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FINITE NON-SOLVABLE GROUPS WITH FEW SUM OF NUMBERS OF SYLOW SUBGROUPS

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Abstract. Let G be a finite group and $n_p(G)$ be the number of Sylow p-subgroups of G. Let $S(G) = \{p \in \pi(G) : n_p(G) > 1\}$ and define $\delta_0(G) = \sum_{p \in S(G)} n_p(G)$. Denote by $Sol(G)$ the solvable radical. In this paper, if G is non-solvable and $\delta_0(G) \leq 1000$, we classify $G/Sol(G)$ completely.

1. Introduction

Let G be a finite group and $\pi(G)$ be the set of prime divisors of $|G|$. For every prime $p \in \pi(G)$, we set $n_p(G)$ to be the number of Sylow p-subgroups of G and call it the Sylow p-number. In [1] Gao, Lima and Shen studied the number of Sylow subgroups $n_p(G)$ in finite groups, particularly when $n_p(G) < p^2$. In [2], Anabanti, Moretó and Zarrin focus on the relationship between the number of Sylow subgroups of a finite group and the solvability of the group, and by studying the number of Sylow subgroups of a particular prime, they give conditions for determining whether the group is solvable or not. In [3], Wu focuses on the relationship between the number of Sylow subgroups in finite simple groups and their structure, centered on the universality of the $DivSyl(p)$ property. In [4], Zhang studied the connection between Sylow numbers and Sylow graphs, systematically investigating the influence of Sylow numbers on the group structure. Set $S(G) = \{p \in \pi(G) : n_p(G) > 1\}$ and define $\delta_0(G) = \sum_{p \in S(G)} n_p(G)$. As in [9], the average number $\delta_0(G)/|S(G)|$ is denoted by $asn(G)$. In [5], Asboei and Amiri proved that if G is a finite nonsolvable group with $asn(G) < 39/4$ and $asn(G) \neq 29/4$, then $G/F(G) \cong A_5$. In [6] Hall showed that certain natural numbers cannot be Sylow p-numbers for an odd prime p. Lu and Meng in [8] proved that if G is a finite group with $asn(G) < 7$, then G is solvable. Liu and Zhang in [7] shows that if G is nonsolvable and $\delta_0(G) \leq 56$, then $G/N \cong A_5$ or S_5, where $Sol(G)$ is the solvable radical, that is, the largest normal solvable subgroup of G. In this paper, we extend the result and consider the non-solvable group and $\delta_0(G) \leq 1000$.

Theorem 1.1. *Let G be a nonsolvable group and N be maximal normal solvable subgroup of G. If $\delta_0(G) \leq 1000$, then G/N is one of the following groups.*

(1) A_5, A_6, A_7, S_5, S_6, S_7, $\mathrm{PSL}(3,3)$, $\mathrm{PSL}(3,3)$, $\mathrm{PSU}(3,3)$, $P\Sigma L(2,16)$,

(2) $\mathrm{PSL}(2,q)$ *where* $q = 7, 8, 11, 13, 16, 17, 19$,

(3) $\mathrm{PGL}(2,q)$ *where* $q = 7, 11, 13, 17$

(4) $\mathrm{P\Gamma L}(2,q)$ *where* $q = 8, 9, 16$;

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(5) $S_5 \times S_5$, $S_5 \times S_6$, $A_m \times A_n$, $A_m \times S_n$, *where* $m, n \in \{5, 6\}$,

(6) $A_5 \times PSL(2, q)$, $S_5 \times PSL(2, q)$, *where* $q = 7, 8$;

(7) $A_5 \times PGL(2, q)$, $S_5 \times PGL(2, q)$, *where* $q = 7, 9$;

(8) $A_5 \times P\Gamma L(2, q)$, $S_5 \times P\Gamma L(2, q)$, *where* $q = 8, 9$;

(9) $A_5 \wr C_2$, $S_5 \wr C_2$, $PSL(3, 3) \rtimes_\phi C_2$, $PSU(3, 3) \rtimes_\varphi C_2$, *where* ϕ *and* φ *are the graph automorphisms.*

Moreover, the values of $\delta_0(G/N)$ *are contained in Table 6.*

2. Finite Simple Groups with $n_2(G) \leq 1000$

In this section, we will classify non-abelian simple groups G with $n_2(G) \leq 1000$. The notation and terminology in this paper are standard and can be found in [10].

Lemma 2.1. *[7, Lemma 3.2] Let* G *be a finite non-abelian simple group and* S *a Sylow* 2*-subgroup of* G*. Then the number of Sylow* 2*-subgroups* $n_2(G)$ *is* $|G|_{2'}$ *except the following cases:*

(a) If G *is a group of Lie type over a field of characteristic* 2*, then* $n_2(G)$ *is in Table 1.*

(b) If $G \cong \mathrm{PSL}(2, q)$, $3 < q \equiv \pm 3 \pmod 8$*, then* $n_2(G) = q(q^2 - 1)/24$*.*

(c) $G \cong \mathrm{PSL}_{n+1}(q)$ *or* $\mathrm{PSU}_{n+1}(q)$, $n \geq 2$, q *is odd, and there exists the* 2*-adic expansion* $n + 1 = 2^{s_1} + \ldots + 2^{s_t}$ *such that* $s_1 > \ldots > s_t \geq 0$ *and* $t \geq 2$*.*

(1) If $G \cong \mathrm{PSL}_{n+1}(q)$*, then* $n_2(G) = q^{n(n+1)/2}\left(\prod_{i=2}^{n+1} \frac{q^i-1}{(q-1)^{t-1}}\right)_{2'}$*.*

(2) If $G \cong \mathrm{PSU}_{n+1}(q)$*, then* $n_2(G) = q^{n(n+1)/2}\left(\prod_{i=2}^{n+1} \frac{q^i-(-1)^i}{(q+1)^{t-1}}\right)_{2'}$*.*

(d) If $G \cong \mathrm{PSp}_{2n}(q)$, $n \geq 2$, $q \equiv \pm 3 \pmod 8$*, and there exists the* 2*-adic expansion* $n = 2^{s_1} + \ldots + 2^{s_t}$, $s_1 > \ldots > s_t \geq 0$*, then* $n_2(G) = \frac{|G|_{2'}}{3^t}$*.*

(e) If $G \cong E_6(q)$ *and* q *is odd, then* $n_2(G) = \left(\frac{|G|(q-1,3)}{q-1}\right)_{2'}$*.*

(f) If $G \cong {}^2E_6(q)$ *and* q *is odd, then* $n_2(G) = \left(\frac{|G|(q+1,3)}{q+1}\right)_{2'}$*.*

(g) $G \cong {}^2G_2(q)$ *or* J_1*, then* $n_2(G) = \frac{|G|}{168}$*.*

(h) $G \cong J_2$, J_3, Suz*, or* HN*, then* $n_2(G) = \left(\frac{|G|}{3}\right)_{2'}$*.*

TABLE 1. Sylow 2-numbers of simple groups of Lie type where q is even.

Group G	$n_2(G)$
$L_{n+1}(q)$	$\prod_{i=2}^{n+1}(q^i-1)/(q-1)^n$
$U_{n+1}(q)$, n is even	$\prod_{i=2}^{n+1}(q^i-(-1)^i)/[(q-1)^{n/2}(q+1)^{n/2}]$
$U_{n+1}(q)$, n is odd	$\prod_{i=2}^{n+1}(q^i-(-1)^i)/[(q-1)^{(n+1)/2}(q+1)^{(n-1)/2}]$
$\Omega_{2n+1}(q), PSp(2n,q)$	$\prod_{i=1}^{n}(q^{2i}-1)/(q-1)^n$
$P\Omega^+(2n,q)$	$(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1)/(q-1)^n$
$P\Omega^-(2n,q)$	$(q^n+1)\prod_{i=1}^{n-1}(q^{2i}-1)/[(q+1)(q-1)^{n-1}]$
$G_2(q)$	$(q+1)(q^6-1)/(q-1)$
$F_4(q)$	$(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)/(q-1)^4$
$E_6(q)$	$(q^2-1)(q^5-1)(q^6-1)(q^8-1)(q^9-1)(q^{12}-1)/(q-1)^6$
$E_7(q)$	$(q^2-1)(q^6-1)(q^8-1)(q^{10}-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)/(q-1)^7$
$E_8(q)$	$(q^2-1)(q^8-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)(q^{20}-1)(q^{24}-1)(q^{30}-1)/(q-1)^8$
${}^2B_2(q^2)$	q^4+1, $q^2=2^{2m+1}$
${}^3D_4(q^3)$	$(q+1)(q^3+1)(q^8+q^4+1)$
${}^2F_4(q^2)$	$(q^2+1)(q^4+1)(q^6+1)(q^{12}+1)$, $q^2=2^{2m+1}>2$
${}^2E_6(q)$	$(q^2-1)(q^5+1)(q^6-1)(q^8-1)(q^9+1)(q^{12}-1)/[(q-1)^4(q+1)^2]$
${}^2F_4(2)'$	$3^3\cdot5^2\cdot13$

Lemma 2.2. *[7, Page 8] Let G be a sporadic simple group or an alternating group A_n with $n>6$. Then the number of Sylow 2-subgroups of G is provided in Table 2.*

TABLE 2. Sylow 2-numbers of sporadic simple groups and alternating group.

Group G	$n_2(G)$	G	$n_2(G)$
M_{11}	$3^2\cdot5\cdot11$	$O'N$	$3^4\cdot5\cdot7^3\cdot11\cdot19\cdot31$
M_{12}	$3^3\cdot5\cdot11$	Co_3	$3^7\cdot5^3\cdot7\cdot11\cdot23$
J_1	$5\cdot11\cdot19$	Co_2	$3^6\cdot5^3\cdot7\cdot11\cdot23$
M_{22}	$3^2\cdot5\cdot7\cdot11$	Fi_{22}	$3^9\cdot5^2\cdot7\cdot11\cdot13$
J_2	$3^2\cdot5^2\cdot7$	HN	$3^5\cdot5^6\cdot7\cdot11\cdot19$
M_{23}	$3^2\cdot5\cdot7\cdot11\cdot23$	Ly	$3^7\cdot5^6\cdot7\cdot11\cdot31\cdot37\cdot67$
HS	$3^2\cdot5^3\cdot7\cdot11$	Th	$3^{10}\cdot5^3\cdot7^2\cdot13\cdot19\cdot31$
J_3	$3^4\cdot5\cdot17\cdot19$	Fi_{23}	$3^{13}\cdot5^2\cdot7\cdot11\cdot13\cdot17\cdot23$
M_{24}	$3^3\cdot5\cdot7\cdot11\cdot23$	Co_1	$3^9\cdot5^4\cdot7^2\cdot11\cdot13\cdot23$
McL	$3^6\cdot5^3\cdot7\cdot11$	J_4	$3^3\cdot5\cdot7\cdot11^3\cdot23\cdot29\cdot31\cdot37\cdot43$
He	$3^3\cdot5^2\cdot7^3\cdot17$	Fi'_{24}	$3^{16}\cdot5^2\cdot7^3\cdot11\cdot13\cdot17\cdot23\cdot29$
Ru	$3^3\cdot5^3\cdot7\cdot13\cdot29$	B	$3^{13}\cdot5^6\cdot7^2\cdot11\cdot13\cdot17\cdot19\cdot23\cdot31\cdot47$
Suz	$3^6\cdot5^2\cdot7\cdot11\cdot13$	M	$3^{20}\cdot5^9\cdot7^6\cdot11^2\cdot13^3\cdot17\cdot19\cdot23\cdot29\cdot31\cdot41\cdot47\cdot59\cdot71$
A_5	5	A_6	$3^2\cdot5$
A_n	$(n!/2)_{2'}(n>6)$		

Lemma 2.3. *Let G be a non-abelian simple group such that $n_2(G) \leq 1000$. Then G is isomorphic to one of the following groups:* M_{11}, $A_n(n = 5, 6, 7, 8$, $\mathrm{PSL}(2, q)(q = 7, 8, 11, 13, 16, 17, 19, 23, 25, 27, 31, 32, 64, 128, 256, 512)$, $\mathrm{PSL}(3, q)(q = 3, 4, 8)$, $PSU(3, q)(q = 3, 4, 8)$, $PSU(4, 2)$, $PSp(4, 4)$, $P\Omega(5, 3)$.

Proof. Let G be a sporadic simple group. By Table 2, we have $n_2(G) \leq 1000$ if and only if $G \cong M_{11}$. Let G be an alternating group. From Table 2, it can be seen that $|A_n| = n!/2$ and $n_2(G) = (n!/2)_{2'}$. If $n = 9$, then $n_2(A_9) = 3^4 \cdot 5 \cdot 7 > 1000$. Since $|A_9| \mid |A_n|$ for $n \geq 9$, we have $n_2(A_9) \mid n_2(A_n)$. It leads to $n_2(A_n) > 1000$. Obviously, $n_2(A_n) \leq 1000$ for $5 \leq n \leq 8$.

Next, we consider the simple group of Lie type.

(a) Suppose that G is not an exceptional case in Lemma 2.2, the number of Sylow 2-subgroups $n_2(G)$ is $|G|_{2'}$ which can be calculated by the order of simple groups of Lie type (see [7, Page 7]).

(1) Let $G \cong L^{\pm}_{n+1}(q)$, where $\lambda = 1$ corresponds to $L^{+}_{n+1}(q)$ and $\lambda = -1$ corresponds to $L^{-}_{n+1}(q)$. Therefore

$$|G| = \frac{1}{(n+1, q \pm 1)} q^{n(n+1)/2} \prod_{i=2}^{n+1} (q^i - (\lambda)^i),$$

which leads to

$$n_2(G) = \frac{1}{(n+1, q \pm 1)} q^{n(n+1)/2} [\prod_{i=2}^{n+1} (q^i - (\lambda)^i)]_{2'},$$

and so $q^{n(n+1)/2} \leq n_2(G)$. Since $n_2(G) \leq 1000$, it follows that $q^{n(n+1)/2} \leq 1000$. We obtain $n \leq 3$. If $n = 1$, then $n_2(G) = q(q^2-1)_{2'}/(2, q-1) \leq 1000$. Therefore $q = 5, 7, 9, 11, 13, 17, 19, 23, 25, 31, 4, 8, 16, 32, 64, 128, 256, 512$. If $n = 2$, then $n_2(G) = q^3[(q^2-1)(q^3 \pm 1)]_{2'}/(3, q \pm 1) \leq 1000$. Therefore $q = 2, 3, 4, 8$. If $n = 3$, then $n_2(G) = q^6[(q^2 - 1)(q^3 \pm 1)(q^4 - 1)]_{2'}/(4, q \pm 1) \leq 1000$. Thus $q = 2$.

By above arguments, $G \cong L_2(q)(q = 5, 7, 9, 11, 13, 17, 19, 23, 25, 31, 4, 8, 16, 32, 64, 128, 256, 512)$, $L_3(q)(q = 2, 3, 4, 8)$, $U_3(q)(q = 3, 4, 8)$, $U_4(2)$, $L_4(2)$.

(2) Let $G \cong \Omega_{2n+1}(q)$ or $PSp(2n, q)$. Then

$$|G| = \frac{1}{(2, q-1)} q^{n^2} \prod_{i=1}^{n} (q^{2i} - 1),$$

we have

$$n_2(G) = \frac{1}{(2, q-1)} q^{n^2} [(q^2 - 1) \cdots (q^{2n} - 1)]_{2'},$$

and so $q^{n^2} \leq n_2(G)$. Since $n_2(G) \leq 1000$, it follows that $q^{n^2} \leq 1000$. Thus, we obtain $n \leq 2$. If $n = 2$, then $n_2(G) = q^4[(q^2 - 1)(q^4 - 1)]_{2'}/(2, q - 1) \leq 1000$. Therefore $q = 3, 4$. Hence $G \cong \Omega_5(3)$, $PSp(4, 3)$, $PSp(4, 4)$.

(3) Let $G \cong P\Omega^{+}(2n, q)$. Then

$$|G| = \frac{1}{(4, q^n - 1)} q^{n(n-1)} \prod_{i=1}^{n-1} (q^{2i} - 1)(q^n - 1),$$

we obtain

$$n_2(G) = \frac{1}{(4, q^n - 1)} q^{n(n-1)} [\prod_{i=1}^{n-1} (q^{2i} - 1)(q^n - 1)]_{2'},$$

and so $q^{n(n-1)} \leq n_2(G)$. Since $n_2(G) \leq 1000$, it follows that $q^{n(n-1)} \leq 1000$. We obtain $n \leq 3$. If $n = 3$, then $n_2(G) = q^6[(q^2-1)(q^3-1)(q^4-1)]_{2'}/(4, q^3-1) \leq 1000$. Thus $q = 2$. Hence, $G \cong P\Omega^+(6, 2)$.

(4) Let $G \cong P\Omega^-(2n, q)$. Then

$$|G| = \frac{1}{(4, q^n+1)} q^{n(n-1)} \prod_{i=1}^{n-1} (q^{2i}-1)(q^n+1),$$

we have

$$n_2(G) = \frac{1}{(4, q^n+1)} q^{n(n-1)} [\prod_{i=1}^{n-1} (q^{2i}-1)(q^n+1)]_{2'},$$

and so $q^{n(n-1)} \leq n_2(G)$. Since $n_2(G) \leq 1000$, it follows that $q^{n(n-1)} \leq 1000$. We obtain $n \leq 3$. If $n = 2$, then $n_2(G) = q^2[(q^2-1)(q^2+1)]_{2'}/(4, q^4+1) \leq 1000$. Thus $q = 3, 4, 5, 8, 16$. Moreover, if $n = 3$, then $n_2(G) = q^6[(q^2-1)(q^4-1)(q^3+1)]_{2'}/(4, q^3+1) \leq 1000$. Thus $q = 2$. Hence $G \cong P\Omega^-(4, q)(q = 3, 4, 5, 8)$, $P\Omega^-(4, 16)$, $P\Omega^-(6, 2)$. For the remaining cases, it is obvious that $n_2(G) > 1000$.

(b)Next, we consider the special cases of $n_2(G) \neq |G|_{2'}$ in Lemma 2.2.

(1) Let $G \cong L_2(q)$, $3 < q \equiv \pm 3 (mod\ 8)$. Then $n_2(G) = q(q^2-1)/24$. Since $n_2(G) = q(q^2-1)/24 \leq 1000$, we have $q = 5$, 11, 13, 19 or 27. Thus $G \cong L_2(q)(q = 5, 11, 13, 19, 27)$.

(2) Let $G \cong L_{n+1}(q)$, $n \geq 2$, and there exists the 2-*adic* expansion $n+1 = 2^{s_1} + \ldots + 2^{s_t}$ such that $s_1 > \ldots > s_t \geq 0$ and $t \geq 2$. Then

$$n_2(G) = q^{n(n+1)/2} (\prod_{i=2}^{n+1} (q^i - 1)/(q-1)^{t-1})_{2'},$$

and so $q^{n(n+1)/2} \leq n_2(G)$. Since $n_2(G) \leq 1000$, we have $q^{n(n+1)/2} \leq 1000$, then $n \leq 3$. If $n = 2$, we can obtain $n+1 = 2^0 + 2^1$, then $t = 2$ and $n_2(G) = q^3[(q+1)(q^3-1)]_{2'} \leq 1000$. Thus $q = 3$. If $n = 3$, then $n+1 = 2^2$. Thus, $t = 1$, a contradiction. Therefore $G \cong L_3(3)$.

(3) Let $G \cong U_{n+1}(q)$, $n \geq 2$, and there exists the 2-*adic* expansion $n+1 = 2^{s_1} + \ldots + 2^{s_t}$ such that $s_1 > \ldots > s_t \geq 0$ and $t \geq 2$. Then

$$n_2(G) = q^{n(n+1)/2} (\prod_{i=2}^{n+1} (q^i - (-1)^i)/(q+1)^{t-1})_{2'},$$

and so $q^{n(n+1)/2} \leq n_2(G)$. Since $n_2(G) \leq 1000$, it follows that $q^{n(n+1)/2} \leq 1000$. We obtain $n \leq 3$. If $n = 2$, then $n+1 = 2^0 + 2^1$. Thus $t = 2$. When $n_2(G) = q^3[(q-1)(q^3+1)]_{2'} \leq 1000$, $q = 3$. Moreover, if $n = 3$, then $n+1 = 2^2$. Thus $t = 1$, a contradiction. Therefore $G \cong U_3(3)$.

(4) Let $G \cong PSp_{2n}(q)$, $n \geq 2$, $q \equiv \pm 3 (mod\ 8)$, and there exists the 2-*adic* expansion $n = 2^{s_1} + \cdots + 2^{s_t}$, $s_1 > \cdots > s_t \geq 0$. Then

$$n_2(G) = |G|_{2'}/3^t = \frac{1}{3^t(2, q-1)} q^{n^2} [(q^2-1) \cdots (q^{2n}-1)]_{2'},$$

and so $q^{n^2}/3^t \leq n_2(G)$. Since $n_2(G) \leq 1000$, it follows that $q^{n^2}/3^t \leq 1000$. However, the conditions $q \geq 5$, $n \geq 2$, and $t \geq 1$ imply $n_2(G) \geq 24375 > 1000$, a contradiction.

(5) Let $G \cong E_6(q)$ and q is odd. Then $n_2(G) = (|G|(q-1, 3)/(q-1))_{2'}$, and so

$$n_2(G) = [q^{36}(q+1)(q^5-1)(q^6-1)(q^8-1)(q^9-1)(q^{12}-1)]_{2'}$$

Obviously, $n_2(G) > 1000$, a contradiction.

(6) Let $G \cong {}^2E_6(q)$ and q is odd. Then $n_2(G) = (|G|(q+1,3)/(q+1))_{2'}$, and so

$$n_2(G) = [q^{36}(q-1)(q^5+1)(q^6-1)(q^8-1)(q^9+1)(q^{12}-1)]_{2'}$$

Clearly, $n_2(G) > 1000$, a contradiction.

(7) Let $G \cong {}^2G_2(q)$. Then $n_2(G) = |G|/168 > 1000$, and so

$$n_2(G) = q^6(q^2-1)(q^6+1)/168$$

Obviously, $n_2(G) > 1000$, a contradiction.

(8) Let $G \cong J_2$, J_3, Suz or HN. Then $n_2(G) = (|G|/3)_{2'}$. By Table 3, $n_2(G) > 1000$, a contradiction. So the lemma holds. □

By calculation, all non-abelian simple groups with $n_2(G) \leq 1000$ are listed in Table 3.

Table 3. All simple groups where $n_2(G) \leq 1000$.

Group	$n_2(G)$	Group	$n_2(G)$
M_{11}	$3^2 \cdot 5 \cdot 11 = 495$	$L_2(32)$	$3 \cdot 11 = 33$
A_5	5	$L_2(64)$	$5 \cdot 13 = 65$
A_6	$3^2 \cdot 5 = 45$	$L_2(128)$	$3 \cdot 43 = 129$
A_7	$3^2 \cdot 5 \cdot 7 = 315$	$L_2(256)$	257
A_8	$3^2 \cdot 5 \cdot 7 = 315$	$L_2(512)$	$3^3 \cdot 19 = 513$
$L_2(7)$	$3 \cdot 7 = 21$	$L_3(3)$	$3^3 \cdot 13 = 351$
$L_2(8)$	$3^2 = 9$	$U_3(3)$	$3^3 \cdot 7 = 189$
$L_2(11)$	$5 \cdot 11 = 55$	$L_2(25)$	$3 \cdot 5^2 \cdot 13 = 975$
$L_2(13)$	$7 \cdot 13 = 91$	$L_3(4)$	$3 \cdot 5 \cdot 7 = 105$
$L_2(16)$	17	$L_3(8)$	$3^2 \cdot 73 = 657$
$L_2(17)$	$3^2 \cdot 17 = 153$	$U_3(4)$	$5 \cdot 13 = 65$
$L_2(19)$	$3 \cdot 5 \cdot 19 = 285$	$U_3(8)$	$3^3 \cdot 19 = 513$
$L_2(23)$	$3 \cdot 11 \cdot 23 = 759$	$U_4(2)$	$3^3 \cdot 5 = 135$
$L_2(27)$	$3^2 \cdot 7 \cdot 13 = 819$	$PSp_4(4)$	$5^2 \cdot 17 = 425$
$L_2(31)$	$3 \cdot 5 \cdot 31 = 465$	$\Omega_5(3)$	$3^4 \cdot 5 = 405$

3. Proof of Theorem 1.1

Recall the previous definitions of $S(G)$ and $\delta_0(G)$, $S(G) = \{p \in \pi(G) \colon n_p(G) > 1\}$ and $\delta_0(G) = \sum_{p \in S(G)} n_p(G)$. Next, we prove the finite non-abelian simple groups G that satisfy $\delta_0(G) \leq 1000$. First, we cite a lemma (see Lemma 2.1 of [7]).

Lemma 3.1. *[7, Lemma 2.1] Let G be a finite group and N be a normal subgroup of G. Then the product $n_p(N) \cdot n_p(G/N)$ divides $n_p(G)$.*

By Lemma 3.1, we have $n_p(G/N) \mid n_p(G)$, and then $n_p(G/N) \leq n_p(G)$. Moreover, by the definitions of $S(G)$ and $\delta_0(G)$, we can obtain $S(G/N) \subseteq S(G)$, which implies $\delta_0(G/N) \leq \delta_0(G)$. From Lemma 2.3 and Table 3, we know all finite non-abelian simple groups with $n_2(G) \leq 1000$. By using GAP, the finite non-abelian simple groups satisfying $\delta_0(G) \leq 1000$ are determined and listed in Table 4.

Table 4. Non-Abelian simple groups with $\delta_0(G) \leq 1000$

Non-Abelian simple group	$\delta_0(G)$
A_5	$5+6+10=21$
A_6	$45+10+36=91$
A_7	$315+70+126+120=631$
$PSL(2,7)$	$21+28+8=57$
$PSL(2,8)$	$9+28+36=73$
$PSL(2,11)$	$55+55+66+12=188$
$PSL(2,13)$	$91+91+78+14=274$
$PSL(2,17)$	$153+136+18=307$
$PSL(2,19)$	$285+190+171+20=666$
$PSL(2,16)$	$17+136+136+120=409$
$PSL(3,3)$	$351+52+144=547$
$PSU(3,3)$	$189+28+288=505$

Among all isomorphic non-abelian simple groups, such as $A_5 \cong PSL(2,4) \cong PSL(2,5) \cong PSU(2,4) \cong PSU(2,5)$, $A_8 \cong PSL(2,8)$, we will consider only one of them.

Lemma 3.2. *Let H be a subgroup of G. Then $n_p(H) \leq n_p(G)$.*

Let N be the maximal normal solvable subgroup of G. Let K be a normal subgroup of G containing N and let K/N be the direct product of all minimal normal groups of G/N. Assume that $K_1/N, \cdots, K_t/N$ are all minimal normal subgroups of G/N. Then $K/N \cong K_1/N \times \cdots \times K_t/N$. Set $K_i/N \cong S_i^{l_i}$ is a direct product of non-abelian simple groups. By Lemma 3.1 and Lemma 3.2, we have $\delta_0(K/N) \leq \delta_0(G)$. So $\delta_0(K/N) \leq 1000$, that is,

$$\delta_0(S_1^{l_1} \times \cdots \times S_t^{l_t}) \leq 1000. \tag{$*1$}$$

Note that $\delta_0(S_i) \leq 1000$. Hence S_i is isomorphic to the finite non-abelian simple groups in Table 4. We check (*1) and lead K/N as follows, except for the case of simple groups. Otherwise, K/N is isomorphic to the direct product of the finite non-abelian simple groups in Table 5.

Table 5. K/N is not a simple group and $\delta_0(K/N) \leq 1000$.

Non-abelian simple group	$\delta_0(G)$
$A_5 \times A_5$	$25+100+36=161$
$A_5 \times A_6$	$225+100+216=541$
$A_5 \times PSL(2,7)$	$105+280+6+8=399$
$A_5 \times PSL(2,8)$	$45+280+6+36=367$

By the above arguments, if $\delta_0(G) \leq 1000$, then there are 16 possibilities for K/N and $t \leq 2$. If $t=1$, then G/N has the unique minimal normal subgroup K_1/N, and so K_1/N is a simple group in Table 4 or $A_5 \times A_5$. It follows that $S_1 \leq G/N \leq Aut(S_1)$ with S_1 in Table 4 or $A_5 \times A_5 \leq G/N \leq Aut(A_5 \times A_5)$. If $t=2$, then $K_1/N \cong S_1 \cong A_5$ and $K_2/N \cong S_2 \cong A_5$, A_6,

$PSL(2,7)$ or $PSL(2,8)$. So $S_1 \times S_2 \leq G/N \leq Aut(S_1) \times Aut(S_2)$ except for $S_1 \times S_2 \cong A_5 \times A_5$. We compute by GAP or check by ATLAS to get all possibilitis of G/N as follows:

Table 6. The groups G/N and $\delta_0(G/N)$

Group	$\delta_0(G)$	Group	$\delta_0(G)$	Group	$\delta_0(G)$
A_5	21	S_5	31	A_6	91
$PGL(2,9)$	91	$P\Gamma L(2,9)$	91	$PSL(2,7)$	57
$PGL(2,7)$	57	$PSL(2,8)$	73	$P\Gamma L(2,8)$	73
$PSL(2,11)$	188	$PGL(2,11)$	298	$PSL(2,13)$	274
$PGL(2,13)$	456	$PSL(2,17)$	307	$PGL(2,17)$	307
A_7	631	S_7	631	$PSL(2,19)$	666
$PSL(2,16)$	409	$P\Sigma L(2,16)$	477	$P\Gamma L(2,16)$	647
$PSL(3,3)$	547	$PSL3,3) \rtimes_\phi C_2$	547	$PSU(3,3)$	505
$PSU(3,3) \rtimes_\varphi C_2$	505	$A_5 \times A_5$	161	$A_5 \wr C_2$	211
$A_5 \times S_5$	211	$S_5 \times S_5$	361	$S_5 \wr C_2$	361
$A_5 \times A_6$	541	$A_5 \times PGL(2,9)$	541	$A_5 \times P\Gamma L(2,9)$	541
$S_5 \times A_6$	991	$S_5 \times PGL(2,9)$	991	$S_5 \times P\Gamma L(2,9)$	991
$A_5 \times PSL(2,7)$	399	$S_5 \times PSL(2,7)$	609	$A_5 \times PGL(2,7)$	399
$S_5 \times PGL(2,7)$	609	$A_5 \times PSL(2,8)$	367	$A_5 \times P\Gamma L(2,8)$	367
$S_5 \times PSL(2,8)$	457	$S_5 \times P\Gamma L(2,8)$	457		

References

[1] X. Gao, I. Lima, R. Shen, On small Sylow numbers of finite groups, Indian J. Pure Appl. Math. 54 (2023), 485–498.

[2] C.S. Anabanti, A. Moretó, M. Zarrin, Influence of the number of Sylow subgroups on solvability of finite groups, C. R. Math. Acad. Sci. Paris 358 (2020), 1227–1230.

[3] Z. Wu, On the number of Sylow subgroups in finite simple groups, J. Algebra Appl. 20 (2021), 2150115.

[4] J.P. Zhang, Sylow numbers of finite groups, J. Algebra 176 (1995), 111–123.

[5] A. Khalili Asboei, S.S. Salehi Amiri, On the average number of Sylow subgroups in finite groups, Czechoslovak Math. J. 72 (2022), 747–750.

[6] M. Hall, On the number of Sylow subgroups in a finite group, J. Algebra 7 (1967), 363–371.

[7] Y. Liu, J. Zhang, Some results on Sylow numbers of finite groups, Czechoslovak Math. J. 74 (2024), 1083–1095.

[8] J. Lu, W. Meng, A. Moretó, Notes on the average number of Sylow subgroups of finite groups, Czechoslovak Math. J. (2021), 1–4.

[9] A. Moretó, The average number of Sylow subgroups of a finite group, Math. Nachr. 287 (2014), 1183–1185.

[10] R. Steinberg, J.H. Conway, R.T. Curtis, et al., Atlas of finite groups: Maximal subgroups and ordinary characters for simple groups, Math. Comp. 48 (2014), 123–132.

[11] A.S. Kondrat'ev, Normalizers of the Sylow 2-subgroups in finite simple groups, Math. Notes 78 (2005).

[12] The GAP Group, Algorithms, and Programming, Version 4.4 (2005).