X$ satisfies the following conditions

i) $f : \mathcal{B}_\alpha \to X$ is continuous.

ii) There exists a continuous nondecreasing function $\beta : [0, b] \to [0, +\infty[$ such that $\|f(\varphi)\| \leq \beta(t)\|\varphi\|_{\mathcal{B}_\alpha}$ for $\varphi \in \mathcal{B}_\alpha$.

$(\mathbf{H}_2)$ The semigroup $((T(t))_{t\geq 0}$ is compact on X for $t > 0$.

$(\mathbf{H}_3)$ $A^{-\alpha}\varphi \in \mathcal{B}$ for $\varphi \in \mathcal{B}$, where the function $A^{-\alpha}\varphi$ is defined by

$$(A^{-\alpha}\varphi)(\theta) = A^{-\alpha}(\varphi(\theta)) \;\text{ for }\; \theta \leq 0.$$

Theorem 3.5. *Assume that* $(\boldsymbol{H}_1)$, $(\boldsymbol{H}_2)$ *and* $(\boldsymbol{H}_3)$ *hold. Let* $\varphi \in \mathcal{B}_\alpha$ *such that* $\varphi(0) \in X_\alpha$ *and assume that*

$$K_\alpha \left(K_b + M_b\right) \int_0^b \frac{\beta(s)}{(b-s)^\alpha} ds < 1,$$

where $K_b = \sup\limits_{0\le t\le b} K(t)$ *and* $M_b = \sup\limits_{0\le t\le b} \mathcal{M}(t)$. *Then equation* (1.1) *has at least one mild solution on* $[0,b]$.

Proof. Let $k > \|\varphi\|_{\mathcal{B}_\alpha}$, we define the following set

$$Z_k = \{x \in C([0,b], X_\alpha):\ x(0) = \varphi(0) \text{ and } |x|_\infty \le k\},$$

where $|x|_\infty = \sup\limits_{t\in[0,b]} |x(t)|_\alpha$. For $x \in Z_k$, define the $\widetilde{x}(t) : [0,b] \to X_\alpha$ by

$$\widetilde{x}(t) = \begin{cases} x(t) \text{ for } t \in [0,b] \\ \\ \varphi(t) \text{ for } t \in]-\infty, 0] \end{cases}$$

The function $\ell : t \to \widetilde{x}_t$ is continuous from $[0,b]$ to $\mathcal{B}_\alpha$. Now, define the operator $\mathcal{H}$ on Z_k by

$$\mathcal{H}(x)(t) = R(t)\varphi(0) + \int_0^t R(t-s)f(u_s)ds \text{ for } t \in [0,b].$$

It is sufficient to show that $\mathcal{H}$ has a fixed point in Z_k. In fact if $\mathcal{H}$ has at least a fixed point in Z_k, the eventual fixed points u in Z_k satisfy $\mathcal{H}(u)(t) = u(t)$ for $t \ge 0$. Consequently, we find $u \in Z_k$ such that

$$u(t) = \begin{cases} R(t)\varphi(0) + \displaystyle\int_0^t R(t-s)f(u_s)ds \text{ for } t \ge 0 \\ \\ \varphi(t) \text{ for } t \in]-\infty, 0], \end{cases}$$

which according to Definition 3.4 is solution of equation (1.1). We give the proof in several steps.

Step 1: There is a positive $k > \|\varphi\|_{\mathcal{B}_\alpha}$ such that $\mathcal{H}(Z_k) \subset Z_k$.
If not, then for each $k > \|\varphi\|_{\mathcal{B}_\alpha}$, there exist $x_k \in B_k$ and $t_k \in [0,b]$ such that $|(\mathcal{H}x_k)(t_k)|_\alpha > k$. Let

$$Q(t)x = \int_0^t B'(t-s)\Big(\int_0^s R(\tau)xd\tau\Big)ds + B(0)\int_0^t R(s)xds,$$

By Lemma 2.7 we deduce that the operators $(Q(t))_{t\ge 0}$ are uniformly bounded for t in a bounded interval and that

$$R(t)x = T(t)x + \int_0^t T(t-s)Q(s)xds.$$

Moreover let us define $N_b = \sup\{\|Q(t)\| :\ \ t \in [0,b]\}$. Thus by Proposition 2.1 and equation (2.2), we have

$$\begin{aligned} k &< |(\mathcal{H}x_k)(t_k)|_\alpha \le |R(t)\varphi(0)|_\alpha + \int_0^t |R(t-s)f(\widetilde{x}_s)|_\alpha ds \\ &< |T(t)\varphi(0)|_\alpha + \int_0^t |T(t-s)Q(s)\varphi(0)|_\alpha ds + \int_0^t |R(t-s)f(\widetilde{x}_s)|_\alpha ds \\ &< M_\alpha e^{-\omega t}t^{-\alpha}\|\varphi(0)\| + M_\alpha N_b\|\varphi(0)\| \int_0^{+\infty} \frac{e^{-\omega(t-s)}}{(t-s)^\alpha} + K_\alpha \int_0^t \frac{\beta(s)}{(t-s)^\alpha}\|\widetilde{x}_s\|_{\mathcal{B}_\alpha} ds. \end{aligned}$$

By axiom (**A**), one has

$$\|\widetilde{x}_s\|_{\mathcal{B}_\alpha} \leq K_b \sup_{0\leq\tau\leq s} |x(\tau)|_\alpha + M_b\|\varphi\|_{\mathcal{B}_\alpha},$$

then we have

$$< \quad M_\alpha e^{-\omega t}t^{-\alpha}\|\varphi(0)\| + \frac{M_\alpha N_b}{\omega^{1-\alpha}}\|\varphi(0)\| \int_0^{+\infty} e^{-s}s^{-\alpha}ds + (K_\alpha K_b|x|_\infty + K_\alpha M_b\|\varphi\|_{\mathcal{B}_\alpha}) \int_0^t \frac{\beta(s)}{(t-s)^\alpha}ds,$$

where $|x|_\infty = \sup\limits_{t\in[0,b]} |x(t)|_\alpha$. Since

$$\int_0^{+\infty} e^{-\omega t}t^{-\alpha}dt = \frac{\Gamma(1-\alpha)}{\omega^{1-\alpha}} < \infty,$$

then there exists $\eta > 0$ such that $e^{-\omega t}t^{-\alpha} < \eta$ on $]0, +\infty[$. It follows that

$$k \quad < \quad M_\alpha\eta\|\varphi(0)\| + \frac{M_\alpha N_b}{\omega^{1-\alpha}}\Gamma(1-\alpha)\|\varphi(0)\| + (K_\alpha K_b|x|_\infty + K_\alpha M_b\|\varphi\|_{\mathcal{B}_\alpha}) \int_0^t \frac{\beta(s)}{(t-s)^\alpha}ds.$$

Moreover $|x|_\infty \leq k$ for all $s \in [0, b]$ and $x \in Z_k$. Then, we obtain

$$k \quad < \quad M_\alpha\Big(\eta + \frac{N_b\Gamma(1-\alpha)}{\omega^{1-\alpha}}\Big)\|\varphi(0)\| + K_\alpha k\,(K_b + M_b) \int_0^t \frac{\beta(s)}{(t-s)^\alpha}ds,$$

On the other hand, we shall show that the function $g : t \mapsto \int_0^t \frac{\beta(s)}{(t-s)^\alpha}ds$ is nondecreasing on $[0, b]$. In fact, let $t, t' \in [0, b]$ be such that $t < t'$. Then we have

$$\int_0^t \frac{\beta(s)}{(t-s)^\alpha}ds = \int_0^t \frac{\beta(t-s)}{s^\alpha}ds \leq \int_0^t \frac{\beta(t'-s)}{s^\alpha}ds \leq \int_0^{t'} \frac{\beta(t'-s)}{s^\alpha}ds = g(t').$$

Consequently

$$1 \quad < \quad \frac{M_\alpha\Big(\eta + \frac{N_b\Gamma(1-\alpha)}{\omega^{1-\alpha}}\Big)\|\varphi(0)\|}{k} + K_\alpha\,(K_b + M_b) \int_0^b \frac{\beta(s)}{(b-s)^\alpha}ds.$$

It follows that when $k \to +\infty$ that

$$1 \quad \leq \quad K_\alpha\,(K_b + M_b) \int_0^b \frac{\beta(s)}{(b-s)^\alpha}ds,$$

which gives a contradiction.

Step 2: $\mathcal{H}$ is continuous

Let $(x^n)_n \in Z_k$ with $x^n \to x$ in Z_k. Then, the set $\Delta = \{(s, \widetilde{x}^n_s),\ (s, \widetilde{x}_s) :\ s \in [0, b],\ n \geq 1\}$ is compact in $[0, b] \times \mathcal{B}_\alpha$. Heine's Theorem implies that f is uniformly continuous in Δ and

$$\begin{aligned} |\mathcal{H}(x^n)(t) - \mathcal{H}(x)(t)|_\infty \quad &\leq \quad \sup_{t\in[0,b]} \int_0^t \|A^\alpha R(t-s)(f(x^n_s) - f(x_s))\|ds \\ &\leq \quad K_\alpha \sup_{s\in[0,b]} \|f(x^n_s) - f(x_s))\| \Big(\int_0^b \frac{ds}{s^\alpha}ds\Big) \to 0 \text{ as } n \to +\infty, \end{aligned}$$

and this yield the continuity of $\mathcal{H}$ on Z_k.

Step 3: The set $\{\mathcal{H}(x)(t): \ x \in Z_k\}$ is relatively compact for each $t \in]0,b]$.
Let $t \in]0,b]$ be fixed, and $\gamma > 0$ be such that $\alpha < \gamma < 1$. Then, it follows that

$$\begin{aligned}\|(A^\gamma \mathcal{H}(x))(t)\| &\leq \|A^\gamma R(t)\varphi(0)\| + \int_0^t \|A^\gamma R(t-s) f(\widetilde{x}_s))\| ds \\ &\leq M_\gamma \Big(\eta + \frac{N_b \Gamma(1-\gamma)}{\omega^{1-\gamma}}\Big) \|\varphi(0)\| + K_\gamma k\,(K_b + M_b) \int_0^t \frac{\beta(s)}{(t-s)^\gamma} ds < +\infty.\end{aligned}$$

Then for $t \in]0,b]$ fixed the set $\{A^\gamma \mathcal{H}(x)(t): \ x \in Z_k\}$ is bounded in X. By $(\mathbf{H}_2)$, we deduce that $A^{-\gamma}: X \to X_\alpha$ is compact. It follows that the set $\{\mathcal{H}(x)(t): \ x \in Z_k\}$ is relatively compact for each $t \in]0,b]$ in X_α.

Step 4: The set $\{\mathcal{H}(x): \ x \in Z_k\}$ is an equicontinuous family of functions.
Let $x \in Z_k$, $0 \leq \tau_1 < \tau_2 \leq b$ and set $\varepsilon = \tau_2 - \tau_1$, we have

$$\begin{aligned}& (\mathcal{H}x)(\tau_2) - (\mathcal{H}x)(\tau_1) \\ = \;& (R(\tau_2) - R(\tau_1))\varphi(0) + \int_0^{\tau_2} R(\tau_2 - s) f(\widetilde{x}_s) ds - \int_0^{t_0} R(\tau_1 - s) f(\widetilde{x}_s) ds \\ = \;& (R(\tau_2) - R(\tau_1))\varphi(0) + \int_{\tau_1}^{\tau_2} R(\tau_2 - s) f(\widetilde{x}_s) ds \\ & + \int_0^{\tau_1} \left[T(\tau_2 - s) f(\widetilde{x}_s) ds + \int_0^{\tau_2 - s} T(\tau_2 - s - \tau) Q(\tau) f(\widetilde{x}_\tau) d\tau \right] ds \\ & - \int_0^{\tau_1} \left[T(\tau_1 - s) f(\widetilde{x}_s) ds + \int_0^{\tau_1 - s} T(t_0 - s - \tau) Q(\tau) f(\widetilde{x}_\tau) d\tau \right] ds \\ = \;& (R(\tau_2) - R(\tau_1))\varphi(0) + \int_{\tau_1}^{\tau_2} R(\tau_2 - s) f(\widetilde{x}_s) ds \\ & + \left[\int_0^{t_0} [T(\tau_1 - s) - T(\tau_1 - s)] f(\widetilde{x}_s) ds \right] \\ & + \int_{\tau_1}^{\tau_2} \left[\int_{\tau_1 - s}^{\tau_2 - s} T(\tau_2 - s - \tau) Q(\tau) f(\widetilde{x}_\tau) d\tau \right] ds \\ & + \int_0^{\tau_1} \left[\int_0^{\tau_1 - s} [T(\tau_2 - s - \tau) - T(\tau_1 - s - \tau)] Q(\tau) f(\widetilde{x}_\tau) d\tau \right] ds.\end{aligned}$$

This implies that

$$\begin{aligned}& |(\mathcal{H}x)(\tau_2) - (\mathcal{H}x)(\tau_1)|_\alpha \\ \leq \;& |(R(\tau_2) - R(\tau_1))\varphi(0)|_\alpha + \left\| (T(\tau_2 - \tau_1) - I) \int_0^{\tau_1} A^\alpha T(\tau_1 - s) f(\widetilde{x}_s) ds \right\| \\ & + K_\alpha k (K_b + M_b) \int_{\tau_2 - \varepsilon}^{\tau_2} \frac{\beta(s)}{(\tau_2 - s)^\alpha} ds \\ & + K_\alpha N_b k (K_b + M_b) \int_{\tau_2 - \varepsilon}^{\tau_2} \left[\int_{\tau_1 - s}^{\tau_2 - s} \frac{\beta(\tau)}{(s - \tau)^\alpha} d\tau \right] ds\end{aligned}$$

$$
\begin{aligned}
&+\left\|(T(\tau_2-\tau_1)-I)\int_0^{\tau_1}\left[\int_0^{t_0-s}A^\alpha T(t_0-s-\tau)Q(\tau)f(\widetilde{x}_\tau)d\tau\right]ds\right\| \\
\leq\ & |(R(\tau_2)-R(\tau_1))\varphi(0)|_\alpha+\left\|(T(\tau_2-\tau_1)-I)\int_0^{\tau_1}A^\alpha T(\tau_1-s)f(\widetilde{x}_s)ds\right\| \\
&+K_\alpha k(K_b+M_b)\left(\int_{b-\varepsilon}^{b}\frac{\beta(s)}{(\tau_2-s)^\alpha}ds+N_b\int_{b-\varepsilon}^{b}\left[\int_{\tau_1-s}^{\tau_2-s}\frac{\beta(\tau)}{(s-\tau)^\alpha}d\tau\right]ds\right) \\
&+\left\|(T(\tau_2-\tau_1)-I)\int_0^{t_0}\left[\int_0^{\tau_1-s}A^\alpha T(\tau_1-s-\tau)Q(\tau)f(\widetilde{x}_\tau)d\tau\right]ds\right\|,
\end{aligned}
$$

where $N_b=\sup\{\|Q(t)\|:\ \ t\in[0,b]\}$. Using the continuity of $(R(.)x)$, we claim that the first part tend to zero as $|\tau_2-\tau_1|\to 0$. On the other hand, Let $t\in]0,b]$ fixed and $\beta>0$ such that $\alpha<\beta<1$, from Propositon 2.1, we have

$$
\left\|A^\beta\int_0^t T(t-s)f(\widetilde{x}_s)ds\right\|\leq K_\beta k(K_b+M_b)\int_0^b\frac{\beta(s)}{(b-s)^\alpha}ds.
$$

Then for $t\in]0,b]$ fixed the set

$$
\left\{A^\beta\int_0^t R(t-s)f(\widetilde{x}_s)ds,\ \ x\in Z_k\right\}
$$

is bounded in X. ($\mathbf{H}_2$) and Remark 2.2 imply that $A^{-\beta}:X\to X_\alpha$ is compact. Consequently

$$
\left\{\int_0^t T(t-s)f(\widetilde{x}_s)ds,\ \ x\in Z_k\right\}
$$

is relatively compact set in X_α.

Like previously, we can prove that for $\tau_1>0$ the set

$$
\left\{\int_0^{\tau_1}\left[\int_0^{\tau_1-s}A^\alpha T(\tau_1-s-\tau)Q(\tau)f(\widetilde{x}_\tau)d\tau\right]ds,\ \ x\in Z_k\right\}
$$

is also relatively compact in X. There exist compact sets Ω_1 and Ω_2 in X such that

$$
\left\{\int_0^{\tau_1}A^\alpha T(\tau_1-s)f(\widetilde{x}_s)ds,\ \ x\in Z_k\right\}\subset\Omega_1
$$

and

$$
\left\{\int_0^{\tau_1}\left[\int_0^{\tau_1-s}A^\alpha T(\tau_1-s-\tau)Q(\tau)f(\widetilde{x}_\tau)d\tau\right]ds,\ x\in Z_k\right\}\subset\Omega_2
$$

By Banach-Steinhaus's Theorem, we have

$$
\left\|(T(\tau_2-\tau_1)-I)\int_0^{t-\varepsilon}A^\alpha T(\tau_1-s)f(s,x_s)ds\right\|\to 0\ \ \text{as}\ \ \tau_1\to\tau_2
$$

and

$$
\left\|(T(\tau_2-\tau_1)-I)\int_0^{t_0}\left[\int_0^{t_0-s}A^\alpha T(t_0-s-\tau)Q(\tau)f(\widetilde{x}_\tau)d\tau\right]ds\right\|\to 0\ \ \text{as}\ \ \tau_1\to\tau_2
$$

Thus, $\mathcal{H}$ maps Z_k into an equicontinuous family of functions.
The equicontinuities for the cases $\tau_1<\tau_2\leq 0$ and $\tau_1<0<\tau_2$ are obvious.
So from the above **step 1** to **step 4** and the Ascoli-Arzelá theorem, we can conclude that $\mathcal{H}:Z_k\to Z_k$ is completely continuous. Hence by the Schauder fixed point theorem, $\mathcal{H}$ has at least one fixed point $\overline{x}$ in Z_k which is a mild solutions of equation (1.1). $\square$

In the following, we prove the uniqueness of a local existence of mild solutions of equation (1.1). For this purpose, we make the following assumption.

$(\mathbf{H}_4)$ f is locally lipschitz, that is, for each $\delta > 0$ there is a constant $c_0(\delta) > 0$ such that if $\varphi_1, \varphi_2 \in \mathcal{B}_\alpha$ with $\|\varphi_1\|_{\mathcal{B}_\alpha}, \|\varphi_2\|_{\mathcal{B}_\alpha} \leq \delta$ then

$$\|f(\varphi_1) - f(\varphi_2)\| \leq c_0(\delta)\|\varphi_1 - \varphi_2\|_{\mathcal{B}_\alpha} \text{ for } t \in [0, \delta].$$

Theorem 3.6. *Assume that $(\boldsymbol{H}_2)$, $(\boldsymbol{H}_3)$ and $(\boldsymbol{H}_4)$ hold. Let $\varphi \in \mathcal{B}_\alpha$ such that $\varphi(0) \in X_\alpha$. Then, there exists a maximal interval of existence $]-\infty, b_\varphi[$ and a unique mild solution $u(., \varphi)$ of equation (1.1) defined on $[-r, b_\varphi[$ and either*

$$b_\varphi = +\infty \text{ or } \overline{\lim}_{t \to b_\varphi^-} |u(t, \varphi)|_\alpha = +\infty.$$

Moreover, $u(t, \varphi)$ is a continuous function of φ in the sense that if $\varphi \in \mathcal{B}_\alpha$ and $t \in [0, b_\varphi[$, then there exist positive constants k and ε such that, for $\psi \in \mathcal{B}_\alpha$ and $|\varphi - \psi| < \varepsilon$, we have

$$t \in [0, b_\psi[\text{ and } |u(s, \varphi) - u(s, \psi)|_\alpha \leq k\|\varphi - \psi\|_{\mathcal{B}_\alpha} \text{ for all } s \in [-r, t].$$

Proof. The local lipschitz condition on f implies that for each $\delta > 0$, there exists $c_0(\delta)$ such that for $\varphi \in \mathcal{B}_\alpha$ with $\|\varphi\|_{\mathcal{B}_\alpha} < \delta$, we have

$$\|f(\varphi)\| \leq c_0(\delta)\|\varphi\|_{\mathcal{B}_\alpha} + \|f(0)\| \leq c_0(\delta)\delta + \|f(0)\|.$$

Let $\varphi \in \mathcal{B}_\alpha$, $\delta = \|\varphi\|_{\mathcal{B}_\alpha} + 1$ and $c_1(\delta) = c_0(\delta)\delta + \|f(0)\|$. Define the function

$$y(t) = \begin{cases} R(t)\varphi(0) & \text{for } t > 0 \\ \\ \varphi(t) & \text{for } t \in]-\infty, 0] \end{cases}$$

By virtue of axioms $(\mathbf{A}_1 - i)$ and $(\mathbf{A}_2)$, we deduce that $y_t \in \mathcal{B}_\alpha$ and the map $t \to y_t$ is continuous. Then for $b_1 \in]0, 1[$, there exists $b_2 \in]0, 1[$ such that

$$|y_t - \varphi|_{\mathcal{B}_\alpha} \leq b_1 \quad \text{for} \quad t \in [0, b_2].$$

Let $0 \leq b \leq b_2$ be such that

$$\frac{K_b K_\alpha c_1(\delta) b^{1-\alpha}}{1-\alpha} < 1 - b_1. \tag{3.2}$$

For $x \in C([0, b]; X_\alpha)$ such that $x(0) = \varphi(0)$, we define its extension on $]-\infty, b]$ by

$$\widetilde{x}(t) = \begin{cases} x(t) & \text{for } t \in [0, b] \\ \\ \varphi(t) & \text{for } t \in]-\infty, 0] \end{cases}$$

Consider the following set

$$\mathbb{F}_b = \Big\{x \in C([0, b]; X_\alpha) : \quad x(0) = \varphi(0) \;\text{ and }\; \sup_{s \in [0,b]} \|\widetilde{x}(s) - \varphi\|_{\mathcal{B}_\alpha} \leq 1\Big\},$$

provided with the uniform norm topology. Consider the mapping $\mathcal{H}$ on $\mathbb{F}_b$ defined by

$$(\mathcal{H}x)(t) = R(t)\varphi(0) + \int_0^t R(t-s) f(\widetilde{x}_s) ds \text{ for } t \in [0, b]$$

$\mathcal{H}$ maps elements of $\mathbb{F}_b$ into $\mathbb{F}_b$. In fact, by $(\mathbf{H}_4)$ and axiom $(\mathbf{A}_1)$, for every $x \in \mathbb{F}_b$, the function $s \mapsto f(\widetilde{x}_s)$ is continuous on $[0,b]$. Then, Definition 2.3 implies that

$$s \mapsto \int_0^t R(t-s)f(\widetilde{x}_s)ds$$

is continuous on $[0,b]$. From this, $v = \mathcal{F}x$ is continuous on $[0,b]$ and $v \in \mathbb{F}_b$. In fact, for any $t \in [0,b]$, we have

$$|\widetilde{v}_t - \varphi|_{\mathcal{B}_\alpha} \le |\widetilde{v}_t - y_t|_{\mathcal{B}_\alpha} + |y_t - \varphi|_{\mathcal{B}_\alpha} \le |\widetilde{v}_t - y_t|_{\mathcal{B}_\alpha} + b_1.$$

Since $|\widetilde{x}_s - \varphi|_{\mathcal{B}_\alpha} \le 1$, for $s \in [0,b]$ and $r = |\varphi|_{\mathcal{B}} + 1$, we deduce that $|\widetilde{x}_s|_{\mathcal{B}_\alpha} \le \delta$ for $s \in [0,b]$. Then

$$\|f(\widetilde{x}_s)\| \le c_0(\delta)|\widetilde{x}_s|_{\mathcal{B}} + \|f(0)\| \le c_1(\delta).$$

On the one hand, by axiom $(\mathbf{A}_1 - \mathbf{iii})$, we see that for any $t \in [0,b]$

$$|\widetilde{v}_t - y_t|_{\mathcal{B}_\alpha} \le K_b \sup_{0\le s\le t} |v(s) - y(s)|_\alpha$$

and we have

$$\begin{aligned}
|v(t) - y(t)|_\alpha &= \left| \int_0^t R(t-s)f(\widetilde{x}_s)ds \right|_\alpha \\
&\le \int_0^t \frac{K_\alpha}{(t-s)^\alpha}\|f(\widetilde{x}_s)\|ds \\
&< K_\alpha c_1(\delta)\int_0^t \frac{ds}{(t-s)^\alpha}ds \\
&< \frac{K_\alpha c_1(\delta)b^{1-\alpha}}{1-\alpha} \\
&< \frac{1}{K_b}(1-b_1).
\end{aligned}$$

Hence $|\widetilde{v}_t - y_t|_{\mathcal{B}_\alpha} < 1 - b_1$, then we deduce that $|v_t - \varphi|_{\mathcal{B}_\alpha} < 1$ for any $t \in [0,b]$. Which implies that $v \in \mathbb{F}_b$.

On the one hand, we show that $\mathcal{H}$ is a strict contraction in $\mathbb{F}_b$. Let $u, v \in \mathbb{F}_b$ and $t \in [0,b_1]$. Then

$$\begin{aligned}
|\mathcal{H}(u)(t) - \mathcal{H}(v)(t)|_\alpha &= \Big| \int_0^t R(t-s)(f(\widetilde{u}_s) - f(\widetilde{v}_s))ds \Big|_\alpha \\
&\le \int_0^t \frac{K_\alpha}{(t-s)^\alpha}\|f(u_s) - f(v_s)\|ds \\
&\le c_0(\delta)\int_0^t \frac{K_\alpha}{(t-s)^\alpha}\|u_s - v_s\|_{\mathcal{B}_\alpha}ds \\
&\le \frac{c_0(\delta)K_\alpha K_b b^{1-\alpha}}{1-\alpha}|u-v|_\infty.
\end{aligned}$$

Then

$$|\mathcal{F}v_1 - \mathcal{F}v_2|_\infty \le Ne^{\beta b}c_0(r)K_b b|v_1 - v_2|.$$

Since by inequality 3.2, we have

$$\frac{c_0(\delta)K_\alpha K_b b^{1-\alpha}}{1-\alpha} \leq \frac{c_1(\delta)K_\alpha K_b b^{1-\alpha}}{1-\alpha} < 1,$$

it follows that $\mathcal{H}$ is a strict contraction in $\mathbb{F}_b$. Thus by a fixed point theorem, $\mathcal{H}$ has a unique fixed point u in $\mathbb{F}_b$. We conclude that equation (1.1) has one and only one mild solution which is defined on $]-\infty, b]$ and denoted by $u(.,\varphi)$. Using the same arguments, we can show that the solution $u(.,\varphi)$ can be extended to a maximal interval of existence $[0, b_\varphi[$. If we assume that $b_\varphi < +\infty$ and $\overline{\lim}_{t\to b_\varphi^-}|u(t,\varphi)|_\alpha < +\infty$, then there exists a constant $\delta > 0$ such that $|u(t,\varphi)|_\alpha \leq \delta$ for all $t \in [0, b_\varphi[$. We claim that $u(.,\varphi)$ is uniformly continuous. Consequently

$$\lim_{t\to b_\varphi^-} u(t,\varphi) \text{ exists,}$$

which contradicts the maximality of $[0, b_\varphi[$. We prove now the uniform continuity of $u(.,\varphi)$. Let $t,\ t+h \in [0, b_\varphi[$ with $h > 0$, we have

$$\begin{aligned}
u(t+h,\varphi) - u(t,\varphi) &= R(t+h)\varphi(0) - R(t)\varphi(0) + \int_0^{t+h} R(t+h-s)f(u_s(.,\varphi))ds \\
&\quad - \int_0^t R(t-s)f(u_s(.,\varphi))ds \\
&= R(t+h)\varphi(0) - R(t)\varphi(0) \\
&\quad + \int_{-h}^t R(t-s)f(s+h, u_{s+h}(.,\varphi))ds \\
&\quad - \int_0^{ta} R(t-s)f(u_s(.,\varphi))ds \\
&= R(t+h)\varphi(0) - R(t)\varphi(0) + \int_{-h}^0 R(t-s)f(s+h, u_{s+h}(.,\varphi))ds \\
&\quad + \int_0^t R(t-s)\Big[f(s+h, u_{s+h}(.,\varphi)) - f(u_s(.,\varphi))\Big]ds \\
&= R(t+h)\varphi(0) - R(t)\varphi(0) + \int_0^h R(t+h-s)f(u_s(.,\varphi))ds \\
&\quad + \int_0^t R(t-s)\Big[f(s+h, u_{s+h}(.,\varphi)) - f(u_s(.,\varphi))\Big]ds.
\end{aligned}$$

Thus

$$\begin{aligned}
|u(t+h,\varphi) - u(t,\varphi)|_\alpha &\leq |R(t+h)\varphi(0) - R(t)\varphi(0)|_\alpha + \int_0^h \frac{K_\alpha c_1(\delta)}{(t+h-s)^\alpha}ds \\
&\quad + \int_0^{t+\theta} \frac{K_\alpha c_0(\delta)}{(t-s)^\alpha}\|u_{s+h}(.,\varphi) - u_s(.,\varphi)\|_{\mathcal{B}_\alpha}ds \\
&\leq |R(t+h)\varphi(0) - R(t)\varphi(0)|_\alpha + \int_0^h \frac{K_\alpha c_1(\delta)}{(h-s)^\alpha}ds \\
&\quad + \int_0^t \frac{K_\alpha c_0(\delta)K_b}{(t-s)^\alpha}\sup_{0\leq\sigma\leq s}|u_{\sigma+h}(.,\varphi) - u_\sigma(.,\varphi)|_\alpha ds
\end{aligned}$$

$$+\int_0^t \frac{K_\alpha c_0(\delta)M_b}{(t-s)^\alpha}\|u_h(.,\varphi)-\varphi\|_{\mathcal{B}_\alpha}ds$$

$$\leq |R(t+h)\varphi(0)-R(t)\varphi(0)|_\alpha+\frac{c_1(\delta)K_\alpha}{1-\alpha}h^{1-\alpha}$$

$$+\frac{c_0(\delta)K_\alpha M_b b^{1-\alpha}}{1-\alpha}\|u_h(.,\varphi)-\varphi\|_{\mathcal{B}_\alpha}$$

$$+\int_0^t \frac{K_\alpha c_0(\delta)K_b}{(t-s)^\alpha}\sup_{0\leq\sigma\leq s}|u_{\sigma+h}(.,\varphi)-u_\sigma(.,\varphi)|_\alpha ds.$$

It follows that

$$|u(t+h,\varphi)-u(t,\varphi)|_\alpha \leq \sup_{\substack{t such that\\ t+h\in[0,b_\varphi[}}|R(t+h)\varphi(0)-R(t)\varphi(0)|_\alpha+\frac{c_1(\delta)K_\alpha}{1-\alpha}h^{1-\alpha}$$

$$+\frac{c_0(\delta)K_\alpha M_b b^{1-\alpha}}{1-\alpha}\|u_h(.,\varphi)-\varphi\|_{\mathcal{B}_\alpha}+\int_0^t \frac{K_\alpha c_0(\delta)K_b}{(t-s)^\alpha}\sup_{0\leq\sigma\leq s}|u_{\sigma+h}(.,\varphi)-u_\sigma(.,\varphi)|_\alpha ds.$$

Since the map $t\mapsto R(t)\varphi(0)$ is uniformly continuous, consequently,

$$\sup_{\substack{t \text{ such that}\\ t+h\in[0,b_\varphi[}}|R(t+h)\varphi(0)-R(t)\varphi(0)|_\alpha \to 0 \quad \text{as} \quad h\to 0$$

By Gronwall's lemma, it follows that

$$|u_{t+h}(.,\varphi)-u_t(.,\varphi)|_\alpha \to 0 \quad \text{as} \quad h\to 0 \tag{3.3}$$

with

$$\gamma(h)=\delta_1(h)+\delta_2(h)+c_1(\delta)\frac{K_\alpha c_1(\delta)h^{1-\alpha}}{1-\alpha}.$$

uniformly for $t\geq 0$ such that $t+h\in[0,b_\varphi[$. Using the same reasoning, one can show a similar result for $h<0$. This implies that $u(.,\varphi)$ is uniformly continuous and can be extended to $[0,b_\varphi+\eta]$, for some $\eta>0$, which contradicts the maximality of $[0,b_\varphi[$. Using the same reasoning, one can show a similar result for $h<0$.

Next, we prove that the solution depends continuously on initial data. Let $\varphi\in\mathcal{B}_\alpha$ and $t\in[0,b_\varphi[$ be fixed. Set

$$\delta=1+\sup_{0\leq s\leq t}|u_s(.,\varphi)|_\alpha$$

and

$$c(t)=\left[M_t+K_t\left(M_\alpha\left(\eta+\frac{N_t\Gamma(1-\alpha)}{\omega^{1-\alpha}}\right)+\frac{c_0(\delta)K_\alpha M_t t^{1-\alpha}}{1-\alpha}\right)\right]\exp\left[K_tK_\alpha c_0(\delta)\int_0^t\frac{ds}{s^\alpha}\right].$$

Let $\varepsilon\in]0,1[$ be chosen such that $\varepsilon c<1$ and take $\psi\in\mathcal{B}_\alpha$ such that $\|\varphi-\psi\|_{\mathcal{B}_\alpha}<\varepsilon$. Then

$$\|\psi\|_{\mathcal{B}_\alpha}\leq\|\varphi\|_{\mathcal{B}_\alpha}+\varepsilon<\delta.$$

We define

$$b_0:=\sup\{s>0:\|u_\sigma(.,\psi)\|_{\mathcal{B}_\alpha}\leq\delta \text{ for } \sigma\in[0,s]\}.$$

We claim that $b_0 \geq t$. Suppose that $b_0 < t$, then for $s \in]0, b_0]$, proceeding as in Step 1 in the proof of Theorem 3.5

$$\begin{aligned}
& \|u_s(.,\varphi) - u_s(.,\psi)\|_{\mathcal{B}_\alpha} \\
\leq\ & M_{b_0}|\varphi - \psi|_{\mathcal{B}_\alpha} + K_{b_0} \sup_{0\leq\xi\leq s} |u(\xi,\varphi) - u(\xi,\psi)| \\
\leq\ & M_{b_0}|\varphi - \psi|_{\mathcal{B}_\alpha} \\
& + K_{b_0} \sup_{0\leq\xi\leq s} \left\{ |R(\xi)[\varphi(0) - \psi(0)]|_\alpha + \int_0^\xi \frac{K_\alpha c_0(\delta)}{(\xi-\sigma)^\alpha} \|u_\sigma(.,\varphi) - u_\sigma(.,\psi)\|_{\mathcal{B}_\alpha} d\sigma \right\} \\
\leq\ & M_t|\varphi - \psi|_{\mathcal{B}_\alpha} + \\
& + K_t \left(M_\alpha \left(\eta + \frac{N_t\Gamma(1-\alpha)}{\omega^{1-\alpha}} \right) \|\varphi(0) - \psi(0)\| + \frac{c_0(\delta)K_\alpha M_t t^{1-\alpha}}{1-\alpha} |\varphi - \psi|_{\mathcal{B}_\alpha} \right) \\
& + K_t \sup_{0\leq\xi\leq s} \left\{ \int_0^\xi \frac{K_\alpha c_0(\delta)}{(\xi-\sigma)^\alpha} \sup_{0\leq\tau\leq\sigma} |u_\tau(.,\varphi) - u_\tau(.,\psi)|_\alpha d\sigma \right\}
\end{aligned}$$

By Gronwall's lemma, we deduce that

$$|u_s(.,\varphi) - u_s(.,\psi)|_\infty \leq c(t)\|\varphi - \psi\|_{\mathcal{B}_\alpha}. \tag{3.4}$$

This implies that

$$\|u_s(.,\psi)\|_{\mathcal{B}_\alpha} \leq c(t)\varepsilon + \delta - 1 < \delta \text{ for all } s \in [0, b_0].$$

It follows that b_0 cannot be the largest number $s > 0$ such that $\|u_s(.,\psi)\|_{\mathcal{B}_\alpha} < \delta$, for $\sigma \in [0, s]$. Thus $b_0 \geq t$ and $t < b_\psi$. Furthermore, $\|u_s(.,\varphi)\|_{\mathcal{B}_\alpha} < \delta$ for $s \in [0, t]$, then using inequality 3.4, we deduce the continuous dependence on the initial data. □

As an immediat consequence, we get the following result.

Corollary 3.7. *Assume that* $(\boldsymbol{H}_2)$, $(\boldsymbol{H}_3)$ *and* $(\boldsymbol{H}_4)$ *hold and* $\alpha \in \left[0, \frac{1}{2}\right[$. *Let* k_1 *be a continuous function on* $\mathbb{R}^+$ *and* $k_2 \in L^2_{loc}(\mathbb{R}^+;\mathbb{R}^+)$ *such that* $\|f(\varphi)\| \leq k_1(t)\|\varphi\|_{\mathcal{B}_\alpha} + k_2(t)$ *for* $t \geq 0$ *and* $\varphi \in \mathcal{B}_\alpha$. *Then equation* (1.1) *has a unique mild solution which is defined for all* $t \geq 0$.

Proof. Let $[-r, b_\varphi[$ denote the maximal interval of existence of the mild solution $u(t,\varphi)$ of equation (1.1). Then

$$b_\varphi = +\infty \text{ or } \overline{\lim}_{t\to b_\varphi^-} |u(t,\varphi)|_\alpha = +\infty.$$

If $b_\varphi < +\infty$, then $\overline{\lim}_{t\to b_\varphi^-} |u(t,\varphi)| = +\infty$. Thus we have

$$\begin{aligned}
|u(t,\varphi)|_\alpha &\leq |R(t)\varphi(0)|_\alpha + \int_0^t |R(t-s)f(u_s(.,\varphi))|_\alpha ds \\
&\leq M_\alpha \left(\eta + \frac{N_{b_\varphi}\Gamma(1-\alpha)}{\omega} \right) \|\varphi(0)\| + \int_0^t \frac{K_\alpha}{(t-s)^\alpha} (k_1(s)\|u_s(.,\varphi)\|_{\mathcal{B}_\alpha} + k_2(s)) ds \\
&\leq M_\alpha \left(\eta + \frac{N_{b_\varphi}\Gamma(1-\alpha)}{\omega} \right) \|\varphi(0)\| + \int_0^t \frac{K_\alpha k_1(s)}{(t-s)^\alpha} \sup_{0\leq\xi\leq s} |u(\xi,\varphi)|_\alpha ds \\
&\quad + \int_0^t \frac{K_\alpha k_1(s)}{(t-s)^\alpha} \|\varphi\|_{\mathcal{B}_\alpha} ds + \int_0^t \frac{K_\alpha k_2(t)}{(t-s)^\alpha} ds
\end{aligned}$$

$$\leq \quad E + F\int_0^t \frac{1}{(t-s)^{\alpha}} \sup_{0\leq\xi\leq s} |u(\xi,\varphi)|_{\alpha} ds,$$

where

$$\begin{aligned} E &= M_{\alpha}\Big(\eta + \frac{N_{b_{\varphi}}\Gamma(1-\alpha)}{\omega}\Big)\|\varphi(0)\| + \frac{K_{\alpha}b_{\varphi}^{1-\alpha}F}{1-\alpha}\|\varphi\|_{\mathcal{B}_{\alpha}} + \frac{K_{\alpha}b_{\varphi}^{1-2\alpha}}{1-2\alpha}\left(\int_0^{b_{\varphi}} k_2^2(t)dt\right)^{\frac{1}{2}} \\ \text{and } F &= K_{\alpha}\sup_{0\leq s\leq b_{\varphi}} k_1(s). \end{aligned}$$

On the other hand, we claim that the function $g : s \mapsto \displaystyle\int_0^s \frac{1}{(s-\xi)^{\alpha}} \sup_{0\leq\xi\leq s} |u(\xi,\varphi)|_{\alpha} d\xi$ is nondecreasing on $[0,t]$. In fact, let $s, s' \in [0,t]$ be such that $s < s'$. Then we have

$$\begin{aligned} \int_0^s \frac{1}{(s-\xi)^{\alpha}} \sup_{0\leq\xi\leq s} |u(\xi,\varphi)|_{\alpha} d\xi &= \int_0^s \frac{1}{\xi^{\alpha}} \sup_{0\leq\xi\leq s-\xi} |u(\xi,\varphi)|_{\alpha} d\xi \\ &\leq \int_0^s \frac{1}{\xi^{\alpha}} \sup_{0\leq\xi\leq s'-\xi} |u(\xi,\varphi)|_{\alpha} d\xi \\ &\leq \int_0^{s'} \frac{1}{\xi^{\alpha}} \sup_{0\leq\xi\leq s-\xi} |u(\xi,\varphi)|_{\alpha} d\xi = g(s'). \end{aligned}$$

Therefore g is nondecreasing on $[0,t]$ and $\sup\limits_{0\leq s\leq t} g(s) = g(t)$. Then we obtain

$$\sup_{0\leq s\leq t} |u(s,\varphi)|_{\alpha} \quad\leq\quad E + F\int_0^t \frac{1}{(t-s)^{\alpha}} \sup_{0\leq\xi\leq s} |u(\xi,\varphi)|_{\alpha} ds.$$

By Lemma 2.10, there is a constant K such that

$$\sup_{0\leq s\leq t} |u(s,\varphi)|_{\alpha} \quad\leq\quad K,$$

which implies that

$$\overline{\lim}_{t\to b_{\varphi}^-} |u(t,\varphi)|_{\alpha} < +\infty,$$

which gives a contradiction. □

Corollary 3.8. *Assume that $(\boldsymbol{H}_2)$, $(\boldsymbol{H}_3)$ hold and f is lipschitzian with respect to the second argument. Then equation (1.1) has a unique mild solution which is defined for all $t \geq 0$.*

4. Existence of strict solutions

Our objective now is to give some sufficient conditions ensuring the regularity of the mild solution. For this, we suppose that $\mathcal{B}$ satifies the following axiom.

C) If $(\varphi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{B}$ and converges compactly to φ on $]-\infty,0]$, then $\varphi\in\mathcal{B}$ and $|\varphi_n - \varphi| \to 0$.

Lemma 4.1. *[12] Assume that $\mathcal{B}$ satifies the axiom (**C**)). Let $\xi : [a,b] \to \mathcal{B}$ be continuous function such that $\xi(t)(\theta)$ is continuous for $t\in[a,b]$ and $\theta\in]-\infty,0]$. Then*

$$\Big(\int_a^b \xi(t)dt\Big)(\theta) = \int_a^b \xi(t)(\theta)dt \;\; \textit{for} \;\; \theta \leq 0$$

Theorem 4.2. *Assume that* $(\boldsymbol{H}_2)$, $(\boldsymbol{H}_3)$ *and* $(\boldsymbol{H}_4)$ *hold. In addition, assume that* $f : \mathcal{B} \to X$ *is continuously differentiable and* $f' : \mathcal{B}_\alpha \to \mathcal{L}(\mathcal{B}_\alpha; X_\alpha)$ *is locally lipschitz continuous. Let* $\varphi \in \mathcal{B}_\alpha$ *be a continuously differentiable function such that*

$$\varphi' \in \mathcal{B}_\alpha, \ \ \varphi(0) \in X_\alpha, \ \varphi'(0) \in X_\alpha \ \textit{and} \ \varphi'(0) = A\varphi(0) + f(\varphi).$$

Then the corresponding mild solution u *becomes a strict solution of equation* (1.1)*.*

Proof. Let $\varphi \in \mathcal{B}_\alpha$ such that $\varphi(0) \in X_\alpha$ and $\dot{\varphi}(0) = A\varphi(0) + f(0, \varphi)$. Let u be the corresponding mild solution of equation (1.1) which is defined on some maximal interval $[0, b_\varphi[$ and let $a < b_\varphi$. Then by using the strict contraction principle, we can show that there exists a unique continuous function v such that

$$\mu(t) = \begin{cases} R(t)(-A\varphi(0) + f(0,\varphi)) + \displaystyle\int_0^t R(t-s)f'(u_s)\mu_s d + \int_0^t R(t-s)B(s)\varphi(0)ds. \\ \\ \varphi'(t) \text{ if } -\infty < t \leq 0. \end{cases}$$

We introduce the function w defined by

$$\rho(t) = \begin{cases} \varphi(0) + \displaystyle\int_0^t \mu(s)ds \text{ if } t \geq 0 \\ \\ \varphi(t) \text{ if } \infty < t \leq 0. \end{cases} \tag{4.1}$$

Using Lemma 4.1, we can see that

$$\rho_t = \varphi + \int_0^t \mu_s ds \ \text{ for } t \in [0, a].$$

We will prove that $u(t) = \rho(t)$. Using the expression of μ, we obtain

$$\begin{aligned} \rho(t) \ &= \ \varphi(0) + \int_0^t R(s)(A\varphi(0) + f(\varphi))ds + \int_0^t \int_0^s R(s-\tau)f'(u_\tau)\mu_\tau d\tau ds \\ &\quad + \int_0^t \int_0^s R(s-\tau)B(\tau)\varphi(0)d\tau ds. \end{aligned}$$

Consequently, the maps $t \to \rho_t$ and $t \to \displaystyle\int_0^t R(t-s)f(\rho_s)ds$ are continuously differentiable and the following formula holds

$$\begin{aligned} \frac{d}{dt}\int_0^t R(t-s)f(\rho_s)ds \ &= \ \frac{d}{dt}\int_0^t R(s)f(\rho_{t-s})ds \\ &= \ R(t)f(\rho_0) + \int_0^t R(t-s)f'(\rho_s)\mu_s ds \\ &= \ R(t)f(\varphi) + \int_0^t R(t-s)f'(\rho_s)\mu_s ds, \end{aligned}$$

which implies

$$\int_0^t R(s)f(\varphi)ds = \int_0^t R(t-s)f(\rho_s)ds - \int_0^t \int_0^s R(s-\tau)f'(\rho_\tau)\mu_\tau d\tau ds.$$

On the one hand, from Definition 2.3, we have

$$R(t)\varphi(0) \ = \ \varphi(0) + \int_0^t R(s)A\varphi(0)ds + \int_0^t \int_0^s R(s-\tau)B(\tau)\varphi(0)d\tau ds. \tag{4.2}$$

This implies that

$$\begin{aligned}
\rho(t) &= R(t)\varphi(0) - \int_0^t R(s)A\varphi(0)\,ds - \int_0^t\int_0^s R(s-\tau)B(\tau)\varphi(0)d\tau ds + \int_0^t R(s)[A\varphi(0)+f(\varphi)]\,ds \\
&\quad + \int_0^t\int_0^s R(s-\tau)f'(u_\tau)\mu_\tau d\tau ds + \int_0^t\int_0^s R(s-\tau)B(\tau)\varphi(0)d\tau ds. \\
&= R(t)\varphi(0) + \int_0^t R(t-s)f(\rho_s)ds + \int_0^t\int_0^s R(s-\tau)[f'(u_\tau)\mu_\tau - f'(\rho_\tau)\mu_\tau]d\tau ds \qquad (4.3)
\end{aligned}$$

Since

$$u(t) = R(t)\varphi(0) + \int_0^t R(t-s)f(u_s)ds \quad \text{for} \quad t \geq 0,$$

then we deduce that

$$u(t) - \rho(t) = \int_0^t R(t-s)[f(u_s) - f(\rho_s] + \int_0^t\int_0^s R(s-\tau)[f'(u_\tau)\mu_\tau - f'(\rho_\tau)\mu_\tau]d\tau ds.$$

Consequently

$$|u(t)-\rho(t)|_\alpha = \int_0^t \frac{K_\alpha}{(t-s)^\alpha}\|f(u_s) - f(\rho_s\| + \int_0^t\int_0^s \frac{K_\alpha}{(s-\tau)^\alpha}\|f'(u_\tau)\mu_\tau - f'(\rho_\tau)\|\ \|\mu_\tau\|_{\mathcal{B}_\alpha} d\tau ds.$$

Let

$$\delta = \max(\sup_{0\leq s\leq a}|u_s|_{\mathcal{B}_\alpha}, \sup_{0\leq s\leq a}|\mu_s|_{\mathcal{B}_\alpha}, \sup_{0\leq s\leq a}|\rho_s|_{\mathcal{B}_\alpha}).$$

There exists $c_0(\delta)$ such that if $\varphi_1, \varphi_2 \in \mathcal{B}$ and $|\varphi_1|_{\mathcal{B}_\alpha}, |\varphi_2|_{\mathcal{B}_\alpha} \leq \delta$, then

$$|f(\varphi_1) - f(\varphi_2)| \leq c_0(\delta)|\varphi_1 - \varphi_2|_{\mathcal{B}_\alpha},$$

$$|f'(\varphi_1) - f'(\varphi_2)| \leq c_{0(\delta)}|\varphi_1 - \varphi_2|_{\mathcal{B}_\alpha}.$$

This implies that we can find a positive constant $k(a)$ such that

$$|u(t)-\rho(t)|_\alpha \leq k(a)\int_0^t \frac{\|u_s-\rho_s\|_{\mathcal{B}_\alpha}}{(t-s)^\alpha}ds \text{ for } s\in[0,a].$$

and by $(\mathbf{A}_1 - \mathbf{iii})$, it follows that

$$\sup_{0\leq s\leq t}|u(s)-\rho(s)|_\alpha \leq k(a)\int_0^t \frac{ds}{(t-s)^\alpha}\sup_{0\leq\sigma\leq s}|u(\sigma)-\rho(\sigma)|_\alpha \text{ for } s\in[0,a].$$

By Gronwall's lemma, $|u_t - \rho_t|_\infty = 0$ for any $t \in [0, b_1]$. Using the axiom $(\mathbf{A}_1 - \mathbf{ii})$, we conclude that $u(t) = z(t)$ for any $t \in]-\infty, a]$. Consequently, the mild solution u is continuously differentiable. According Theorem 2.9, we deduce that u is a strict solution of equation (1.1). □

5. Application

For illustration, we propose to study the existence of solutions for the following model

$$
(5.1)\qquad \begin{cases} \dfrac{\partial}{\partial t}z(t,x) = \dfrac{\partial^2}{\partial x^2}z(t,x) + \displaystyle\int_0^t \eta(t-s)\dfrac{\partial^2}{\partial x^2}z(s,x)ds \\ \qquad\qquad +te^{-|t|}\displaystyle\int_{-\infty}^0 g(\theta, z(t+\theta,\xi))d\theta \text{ for } t\geq 0 \text{ and } x\in[0,\pi] \\ z(t,0) = z(t,\pi) = 0 \text{ for } t\geq 0 \\ z(\theta,x) = \varphi_0(\theta,x) \text{ for } \theta\in\mathbb{R}^- \text{ and } x\in[0,\pi], \end{cases}
$$

where $g:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ is continuous and lipschitzian with respect to the second argument, $\eta:\mathbb{R}^+\to\mathbb{R}$ is uniformly bounded continuous, differentiable and η' is bounded uniformly continuous, the function $\varphi_0 :]-\infty,0]\times[0,\pi]\to]-\infty,+\infty[$ will be specified later. To rewrite equation (5.1) in the abstract form, we introduce the space $X = L^2([0,\pi];]-\infty,+\infty[)$ vanishing at 0 and π, equipped with the L^2 norm that is to say for all $x\in X$,

$$
\|x\|_{L^2} = \Big(\int_0^\pi |x(s)|^2 ds\Big)^{\frac{1}{2}}.
$$

Let $A : X\to X$ be defined by

$$
\begin{cases} D(A) = H^2(0,\pi)\cap H_0^1(0,\pi) \\ Ay = -y''. \end{cases}
$$

Then the spectrum $\sigma(A)$ of A equals to the point spectrum $\sigma_p(A)$ and is given by

$$
\sigma(A) = \sigma_p(A) = \{-n^2 : \ n\geq 1\}
$$

and the associated eigenfunctions $(e_n)_{n\geq 1}$ are given by

$$
e_n(s) = \sqrt{\frac{2}{\pi}}\sin(ns),\ \ s\in[0,\pi].
$$

Then the operator is computed by

$$
Ay = \sum_{n=1}^{+\infty} n^2(y,e_n)e_n,\ \ y\in D(A).
$$

For each $y\in D(A^{\frac{1}{2}}) = \{y\in X : \displaystyle\sum_{n=1}^{+\infty} n(y,e_n)e_n \in X\}$, the operator $A^{\frac{1}{2}}$ is given by

$$
A^{\frac{1}{2}}y = \sum_{n=1}^{+\infty} n(y,e_n)e_n,\ \ y\in D(A).
$$

Lemma 5.1. *[11] If $y\in D(A^{\frac{1}{2}})$, then y is absolutely continuous, $y'\in X$ and*

$$
\|y\|_{\frac{1}{2}} = \|y'\| = \|A^{\frac{1}{2}}y\|.
$$

It is well known that $-A$ is the generator of a compact analytic semigroup $(T(t))_{t\geq 0}$ on X which is given by

$$
T(t)x = \sum_{n=1}^{+\infty} e^{-n^2 t}(x,e_n)e_n,\ \ x\in X.
$$

Here we choose $\alpha = \frac{1}{2}$. Let $\mu > 0$, we define the phase space

$$\mathcal{B} = C_\mu = \{\varphi \in C(] - \infty, 0]; X) : \lim_{\theta \to -\infty} e^{\mu\theta}\varphi(\theta) \text{ exist in } X\},$$

with the norm

$$\|\varphi\|_\mu = \sup_{\theta \le 0} e^{\mu\theta}\|\varphi(\theta)\|, \text{ for } \varphi \in C_\mu$$

This space satisfies axioms $(\mathbf{A}_1)$, $(\mathbf{A}_2)$ and $(\mathbf{B})$. The norm in $\mathcal{B}_{\frac{1}{2}}$ is given by

$$\|\varphi\|_{\mathcal{B}_{\frac{1}{2}}} = \sup_{\theta \le 0} e^{\mu\theta}\|A^{\frac{1}{2}}\varphi(\theta)\| = \sup_{\theta \le 0} e^{\mu\theta}\Big(\int_0^\pi \Big(\frac{\partial}{\partial x}(\varphi)(\theta)(\xi)\Big)^2 d\xi\Big)^{\frac{1}{2}}.$$

Let $B : D(A) \to X$ be defined by

$$B(t)(y) = \eta(t)Ay \; for \; t \ge 0$$

and we define the function f by

$$f(t, \varphi)(\xi) = te^{-|t|}\int_{-\infty}^0 g(\theta)\varphi(\theta)(\xi)d\theta \text{ for } \xi \in [0, \pi] \text{ and } t \ge 0.$$

Let us pose $v(t) = z(t, \xi)$.Then equation (5.1) takes the following abstract form

$$\begin{cases} v(t) = -Av(t) + \displaystyle\int_0^t B(t-s)u(s)ds + f(t, v_t), \ t \ge 0 \\ \\ v_0 = \varphi \in \mathcal{B} \end{cases} \tag{5.2}$$

Assume that

$(\mathbf{H}_5)$: $e^{-2\mu}g \in L^2(\mathbb{R}^-)$.

Moreover, for every $\varphi_1, \varphi_2 \in \mathcal{B}_\alpha$, using Hölder inegality, we have

$$\begin{aligned}
& \|f(t,\varphi_1) - f(t,\varphi_2)\|^2 \\
= \; & \int_0^\pi \Big|te^{-|t|}\int_{-\infty}^0 g(\theta)[\varphi_1(\theta)(\xi) - \varphi_2(\theta)(\xi)]d\theta\Big|^2 d\xi \\
= \; & t^2e^{-2|t|}\int_0^\pi \Big|\int_{-\infty}^0 g(\theta)e^{-2\mu\theta}e^{2\mu\theta}[\varphi_1(\theta)(\xi) - \varphi_2(\theta)(\xi)]d\theta\Big|^2 d\xi \\
\le \; & t^2e^{-2|t|}\int_0^\pi \Big(\int_{-\infty}^0 g^2(\theta)e^{-4\mu\theta}d\theta\Big)\Big(\int_{-\infty}^0 e^{4\mu\theta}[\varphi_1(\theta)(\xi) - \varphi_2(\theta)(\xi)]^2 d\theta\Big)d\xi \\
\le \; & \Big(\int_{-\infty}^0 g^2(\theta)e^{-4\mu\theta}d\theta\Big)t^2e^{-2|t|}\int_0^\pi \Big(\int_{-\infty}^0 e^{2\mu\theta}e^{2\mu\theta}[\varphi_1(\theta)(\xi) - \varphi_2(\theta)(\xi)]^2 d\theta\Big)d\xi \\
\le \; & \Big(\int_{-\infty}^0 g^2(\theta)e^{-4\mu\theta}d\theta\Big)t^2e^{-2|t|}\int_0^\pi \sup_{\theta \le 0} e^{2\mu\theta}[\varphi_1(\theta)(\xi) - \varphi_2(\theta)(\xi)]^2\Big(\int_{-\infty}^0 e^{2\mu\theta}d\theta\Big)d\xi \\
\le \; & \Big(\int_{-\infty}^0 g^2(\theta)e^{-4\mu\theta}d\theta\Big)t^2e^{-2|t|}\sup_{\theta \le 0} e^{2\mu\theta}\int_0^\pi [\varphi_1(\theta)(\xi) - \varphi_2(\theta)(\xi)]^2\Big(\int_{-\infty}^0 e^{2\mu\theta}d\theta\Big)d\xi
\end{aligned}$$

$$\leq \frac{1}{2\mu}\Big(\int_{-\infty}^{0} g^2(\theta)e^{-4\mu\theta}d\theta\Big)t^2e^{-2|t|}\sup_{\theta\leq 0} e^{2\mu\theta}\int_0^{\pi}[\varphi_1(\theta)(\xi)-\varphi_2(\theta)(\xi)]^2 d\xi$$

$$\leq \frac{1}{2\mu}\Big(\int_{-\infty}^{0} g^2(\theta)e^{-2\mu\theta}d\theta\Big)t^2e^{-2|t|}\|\varphi_1-\varphi_2\|^2_{\mathcal{B}_{\frac{1}{2}}}$$

Consequently, we conclude that f is locally Lipschitz continuous. Then for $\varphi \in \mathcal{B}_{\frac{1}{2}}$, equation (1.1) has a unique mild solution.

For the regularity, we make the following assumptions.

(**H**$_6$) $g \in C^1(\mathbb{R}^+ \times \mathbb{R};\mathbb{R})$, such that $\dfrac{\partial g}{\partial t}$ and $\dfrac{\partial g}{\partial x}$ are locally lipschitz continuous.

(**H**$_7$)

$$\begin{cases} \varphi \in C^1(]-\infty,0]\times[0,\pi]) \text{ such that } \varphi(0,.) \in D(A^{\frac{1}{2}}) \\ \dfrac{\partial}{\partial\theta}\varphi(0,x) = \dfrac{\partial^2}{\partial x^2}\varphi(0,x) + \displaystyle\int_{-\infty}^{0} g(\theta,\varphi(\theta,x))d\theta \text{ for } x \in [0,\pi]. \end{cases}$$

Proposition 5.2. *Under the above assumptions, equation (5.1) has a unique strict solution v and the solution u defined by $u(t,x) = v(t)(x)$ for $t \geq 0$ and $x \in [0,\pi]$ becomes a solution equation (5.1).*

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