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STABILITY RESULTS OF POSITIVE SOLUTIONS FOR (p, q)-LAPLACIAN SYSTEM WITH APPLICATIONS

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ABSTRACT. In the present article, we investigate the stability results concerning the weak solutions of the following (p, q)-Laplacian system

$$\left.\begin{array}{ll} -\Delta_p u + \lambda_p |u|^{p-2}u = a(x)f(u,v) & \text{in } \Omega, \\ -\Delta_q v + \lambda_q |v|^{q-2}v = b(x)g(u,v) & \text{in } \Omega, \\ \Sigma u = 0 = \Sigma v & \text{on } \partial\Omega. \end{array}\right\}$$

where $\Delta_p u \equiv div[|\nabla u|^{p-2}\nabla u]$, with $p > 1$, denotes the p-Laplacian operator. Here λ_p, λ_q are positive parameters, $a(x), b(x)$ are continuous functions from Ω to $\mathbf{R}$ and $f, g : [0, \infty) \times [0, \infty) \to \mathbf{R}$ are C^1 functions. $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\Sigma u = \rho l(x)u + (1 - \rho)\frac{\partial u}{\partial n}$ where $\rho \in [0, 1]$, $l : \partial\Omega \to \mathbf{R}^+$ where $l = 1$ when $\rho = 1$. Under certain conditions, we establish that every positive weak solution is either stable or unstable.

1. INTRODUCTION

In the last few years, a great deal of attention has been focused on the study of the stability properties of weak solutions for linear [6], semilinear (see [13, 14, 16]), semipositone (see [4, 5]), nonlinear (see [2, 8–12]) and fractional (see [7]) systems, The practical importance of these systems is evident in reaction-diffusion problems and Newtonian fluids, among others; see [3] and references therein.

In [15], the author hase proved some stability theorems concerning the positive solutions of the following semilinear system

$$-\Delta u = \lambda f(u) \text{ in } \Omega, \ \Sigma u = 0 \text{ on } \partial\Omega, \tag{1}$$

for different values of the function f.

Khafagy in [8] have been studied the stability and instability for the nonlinear problem

$$\left.\begin{array}{ll} -\Delta_{P,p} u + a(x)|u|^{p-2}u = \lambda b(x)u^{\alpha} & \text{in } \Omega, \\ \Sigma u = 0 & \text{on } \partial\Omega. \end{array}\right\} \tag{2}$$

where $\Delta_{P,p} u \equiv div[P(x)|\nabla u|^{p-2}\nabla u]$ with $p > 1$, is the weighted p-Laplacian, $a(x)$ and $P(x)$ are weight functions, $b(x) : \Omega \to \mathbf{R}$ is a continuous function satisfing either $b(x) > 0$ or $b(x) < 0$ $\forall\ x \in \Omega$, λ is a positive parameter, $0 < \alpha < p - 1$ and $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary. The boundary condition is given by $\Sigma u = \rho l(x)u + (1 - \rho)\frac{\partial u}{\partial n}$ where $\rho \in [0, 1]$

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and $l : \partial\Omega \to \mathbf{R}^+$ with $l = 1$ when $\rho = 1$. The author proved that if $0 < \alpha < p - 1$ and $b(x) > 0 (< 0)\ \forall\ x \in \Omega$, then every positive weak solutions u of the problem (2) is linearly stable (unstable) respectively.

In the present paper, we are concerned with the study of the stability results of positive weak solutions for the following (p, q)-Laplacian system

$$\left.\begin{array}{lr} -\Delta_p u + \lambda_p |u|^{p-2}u = a(x)f(u,v) & \text{in } \Omega, \\ -\Delta_q v + \lambda_q |v|^{q-2}v = b(x)g(u,v) & \text{in } \Omega, \\ \Sigma u = 0 = \Sigma v & \text{on } \partial\Omega. \end{array}\right\} \tag{3}$$

where $\Delta_p u \equiv div[|\nabla u|^{p-2}\nabla u]$, with $p > 1$, denotes the p-Laplacian operator, λ_p, λ_q are positive parameters, $a(x)$ and $b(x) : \Omega \to R$ are continuous functions satisfing either $a(x), b(x) > 0$ or $a(x), b(x) < 0$ for all $x \in \Omega$. f and $g : [0,\infty)\times[0,\infty) \to \mathbf{R}$ are C^1 functions, and $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\Sigma u = \rho l(x)u + (1-\rho)\frac{\partial u}{\partial n}$ where $\rho \in [0, 1]$, $l : \partial\Omega \to \mathbf{R}^+$ with $l = 1$ when $\rho = 1$. The condition $\Sigma u = \rho l(x)u + (1-\rho)\frac{\partial u}{\partial n} = 0$ is the Dirichlet condition when $\rho = 1$, the Neumann condition when $\rho = 0$ or the mixed condition for other values of ρ. The so-called p-Laplacian boundary value problems arise in a variety of physical phenomena, such: flow through porous media, reaction-diffusion problems, petroleum extraction, non-Newtonian fluids, etc. So, the study of such problems and their generalizations hase attracted several mathematicians in recent years.

Finally, the plan of our paper is as follows. In the next section, we establish the stability results of the positive weak solutions of system (3). Section 3 is devoted to some applications regarding the stability results for system (3).

As we know, the linearized equation of system (3) about (u, v) is given by

$$\left.\begin{array}{c} -(p-1)[div[|\nabla u|^{p-2}\nabla w] - \lambda_p|u|^{p-2}w] - a(x)f_u(u,v)w - a(x)f_v(u,v)z = \mu w, \ \text{in } \Omega, \\ -(q-1)[div[|\nabla v|^{q-2}\nabla z] - \lambda_q|v|^{q-2}z] - b(x)g_u(u,v)w - b(x)g_v(u,v)z = \mu z, \ \text{in } \Omega, \\ \Sigma w = 0 = \Sigma z, \quad \text{on } \partial\Omega, \end{array}\right\} \tag{4}$$

for any positive weak solution (u, v) of system (3), where μ is the eigenvalue corresponding to the eigenfunction (ϕ, ψ) and $f_u(u, v)$ is the partial derivative of f with respect to u.

Definition 1.1. *We call a solution (u, v) of (3) a linearly stable solution if all eigenvalues of (4) are strictly positive, which can be implied if the principal eigenvalue $\mu_1 > 0$. Otherwise (u, v) is linearly unstable.*

2. Main results

In this section, we assume the following hypotheses:

- (C_1) The function $\dfrac{f(u,v)}{u^{p-1}}$ is a strictly increasing (decreasing) function with respect to the variable u, i.e., $uf_u(u, v) - (p-1)f(u, v) > 0 (< 0)$.
- (C_2) The function $\dfrac{g(u,v)}{v^{q-1}}$ is a strictly increasing (decreasing) function with respect to the variable v, i.e., $vg_v(u, v) - (q-1)g(u, v) > 0 (< 0)$.
- (C_3) uf_v and vg_u have the same sign, i.e., uf_v, $vg_u > 0\ (< 0)$.
- (C_4) $a(x)$ and $b(x)$ have the same sign, i.e., $a(x)$, $b(x) > 0\ (< 0)$.

Under these hypotheses, the stability results of the positive weak solutions (u, v) for system (3) will be discussed in the following theorems.

Theorem 2.1. *If* $\frac{f(u,v)}{u^{p-1}}$ *and* $g(u,v)/v^{q-1}$ *are strictly increasing,* uf_v, $vg_u > 0$ *and* $a(x), b(x) > 0$*, then every positive weak solution* (u,v) *of system (3) is linearly unstable.*

Proof. Let (u_0, v_0) be any positive weak solution of (3), then the linearized equation about (u_0, v_0) is

$$\left.\begin{array}{c} -(p-1)[div[|\nabla u_0|^{p-2}\nabla w] - \lambda_p |u_0|^{p-2}w] - a(x)[f_u w + f_v z] = \mu w, \text{ in } \Omega, \\ -(q-1)[div[|\nabla v_0|^{q-2}\nabla z] - \lambda_q |v_0|^{q-2}z] - b(x)[g_u w + g_v z] = \mu z, \text{ in } \Omega, \\ \Sigma w = 0 = \Sigma z, \qquad \text{on } \partial\Omega. \end{array}\right\} \tag{5}$$

Let μ_1 be the first eigenvalue of (5), and let (ϕ, ψ), with $\phi, \psi \geq 0$, be the corresponding eigenfunctions. Multiplying the first equation of (3) by $(p-1)\phi$ and integrating over Ω, we have

$$(p-1)[\int_\Omega div[|\nabla u_0|^{p-2}\nabla u_0]\phi dx - \lambda_p \int_\Omega |u_0|^{p-2}u_0\phi dx + \int_\Omega a(x)f(u_0, v_0)\phi dx] = 0. \tag{6}$$

Also, multiplying the second equation of (3) by $(q-1)\psi$ and integrating over Ω, we obtain

$$(q-1)[\int_\Omega div[|\nabla v_0|^{q-2}\nabla v_0]\psi dx - \lambda_q \int_\Omega |v_0|^{q-2}v_0\psi dx + \int_\Omega b(x)g(u_0, v_0)\psi dx] = 0. \tag{7}$$

On the other hand, multiplying the first equation of (5) by u_0 and integrating over Ω, we derive

$$\begin{aligned} \mu_1 \int_\Omega u_0\phi dx = &-(p-1)[\int_\Omega div[|\nabla u_0|^{p-2}\nabla \phi]u_0 - \lambda_p \int_\Omega |u_0|^{p-2}u_0\phi]dx \\ &- \int_\Omega a(x)f_u(u_0, v_0)u_0\phi dx - \int_\Omega a(x)f_v(u_0, v_0)u_0\psi dx. \end{aligned} \tag{8}$$

Also, multiplying the second equation of (5) by v_0 and integrating over Ω, we obtain

$$\begin{aligned} \mu_1 \int_\Omega v_0\psi dx = &-(q-1)[\int_\Omega div[|\nabla v_0|^{q-2}\nabla \psi]v_0 - \lambda_q \int_\Omega |v_0|^{q-2}v_0\psi]dx \\ &- \int_\Omega b(x)g_u(u_0, v_0)v_0\phi dx - \int_\Omega b(x)g_v(u_0, v_0)v_0\psi dx. \end{aligned} \tag{9}$$

Now, (6) − (8) give

$$\begin{aligned} -\mu_1 \int_\Omega u_0\phi dx = &(p-1)[\int_\Omega div[|\nabla u_0|^{p-2}\nabla \phi]u_0 - \int_\Omega div[|\nabla u_0|^{p-2}\nabla u_0]\phi]dx \\ &+ \int_\Omega a(x)(u_0 f_u - (p-1)f)\phi dx + \int_\Omega a(x)u_0 f_v \psi dx. \end{aligned} \tag{10}$$

By applying Green's first identity to the first term of the right-hand side of (10), we have get

$$\int_\Omega [div[|\nabla u_0|^{p-2}\nabla \phi]u_0 - div[|\nabla u_0|^{p-2}\nabla u_0]\phi]dx = \int_{\partial\Omega} |\nabla u_0|^{p-2}[u_0\frac{\partial \phi}{\partial n} - \phi\frac{\partial u_0}{\partial n}]ds. \tag{11}$$

Inserting (11) in (10), we obtain

$$-\mu_1 \int_\Omega u_0\phi dx = (p-1)\int_{\partial\Omega} |\nabla u_0|^{p-2}[u_0\frac{\partial \phi}{\partial n} - \phi\frac{\partial u_0}{\partial n}]ds$$
$$+\int_\Omega a(x)(u_0 f_u - (p-1)f)\phi dx + \int_\Omega a(x)u_0 f_v \psi dx. \tag{12}$$

Similarly, (7) − (9) provide

$$-\mu_1 \int_\Omega v_0\psi dx = (q-1)[\int_\Omega div[|\nabla v_0|^{q-2}\nabla\psi]v_0 dx - \int_\Omega div[|\nabla v_0|^{q-2}\nabla v_0]\psi dx]$$
$$+\int_\Omega b(x)(v_0 g_v - (q-1)g)\psi dx + \int_\Omega b(x)v_0 g_u \phi dx. \tag{13}$$

Additionally, using Green's first identity on the first term of the right-hand side of (13), implies

$$\int_\Omega [div[|\nabla v_0|^{q-2}\nabla\psi]v_0 - div[|\nabla v_0|^{q-2}\nabla v_0]\psi]dx = \int_{\partial\Omega} |\nabla v_0|^{q-2}[v_0\frac{\partial\psi}{\partial n} - \psi\frac{\partial v_0}{\partial n}]ds. \tag{14}$$

Next, by applying (14) to (13), we get

$$-\mu_1 \int_\Omega v_0\psi dx = (q-1)\int_{\partial\Omega} |\nabla v_0|^{q-2}[v_0\frac{\partial \psi}{\partial n} - \psi\frac{\partial v_0}{\partial n}]ds$$
$$+\int_\Omega b(x)(v_0 g_v - (q-1)g)\psi dx + \int_\Omega b(x)v_0 g_u \phi dx. \tag{15}$$

Adding (12) and (15), one hase

$$-\mu_1 \int_\Omega [u_0\phi + v_0\psi]dx = (p-1)\int_{\partial\Omega} |\nabla u_0|^{p-2}[u_0\frac{\partial \phi}{\partial n} - \phi\frac{\partial u_0}{\partial n}]ds$$
$$+(q-1)\int_{\partial\Omega} |\nabla v_0|^{q-2}[v_0\frac{\partial \psi}{\partial n} - \psi\frac{\partial v_0}{\partial n}]ds$$
$$+\int_\Omega a(x)(u_0 f_u - (p-1)f)\phi dx + \int_\Omega b(x)(v_0 g_v - (q-1)g)\psi dx$$
$$+\int_\Omega a(x)u_0 f_v\psi dx + \int_\Omega b(x)v_0 g_u\phi dx, \tag{16}$$

Now, when $\rho = 1$, we have $\Sigma u_0 = u_0 = 0$ and $\Sigma v_0 = v_0 = 0$ for $s \in \partial\Omega$. Additionally, $\phi = \psi = 0$ for $s \in \partial\Omega$. Then

$$\int_{\partial\Omega} |\nabla u_0|^{p-2}[u_0\frac{\partial\phi(s)}{\partial n} - \phi\frac{\partial u_0(s)}{\partial n}]ds = 0 \quad \text{and} \quad \int_{\partial\Omega} |\nabla v_0|^{q-2}[v_0\frac{\partial\psi(s)}{\partial n} - \psi\frac{\partial v_0(s)}{\partial n}]ds = 0. \tag{17}$$

Also, when $\rho \neq 1$, we have

$$\frac{\partial u_0}{\partial n} = -\frac{\rho l u_0}{1-\rho}, \quad \frac{\partial \phi}{\partial n} = -\frac{\rho l \phi}{1-\rho},$$

and

$$\frac{\partial v_0}{\partial n} = -\frac{\rho l v_0}{1-\rho}, \quad \frac{\partial \psi}{\partial n} = -\frac{\rho l \psi}{1-\rho},$$

and thus we obtain again the result given by (17).

Consequently, (16) becomes

$$-\mu_1 \int_\Omega [u_0\phi + v_0\psi]dx = \int_\Omega a(x)(u_0 f_u - (p-1)f)\phi dx + \int_\Omega b(x)(v_0 g_v - (q-1)g)\psi dx$$
$$+ \int_\Omega a(x)u_0 f_v \psi dx + \int_\Omega b(x) v_0 g_u \phi dx. \tag{18}$$

Since $\dfrac{f(u,v)}{u^{p-1}}$ and $\dfrac{g(u,v)}{v^{q-1}}$ are strictly increasing, we can deduce from assumption C_1 that

$$[u_0 f_u(u_0,v_0) - (p-1)f(u_0,v_0)] > 0 \quad \text{and} \quad [v_0 g_v(u_0,v_0) - (q-1)g(u_0,v_0)] > 0 \tag{19}$$

Since (19) and the facts that $uf_v > 0$, $vg_u > 0$, $a(x) > 0$ and $b(x) > 0$ for all $x \in \Omega$, then equation (18) becomes

$$-\mu_1 \int_\Omega [u_0\phi + v_0\psi]dx > 0.$$

The result follows if $\mu_1 < 0$.

Theorem 2.2. *If* $\dfrac{f(u,v)}{u^{p-1}}$ *and* $\dfrac{g(u,v)}{v^{q-1}}$ *are strictly increasing,* uf_v, $vg_u > 0$, *and* $a(x), b(x) < 0$ *forall* $x \in \Omega$, *then every weak solutions* (u,v) *for system (3) is linearly stable.*

Proof. Following the approach used in the proof of Theorem 2.1, we have

$$-\mu_1 \int_\Omega [u_0\phi + v_0\psi]dx < 0,$$

which implies that $\mu_1 > 0$ and hence the result follows.

Theorem 2.3. *If* $\dfrac{f(u,v)}{u^{p-1}}$ *and* $\dfrac{g(u,v)}{v^{q-1}}$ *are strictly decreasing,* $uf_v, vg_u < 0$, *and* $a(x), b(x) < 0$ *for all* $x \in \Omega$, *then every weak solution* (u,v) *for system (3) is linearly unstable.*

Proof. The proof of this theorem is similar to that of Theorem 2.1 and Theorem 2.2. We obtain

$$-\mu_1 \int_\Omega [u_0\phi + v_0\psi]dx > 0,$$

and since $\mu_1 < 0$, the result follows.

Theorem 2.4. *If* $\dfrac{f(u,v)}{u^{p-1}}$ *and* $\dfrac{g(u,v)}{v^{q-1}}$ *are strictly decreasing,* uf_v, $vg_u < 0$, *and* $a(x), b(x) > 0$ *for all* $x \in \Omega$, *then every weak solution* (u,v) *for system (3) is linearly stable.*

Proof. The proof of this theorem proceeds in the same way as for the previous Theorems and we can easily obtain that

$$-\mu_1 \int_\Omega [u_0\phi + v_0\psi]dx < 0.$$

Since $\mu_1 > 0$, the result follows.

Remark 2.5. *For the system (3), when* $f(u,v) = u^\beta v^\gamma$, $g(u,v) = u^r v^\delta$, *and* $a(x) = b(x) = \lambda$ *where* $\lambda, \beta, \gamma, \delta$ *and* r *are positive constants with* $\beta > p-1$ *and* $\delta > q-1$, *we have some results in [1].*

3. Applications

Here we introduce some examples to demonstrate the effectiveness of our results.

Example 1. Consider the reaction-diffusion system with unequal diffusion coefficients involving the Laplacian

$$\left.\begin{array}{rl} -\Delta u = ah(u,v) & \text{in } \Omega, \\ -\Delta v = bk(u,v) & \text{in } \Omega, \\ \Sigma u = 0 = \Sigma v & \text{on } \partial\Omega, \end{array}\right\} \tag{20}$$

where a and b are positive constants, h and k are strictly increasing (decreasing) functions with $uh_v,\ vk_u > 0 (< 0)$. Hence, according to Theorem 2.1 and Theorem 2.4 with $p = q = 2$ and $\lambda_p = \lambda_q = 0$, we have

(i) If h and k are strictly increasing functions and $uh_v,\ vk_u > 0$, then every weak solution (u, v) of system (20) is linearly unstable.

(i) If h and k are strictly decreasing functions and $uh_v,\ vk_u < 0$, then every weak solution (u, v) of system (20) is linearly stable.

Example 2. Consider the reaction-diffusion system with unequal diffusion coefficients involving the $(p, q)-$Laplacian

$$\left.\begin{array}{rl} -\Delta_p u = a\ u^\alpha v^\beta & \text{in } \Omega, \\ -\Delta_q v = b\ u^\gamma v^\delta & \text{in } \Omega, \\ \Sigma u = 0 = \Sigma v & \text{on } \partial\Omega. \end{array}\right\} \tag{21}$$

where $a, b, \alpha, \beta, \gamma$ and δ are positive constants, $\alpha > p - 1$ and $\delta > q - 1$. Hence, according to Theorem 2.1 with $\lambda_p = \lambda_q = 0$, every weak solution (u, v) of system (21) is linearly unstable.

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