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MALLIAVIN CALCULUS AND BOOTSTRAP METHODS FOR STOCHASTIC VOLATILITY MODELS

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Abstract. As a forward problem, in this paper we review some aspects of the method of Malliavin calculus, also known as the stochastic calculus of variations, for the Monte Carlo estimation of the sensitivity parameters (Greeks) of financial models. This helps in pricing and hedging of derivative securities. As an inverse problem, we review bootstrap methods for estimation and testing in continuous time stochastic volatility models based on discrete observations. We put special emphasis on jumps and long memory in the volatility process.

1. Introduction

The paper is concerned with the study of statistics, econometrics and financial engineering of high frequency financial data. The development of increasingly complex financial products requires the use of advanced statistical methods. The purpose of the paper is to present generalized bootstrap methods for estimation, calibration and Malliavin calculus methods for pricing, hedging of derivative products (on equities, interest rate, credit risk), and portfolio optimization. Special attention is paid to models in high dimension, models with jumps, models with long-memory in stochastic volatility models.

Since Black and Scholes (1973) established the theory of option pricing which won Scholes a Nobel prize, volatility has played an important role not only in derivative pricing, but also in portfolio selection and risk management. Despite the assumption of constant volatility in Black and Scholes (1973), it is widely recognized that volatility changes over time. The *stylized facts* of volatility are: 1) volatility changes in time, 2) volatility is random, 3) volatility has heavy tails, 4) volatility clusters on high level. Starting with Engle (1982)'s autoregressive conditional heteroskedasticity (ARCH) model which won him a Nobel prize, various stochastic volatility models have been proposed. On the other hand, volatility is often modeled as a parametrized diffusion coefficient of a continuous time diffusion process and then the parameters are estimated by various methods like filtered maximum likelihood or method of moments. One should also focus on nonparametric estimation of volatility process. In principle, the more data we can

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use, the more accurate the estimate will be. However, one will have technological restriction on the amount of data. Recently this kind of restriction have been removed by development of computer power and data recording system. Those kind of data are called *high-frequency data*. Such high frequency data lend the validity of the method based on quadratic variation formula, that is called the realized volatility. Barndorff-Nielsen and Shephard (2001-2007) have suggested modeling the volatility as a Levy driven Ornstein-Uhlenbeck process.

In the stochastic volatility model, the log-price $y^* = \log S$ with S being the asset price, follows

$$dy^*(t) = \alpha(t)dt + \sigma(t)dW(t),$$
$$d\sigma^2(t) = \widetilde{\alpha}(t)dt + \widetilde{\sigma}(t)d\widetilde{W}(t)$$

where σ and α adapted processes, and the standard Brownian motions W and $\widetilde{W}$ are allowed to be correlated with correlation $\rho < 0$ known as the leverage effect.. The process σ is called the *instantaneous volatility* or *spot volatility* and α is called the *mean process*. A simple example of this is

$$\alpha(t) = \mu t + \beta\sigma^{2*}(t) \quad \text{where} \quad \sigma^{2*}(t) = \int_0^t \sigma^2(u)du$$

in which case β is called the *risk premium* and σ^{2*} is called the *integrated variance*.

Over an interval of time length $h > 0$, returns are defined as

$$y_i := y^*(hi) - y^*((i-1)h), \quad i = 1, 2, \cdots, T$$

which implies that

$$y_i|\alpha_i, \sigma_i^2 \ \sim \ \mathcal{N}(\alpha_i, \sigma_i^2)$$

where

$$\alpha_i := \alpha(ih) - \alpha((i-1))h)$$

and

$$\sigma_i^2 := \sigma^{2*}(ih) - \sigma^{2*}((i-1))h) = \int_{(i-1)h}^{ih} \sigma^2(u)du.$$

Here σ_i^2 is called the *actual variance* and α_i is called the *actual mean*. Suppose one is interested in estimating the actual volatility σ_i using M intra-h observations. A natural candidate is the *realized volatility* given by

$$\sqrt{[y_M^*]} := \sqrt{\sum_{i=1}^{M} y_{j,i}^2}$$

where

$$y_{j,i} := y^*\left((i-1)h + \frac{jh}{M}\right) - y^*\left((i-1)h + \frac{(j-1)h}{M}\right), \quad j = 1, 2, \cdots, M.$$

When $h \to 0$, realized volatility converges in L_2 to the integrated volatility. We consider the fixed h case. One could also obtain large deviations and moderate deviations results.

First one should obtain Berry-Esseen type bound on

$$\sup_{x\in\mathbb{R}} \left| P\left(\frac{\sqrt{\frac{M}{h}}([y_M^*] - \sigma_i^2)}{\sqrt{2\sigma_i^{[4]}}} \leq x\right) - \Phi(x)\right| \quad \text{where} \quad \sigma_i^{[4]} := \int_{(i-1)h}^{ih} \sigma^4(u)du$$

is the *actual quarticity* and σ^4 is the *spot quarticity*.

One could use the decomposition technique (Michel and Pfanzagl (1971) developed for the classical i.i.d. case): Decompose the rate of convergence of the numerator to normal distribution and rate of convergence of the denominator to a constant. Note that

$$y_{j,i} = \alpha_j + \epsilon_j$$

where by Itô formula

$$\alpha_j := \int_{(i-1)h+\frac{(j-1)h}{M}}^{(i-1)h+\frac{jh}{M}} \alpha(u)du, \quad \epsilon_j := \int_{(i-1)h+\frac{(j-1)h}{M}}^{(i-1)h+\frac{jh}{M}} \sigma(u)dW(u).$$

Hence

$$y_{j,i}^2 \;=\; \alpha_j^2 + 2\alpha_j\epsilon_j + \epsilon_j^2 = \sigma_j^2 + \alpha_j^2 + 2\alpha_j\epsilon_j + (\epsilon_j^2 - \sigma_j^2) = \sigma_j^2 + e_j$$

where

$$e_j := \alpha_j^2 + 2\alpha_j\epsilon_j + 2\int_{(i-1)h+\frac{(j-1)h}{M}}^{(i-1)h+\frac{jh}{M}} \left(\int_{(i-1)h+\frac{(j-1)h}{M}}^{u} \sigma(s)dW_s\right)\sigma(u)dW_u.$$

Thus

$$[y_M^*] - \sigma_i^2 = \sum_{j=1}^{M} e_j := e_M^*(h).$$

The term e_j is called the *realized volatility error*. Since $E(e_j|\sigma_j^2) = 0$, realized volatility is an unbiased estimator of actual volatility. When $M \to \infty$, $y_M^* \to \sigma_i^2$ almost surely, so it is strongly consistent estimator.

$$\lim_{h\to 0} \frac{E(e_M^*(h))}{\sqrt{h}} = 0$$

since

$$e_M^*(h) = \sum_{j=1}^{M} e_j = \sum_{j=1}^{M} \sigma_j^2(v_j^2 - 1)$$

where $v_j \sim^{iid} N(0,1)$ and independent of σ_j^2. It is clear that e_j is a weak white noise sequence which is uncorrelated with the actual volatility process σ_j^2.

Thus

$$\frac{\sqrt{\frac{M}{h}}([y_M^*] - \sigma_i^2)}{\sqrt{2\sigma_i^{[4]}}} = \frac{\sqrt{\frac{M}{h}}\sum_{j=1}^{M}\sigma_j^2(v_j^2-1)}{\sqrt{2\sigma_i^{[4]}}}.$$

One can apply the characteristic function technique followed by Esseen's lemma for the numerator. Then one can apply the splitting technique of Michel and Pfanzagl (1971) or the more precise squeezing technique of Pfanzagl (1971). Then one should refine the result using the generalized bootstrap techniques.

We will emphasize on model calibration and inference. The implied volatility smile phenomenon has led to the appearance of a large variety of extensions of the Black-Scholes model: the local volatility models, diffusions with stochastic volatility, jump diffusions, and long memory models. The model calibration is the reconstruction of model parameters from the prices of traded options. It is an inverse problem to that of option pricing and as such, typically ill-posed.

We will also emphasize on interest rate models. The calibration problem is yet more complex in the interest rate markets since in this case the empirical data that can be used includes a wider variety of financial products from standard obligations to swaptions. Efficient computational algorithms are thus most needed.

A stochastic volatility model models both the underlier's value and its volatility as stochastic processes. As with a jump-diffusion model, this has the effect of giving the underlier's value a leptokurtic distribution. Heston's (1993) model is a popular stochastic volatility model.

A jump-diffusion model adds random jumps to the geometric Brownian motion that Black-Scholes (1973) assumes for the underlier. Among other things, this has the effect of giving the underlier's value a leptokurtic distribution.

It is well accepted that estimating financial volatility is a central issue in financial theory and practice. To estimate volatility, researchers have given attention to high frequency data. Unfortunately, high-frequency data is often contaminated my movements dictated by institutional factors of the market. Removing these institutionally driven factors, termed *microstructure effects*, is vital to accurately determine the price volatility. Thus high frequency data is contaminated by market microstructure noise. The available market microstructure noise models have used i.i.d. noise with constant variance to describe market microstructure effects. In continuous time, i.i.d. process is a process with extreme volatility and because of this, the observed price process is dominated by the noise process and carries no information on the volatility of the efficient price process. To overcome this problem, we propose fractional Brownian motion driving noise for market microstructure noise which carries long memory of volatility.

The main difficulty in modeling and forecasting financial volatility is the fact that the volatility is latent and not directly observable, like a *hidden Markov model.* Usually researchers have used realized volatility as an observable proxy for latent volatility. Realized volatility is defined as the summation of squared intra-day returns. It is a good estimate to estimate the object of interest, the integrated volatility of an asset, so is optimal to use all the available data. However, it is a common practice in measuring realized volatility to avoid the data at highest frequency available, the tick-by-tick data. Instead, researchers tend to sample intra daily returns at somewhat moderate frequencies such as 5, 10 or 30 minutes. This is due to the suspicion that the market microstructure effects become more severe as sampling frequency increases.

2. Stochastic Derivative and Elements of Malliavin Calculus

Let $(\Omega, \mathcal{F}, P)$ denote the Wiener space, i.e., $\Omega = C_0[0, T]$ is the Wiener space of continuous functions on $[0, T]$ with initial value 0 which is equipped with the supremum norm $\| \, . \, \|, \mathcal{F}$ the Borel σ-algebra of subsets of Ω and P is the standard Wiener measure on $(\Omega, \mathcal{F})$. We will denote by $\{\mathcal{F}_t, t \in [0, T]\}$ the filtration generated by Brownian motion and completed by the P-null sets. We will represent by $W(\omega, t) = \omega(t), 0 \leq t \leq T,\ \omega \in \Omega$ the Wiener process on the canonical probability space $(\Omega, \mathcal{F}, P)$.

On $(\Omega, \mathcal{F}, P)$ consider the homogeneous nonlinear Skorohod stochastic differential equation

$$\begin{aligned} dX(t) &= f(\theta, X(t))dt + g(\sigma, X(t))dW(t),\ \ 0 \leq t \leq T \\ X(0) &= \psi(X(T)) \end{aligned} \tag{2.1}$$

ψ is a known function, f is a known real valued function defined on $\mathbb{R}$ satisfying the existence and uniqueness of solution to (2.1), see Nualart (1995), $\theta \in \Theta \subset \mathbb{R}$ and $\sigma \in \Sigma \subset \mathbb{R}^+$ are the unknown parameters which are to be estimated. Let θ_0 and σ_0 be the true values of the unknown parameters. On the basis of n-independent copies $X_1(t), X_2(t), \dots, X_n(t)$ of $\{X(t), 0 \le t \le T\}$, maximum likelihood estimation was studied in Bishwal (2010), which uses the following Girsanov theorem for likelihood.

Anticipative Girsanov Theorem

The following is the nonadapted (anticipative) extension of the Girsanov theorem proved by Kusuoka (1982, Theorem 6.4). See also Theorem 4.1.2 in Nualart (1995).

Theorem 2.1 *Let $V : \Omega \to \Omega$ be a mapping of the form*

$$V(t, \omega) = \omega(t) + \int_0^t U(s, \omega) ds,$$

where U is a measurable mapping from Ω in to $H = L^2(0, T)$ and suppose that the following conditions are satisfied:
(i) V is bijective.
(ii) For all $\omega \in \Omega$, there exists a Hilbert-Schmidt operator $DU(\omega)$ from H into itself such that:
(a)

$$\|U(\omega + \int_0^\bullet h_s ds) - U(\omega) - DU(\omega)h\|_H = o(\|h\|_H)$$

for all $\omega \in \Omega$ as $\|h\|_H \to 0$,
(b) $h \to DU(\omega + \int_0^\bullet h_s ds)$ is continuous from H into $L^2([0, T]^2)$ the space of Hilbert-Schmidt operators for all ω,
(c) $I + DU(\omega) : H \to H$ is invertible.
Then if Q is the measure on $\Omega, \mathcal{F}$ such that $F = QV^{-1}, Q$ is absolutely continuous with respect to P and

$$\frac{dQ}{dP} = |d_c(-DU)| \exp\left(-\int_0^T U(t)dW_t - \frac{1}{2}\int_0^T U_t^2 dt\right)$$

where $d_c(-DU)$ denotes the Carleman-Fredholm determinant of the Hilbert-Schmidt operator $-DU$ and $\int_0^T U(t)dW(t)$ is the Skorohod integral.

We recall that the Carleman-Fredholm determinant of a Hilbert-Schmidt operator B from $L^2[0, T]$ into itself is defined by the product expansion

$$d_c(B) = \prod_{j=1}^{\infty} (1 - \lambda_j) \exp(\lambda_j)$$

where $\{\lambda_j, j \ge 1\}$ are the nonzero eigenvalues of B counted as many times as its multiplicities, see Simon (1979). In particular, if the operator B is nuclear, then

$$d_c(B) = \det(I - B) \exp\{\text{trace } B\}.$$

Thus if the operator DU is nuclear, then

$$d_c(-DU) = \det(I + DU) \exp\{\text{trace } (-DU)\}.$$

Maximum likelihood estimation of drift parameter of Skorohod stochastic differential equations was studied in Bishwal (2010) using anticipative Girsanov transformation.

Next we introduce some basic tools from Malliavin calculus. The idea of stochastic (or Malliavin) derivative is to define the notion of differentiability within the family of random variables that are equal to or can be approximated by functions of independent increments of Brownian motion. Under suitable assumptions, this family is wide enough to contain solution of stochastic differential equations. Let $C_b^\infty(\mathbb{R}^n)$ be the set of C^∞ functions $g:\mathbb{R}^n\to\mathbb{R}$ which are bounded and have bounded derivatives of all orders. The class of all real random variables of the form $F(\omega)=g(W(t_1),W(t_2),\dots,W(t_n)),\ g\in C_b^\infty(\mathbb{R}^n)$, called Wiener functionals, is denoted by $\mathcal{S}$. The space $\mathbb{D}^{1,p}$ designates the Banach space which is the completion of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p}=\{E|F|^p\}^{1/p}+\left(E\left[\left\{\int_0^T|D_sF|^2ds\right\}^{p/2}\right]\right)^{1/p},$$

where

$$D_sF:=\sum_{i=1}^n\frac{\partial g}{\partial x_i}(W(t_1),W(t_2),\dots,W(t_n))I_{[0,t_i]}(t).$$

More generally, the k-th order derivative of F is defined as the k-parameter process given by $D^{(k)}_{s_1\dots s_k}(F)=D_{s_1}D_{s_2}\dots D_{s_k}F$.

The space $\mathbb{D}^{m,p}$ is defined analogously and its associated norm is denoted by $\|.\|_{m,p}$. That is,

$$\|F\|_{m,p}=\{E|F|^p\}^{1/p}+\left(E\left[\left\{\int_0^T\dots\int_0^T|D^{(m)}_{s_1\dots s_m}F|^2ds_1\dots ds_m\right\}^{p/2}\right]\right)^{1/p}.$$

Wiener-Itô Chaos Expansion

The stochastic Sobolev space $\mathbb{D}^{1,2}$ consists of all $\mathcal{F}_T$- measurable random variables $F\in L^2(P)$ with chaos expansion

$$F=\sum_{n=0}^\infty I_n(f_n),\quad f_n\in L^2([0,T]^n)$$

satisfying the convergence criterion

$$\|F\|^2_{\mathbb{D}^{1,2}}:=\sum_{n=1}^\infty nn!\|f_n\|^2_{L^2([0,T]^n)}.$$

Let $F\in\mathbb{D}^{1,2}\subset L^2(P)$, that is square-integrable. Then the Malliavin derivative operator D_tF of F at time t defined as the expansion

$$D_tF=\sum_{n=1}^\infty nI_{n-1}(f_n(\cdot,t)),\ t\in[0,T].$$

Skorohod Integral

The space $\mathcal{S}$ is dense in $(\mathbb{D}^{1,p},\|.\|_{1,p}),\ p\ge1$. Set $\mathbb{D}^{1,\infty}=\cap_{p\ge1}\mathbb{D}^{1,p}$. The adjoint of the closed unbounded operator $D:L^2(\Omega)\to L^2([0,T]\times\Omega)$ is usually denoted by δ and is called the *Skorohod integral*. The domain of δ is the class of processes $u\in L^2([0,T]\times\Omega$ such that

$$\left|E\left(\int_0^T D_tFu_tdt\right)\right|\le C\|F\|_{L^2(\Omega)},$$

for all $F \in \mathcal{S}$. Here C is a constant which may depend on u. In the case that u is in the domain of δ, then $\delta(u)$ is the square integrable random variable defined by the *duality relation*

$$E(\delta(u)F) = E\left(\int_0^T D_t F u_t dt\right)$$

for all $F \in \mathbb{D}^{1,2}$. The Skorohod integral $\delta(u)$ turns out to be an extension of the classical Itô integral and it allows one to integrate processes u not necessarily adapted. For this reason, one also writes $\delta(u) = \int_0^T u(s)dW(s)$.

For any real number $p \geq 1$ and any integer $n \geq 1$ we set $\mathbb{L}^{n,p} = \mathbb{L}^p([0,T];\mathbb{D}^{n,p})$ and $\mathbb{L}^{1,\infty} = \cap_{p\geq 1}\mathbb{L}^{1,p}$. The processes u of the space $\mathbb{L}^{1,2}$ verify $uI_{[0,t]} \in$ Dom δ for each $t \in [0,T]$, and for those processes one can define the *indefinite Skorohod integral* $Y_t = \delta(uI_{[0,T]})$ denoted also by $\int_0^t u(s)dW(s)$. This process has continuous modification provided $u \in \mathbb{L}^{1,2}$ and $E(\int_0^T(\int_0^T |D_s u_r|^2 ds)^p dr) < \infty$ for some $p > 2$. The trajectory of this process is continuous but highly irregular. Imkeller (1993) showed that, one can represent the Skorohod integral as the composition of a Gaussian semimartingale depending on an infinite dimensional parameter with a Gaussian vector.

The following proposition is a trivial consequence of the fact that Itô integral and thus also iterated Itô integrals have zero expectation.

Proposition 2.1 For any $u \in Dom(\delta)$, the Skorohod integral has zero expectation, that is,

$$E(\delta(u)) = 0.$$

Duality Relation For every $X \in \mathcal{S}$ and $U \in Dom(\delta)$,

$$E\left(\int_0^T (D_t X)U_t dt\right) = E\left(X\int_0^T U_t \diamond dW_t\right).$$

Example 2.1 Let $X = W_t$. Then

$$D_s X = \begin{cases} 1 & : \quad s \leq t \\ 0 & : \quad s > t \end{cases}$$

Example 2.2 Let $X = \int_0^t u_r dW_r$ where u is a deterministic function. Then

$$D_s X = \begin{cases} u_s & : \quad s \leq t \\ 0 & : \quad s > t \end{cases}$$

That is,

$$D_t \int_0^T s^2 dW_s = t^2.$$

Chain Rule Let $\varphi \in C_b^1$ and let $X \in \mathbb{D}^{1,2}$. Then

$$D\varphi(X) = \varphi'(X)DX.$$

$$D_s W_t^2 = 2W_t I_{[0,t}(s).$$

Let

$$X = \int_0^t u_r dW_r.$$

Then

$$D_s \int_0^t u_r dW_r = u_s + \int_s^t D_s u_r dW_r.$$

Fundamental Theorem of Malliavin Calculus

Let $\{u_t, 0 \leq t \leq T\}$ be a stochastic process such that

$$E\left(\int_0^T u_s^2 ds\right) < \infty$$

and assume that for all $s \in [0,T]$, $u_s \in \mathbb{D}^{1,2}$ and that for all $t \in [0,T]$, $D_t u \in \text{Dom}(\Delta)$. Assume also that

$$E\left[\int_0^T (\delta(D_t u))^2 dt\right] < \infty.$$

Then

$$\int_0^T u_s dW_s \in \mathbb{D}^{1,2} \quad \text{and} \quad D_t\left(\int_0^T u_s dW_s\right) = \int_0^T D_t u_s dW_s + u_t.$$

Clark-Haussman-Ocone Formula

The Clark-Haussman-Ocone formula expresses the martingale representation theorem in terms of the Malliavin derivative.

Recall that the martingale representation theorem asserts that every square integrable martingale can be represented as a stochastic integral with respect to Brownian motion, i.e., for every square integrable martingale X, there exists an square integrable process u such that

$$X = E(X) + \int_0^T u_s dW_s.$$

If X is Malliavin differentiable, we have

$$D_t X = u_t + \int_t^T D_t u_s dW_s.$$

Thus

$$E(D_t X | \mathcal{F}_t^W) = u_t.$$

Hence if $X \in \mathbb{D}^{1,2}$, then

$$X = E(X) + \int_0^T E(D_t X | \mathcal{F}_t^W) dW_t.$$

An interesting fact of the Clark-Ocone formula is that if $X \in \mathbb{D}^{1,2}$ and $DX = 0$, then X is almost surely constant. The method based on Clack-Ocone formula has the advantage that it does not depend on a Markovian setup.

Clark-Ocone Formula under Change of Measure

Let

$$\widetilde{W}_t = \int_0^t u_s ds + W_t$$

where u_s is an adapted stochastic process satisfying the Novikov condition

$$E\left(\exp\left(\int_0^T u_s^2 ds\right)\right) < \infty.$$

By Girsanov theorem $Q(d\omega) = Z_T P(d\omega)$ where

$$Z_t = \exp\left(-\int_0^t u_s dW_s - \frac{1}{2}\int_0^t u_s^2 ds\right),\ 0 \le t \le T.$$

Let $X \in \mathbb{D}^{1,2}$ is $\mathcal{F}_T$ measurable and satisfies

$$E_Q\|X\|^2 < \infty,$$

$$E_Q \int_0^T \|D_t X\|^2 dt < \infty,$$

$$E_Q\left[|X| \int_0^T \left(\int_0^T u_s dW_s + \frac{1}{2}\int_0^t (u_s D_t u_s) ds\right)^2\right] < \infty.$$

Then

$$X = E_Q(X) + \int_0^T E_Q[(D_t X - X\int_0^T D_t u_s d\widetilde{W}_s)|\mathcal{F}_t^W] d\widetilde{W}_t.$$

Bayes Rule

Let μ and ν be two probability measures on a measurable space $(\Omega, \mathcal{F})$ such that $\nu(d\omega) = f(\omega)\nu(d\omega)$ for some $f \in L^1(\mu)$. Let X be a random variable on $(\Omega, \mathcal{F})$ such that $X \in L^1(\nu)$. Let $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra. Then

$$E_\mu(fX|\mathcal{G}) = E_\mu(f|\mathcal{G})E_\nu(X|\mathcal{G}).$$

Let

$$Q(d\omega) = Z_T(\omega)P(d\omega)$$

where

$$Z_t = \exp\left\{-\int_0^t u_s dW_s - \frac{1}{2}\int_0^t u_s^2 ds\right\}.$$

Corollary to Bayes Rule

Suppose $G \in L^1(Q)$. Then

$$E_Q[G|\mathcal{F}_t] = \frac{E[Z_T G|\mathcal{F}_t]}{Z_t}.$$

Lemma 2.1 Using Clark-Ocone formula, we have

$$D_t(Z_T F) = Z_T\left[D_t F - F\left(u_t + \int_t^T D_t u_s d\widetilde{W}_s\right)\right].$$

Black-Scholes Delta

Consider the Black-Scholes model for stock price

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

$$= (\mu - \sigma u)S_t dt + \sigma S_t d\widetilde{W}_t$$

$$= rS_t dt + \sigma S_t d\widetilde{W}_t$$

since $\frac{\mu-r}{\sigma}=u$ is the risk premium and $D_t u=0$. Under the risk neutral measure Q,

$$dS_t = rS_t dt + \sigma S_t d\widetilde{W}_t$$

which gives

$$S_t = S_0 \exp\left\{(r-\frac{1}{2}\sigma^2)t + \sigma\widetilde{W}_t\right\}$$

known as the geometric Brownian motion. The derivative of the call option $E_Q[e^{-r(t-T)}(S_T - K)^+|\mathcal{F}_t]$ with respect to the stock price, known as Black-Scholes delta, is given by

$$\Delta_t = e^{-r(t-T)}\sigma^{-1}S_t^{-1}E_Q[D_t(S_T-K)^+|\mathcal{F}_t.]$$

But

$$D_t(S_T-K)^+ = I_{[K,\infty)}(S_T)S_T\sigma.$$

By Markov property of S_t, this is the same as

$$\Delta_t = e^{r(t-T)}S_t^{-1}E_Q^y[S_{T-t}I_{[K,\infty)}(S_{T-t})]|_{y=S_t}$$

where E_Q^y is the expectation under the risk neutral measure Q when $S_0=y$.

Hence

$$\Delta_t = e^{r(t-T)}S_t^{-1}E^y[Y_{T-t}I_{[K,\infty)}(Y_{T-t})]_{|y=S_t}$$

where

$$Y_t = S_0\exp\{(r-\frac{1}{2}\sigma^2)t+\sigma W_t\}.$$

Since the distribution of W_t is known, Δ_t can be explicitly expressed in terms of quantities involving S_t and the normal distribution function.

The quantity Δ_t represents the number of units we must invest in the risky investment at time t in order to be guaranteed to get the payoff $(S_T-K)^+$ at time T. Let V_0^Δ represent the corresponding initial fortune needed to achieve this. Thus V_0^Δ is the unique initial fortune, which makes it possible to establish a self-financing portfolio with the same payoff at time T as the option gives:

$$V_0^\Delta = E_Q[e^{-rT}(S_T-K)^+] = E[e^{-rT}(Y_T-K)^+]$$

which again can be expressed explicitly by the normal distribution.

Financial Interpretation of the Clark-Ocone Formula

Let X be the payoff of an European option on an asset S. The dynamics of the discounted price under the equivalent martingale measure (EMM) is given by

$$d\widetilde{S}_t = \sigma_t\widetilde{S}_t dW_t.$$

If (α,β) is a replication strategy of the option, we have

$$\widetilde{X} = E[\widetilde{X}] + \int_0^T \alpha_t d\widetilde{S}_t = E[\widetilde{X}] + \int_0^T \alpha_t\sigma_t\widetilde{S}_t dW_t.$$

On the other hand, by Clark-Ocone Formula,

$$\widetilde{X} = E[\widetilde{X}] + \int_0^T E[D_t\widetilde{X}|\mathcal{F}_t^W]dW_t.$$

Hence the replication strategy is given by

$$\alpha_t = \frac{E[D_t\widetilde{X}|\mathcal{F}_t^W]}{\sigma_t\widetilde{S}_t}, \quad t \in [0,T].$$

Stochastic Integration by Parts (SIBP)

The technique based on Malliavin calculus can be effective also when poor regularity conditions are assumed on the payoff function F, i.e., direct application of the Monte carlo methods gives unsatisfactory results, even if the underlying asset follows geometric Brownian motion.

The stochastic integration by parts allows removing the derivative of the payoff function, thus improving the numerical approximation. More precisely, let us suppose that we want to determine $\partial_\alpha E[F(S_T)Y]$ where S_T denotes the final price of the underlying asset depending on a parameter α, e.g, α is S_T in the case of Delta, α is volatility in the case of vega and Y is some random variable., e.g, a discount factor. The idea is to try to express $\partial_\alpha E[F(S_T)Y]$ in the form $\int_0^T D_sF(S_T)YU_s ds$ for some adapted integrable process U. By using the duality relation, formally we obtain

$$\partial_\alpha E[F(S_T)Y] = E[F(S_T)D^*(YU)].$$

Let Y be some random variable, e.g., a discount factor. Let $F \in C_b^1$ and let $X \in \mathbb{D}^{1,2}$. Then the following integration by parts holds:

$$E[F'(X)Y] = E\left[F(X)\int_0^T \frac{u_tY}{\int_0^T u_sD_sX ds} \diamond dW_t\right].$$

In general it is not allowed to take out a random variable from Itô integral. This can be made more precise in the case of anticipative stochastic integral.

Let $X \in \mathbb{D}^{1,2}$ and let U be a second-order Skorohod integrable process. Then

$$\int_0^T XU_t \diamond dW_t = X\int_0^T U_t \diamond dW_t - \int_0^T (D_tX)U_t dt.$$

The typical case when U is adapted, the above formula becomes

$$\int_0^T XU_t \diamond dW_t = X\int_0^T U_t dW_t - \int_0^T (D_tX)U_t dt.$$

and so it is possible to express Skorohod integral as the sum of an Itô integral and a Lebesgue integral.

Recall that by Itô formula

$$\int_0^T W_t dW_t = \frac{1}{2}(W_T^2 - T)$$

but by Integration by Parts (IBP) formula in Malliavin calculus, that is, a direct application of the above formula

$$\int_0^T W_T \diamond dW_t = W_T^2 - T.$$

This shows the difference between adapted and nonadapted integrands in stochastic calculus.

Thus even if the integrand does not depend on T, we have

$$\int_0^T W_T \diamond dW_t \neq W_T \int_0^T dW_t.$$

Bismut-Elworthy Formula

Let the diffusion process be given by

$$S_t = x + \int_0^t b(s, S_s)ds + \int_0^t \sigma(s, S_s)dW_s$$

where $x \in \mathbb{R}$ *and* $b, \sigma \in C_b^1$.

Then

$$E[\partial_x F(S_T)G] = \frac{1}{T}E\left[F(S_T)\left(G\int_0^T \frac{\partial_x S_t}{\sigma(t,S_t)}dW_t - \int_0^T D_tG\frac{\partial_x S_t}{\sigma(t,S_t)}dt\right)\right]$$

for every $G \in \mathbb{D}^{1,\infty}$.

Stochastic Flow of Diffeomorphisms and Ocone-Karatzas Hedging

Let

$$S_t = x + \int_0^t b(s, S_s)ds + \int_0^t \sigma(s, S_s)dW_s$$

where $x \in \mathbb{R}$ and $b, \sigma \in C_b^{1,2}$. The stochastic flow (see Kunita (1994)) is given by

$$D_sS_t = \sigma(s, X_s) + \int_s^t \partial_x b(r, S_r)D_sS_r dr + \int_s^t \partial_x\sigma(r, S_r)D_sS_r dr.$$

Assume that the payoff function is F for the European option with maturity T. Then the payoff functional is represented by the following Itô integral:

$$F(S_T) - E(F(S_T)) = \int_0^T \beta(t)dW_t,$$

where

$$\beta(t) = E\left[D_tF(S_T)|\mathcal{F}_t^W\right]$$

and $D_tF(S_T)$ is the stochastic flow of $F(S_T)$.

Further, assume that $F \in C^1$. Then using the Clark-Ocone-Karatzas formula,

$$\beta(t) = E\left[\langle\zeta(t), dF(S_T)\rangle\right] \text{ where } \zeta(t) = J_{T\leftarrow t}b(t, S_t)$$

and $J_{T\leftarrow t}$ is the Jacobian of the stochastic flow (the linearized equation of the SDE driving the flow) which satisfies

$$d_tJ_{T\leftarrow t} = (b'(t, S_t)dt + \sigma'(t, S_t)dW_t)J_{T\leftarrow t}$$

where $'$ denotes derivative with respect to x.

3. Hedging and Greek Estimation

Due to the 2008-2011 financial crisis in the United States, both modeling and valuation of financial products face challenges. Mathematical finance deals with pricing derivatives (that derives its value from the performance of an underlying asset, e.g., a stock, a mutual fund or an interest rate) products, e.g., options on stocks and bonds. For the derivation of option price formula, one calculates mathematical expectation (the population mean) of a discounted payoff function. Monte Carlo methods (based on the philosophy of law of large numbers, which basically says that for large sample, the sample mean converges in an appropriate sense to the population mean) in mathematical finance and financial engineering are alternative methods of pricing derivatives when analytical pricing formula is not available and when it is available it is too difficult to use, see Glasserman (2004). European options are contracts that are signed between two parties (usually a bank and a costumer) that allows to obtain certain monetary benefits if the price of certain asset falls above (known as call option) or falls below (known as put option) a certain fixed value, the strike price, at a certain fixed date, the expiration date. Whereas European options can be exercised only at the expiration date, American options are contracts which give the holder the right to exercise at any time on or before the exercise date. A Greek is the derivative (mathematical differentiation as in calculus) of the option price with respect to a parameter, also known as the *sensitivity parameter*. Greeks or sensitivity parameters, are frequently used for hedging (mitigating risk by taking investment position to offset potential losses/gains suffered by an individual or an organization), market risk management and profit and loss attribution. Greeks measure the stability of a portfolio with respect to change in parameters. For instance, one would be interested to know how sensitive is the change in the option price for a corresponding change in the stock price. It is important to carry out these calculations with high accuracy to prevent future instabilities in the position of a company holding these options. If the Greek does not have a closed form formula then one may think of performing Monte Carlo simulations in order to approximate it. There are 13 sensitivity parameters of a derivative product: *Delta, Gamma, Vega, Theta, Rho, Phi, Speed, Volga, Vanna, Veta, Charm, Color and Zomma.* In an idealized setting of continuous trading in a complete financial market, the payoff of a contingent claim can be manufactured or hedged through trading in underlying assets. The risk in a short position in an option, for example is offset, by a delta-hedging strategy of holding delta units of each underlying asset, where delta is simply the partial derivative of the option price with respect to the current price of that underlying asset. Implementation of the strategy requires the knowledge of these price sensitivities. Sensitivities with respect to other parameters, for example, derivative of the option price with respect to the volatility, called the *vega*, and the second derivative of the option price with respect to the asset price called the *gamma* are also widely used to measure and manage risk. *Theta* is the first derivative with respect to time to maturity. *Rho* is the first derivative with respect to the interest rate. *Phi* is the first derivative with respect to dividend yield. Speed is the third derivative of the option price with respect to the stock price. *Volga*, also called *Vomma*, is the second partial derivative of the option with respect to volatility. *Vanna* is the second partial derivative once with respect to stock and once with respect to volatility. *Veta* is the second partial derivative once with respect to maturity and once with respect to maturity. *Charm* is the second partial derivative of option price, once with respect to stock and once with respect to time to maturity. *Color* is the third partial derivative, twice with respect to stock and once with respect to time to maturity. *Zomma* is the third partial derivative of the option twice with respect to stock and once with respect to volatility.

One of the fundamental problems in finance is that the asset pricing models are continuous but the data are discrete, see Bishwal (2008). Whereas the prices themselves can often be observed in the market, their sensitivities can't, so accurate calculation of sensitivities is even more important than calculation of prices. Thus derivative estimation presents both theoretical and practical challenges to Monte Carlo simulation.

Traditionally, the methods for estimating sensitivities are broadly of two types: methods that involve simulating at two or more values of the parameter of differentiation and methods that do not. The first category- the finite difference approximation methods are easier to implement but produce biased estimates. Methods of the second category produce unbiased estimates. Second category methods are of two types: the pathwise method and the likelihood ratio method. Pathwise method differentiates each simulated outcome with respect to the parameter of interest. The likelihood ratio method differentiates a probability density rather than an outcome. When applicable, pathwise method provides best estimates of the sensitivities. Compared with finite difference methods, pathwise estimates require less computing time. Compared with likelihood ratio method, pathwise estimates usually have smaller variance. The application of the pathwise method requires interchanging the order of differentiation and integration. Pathwise method yields unbiased estimates of the derivative of an option price if the option's discounted payoff is almost surely continuous in the parameter of differentiation. This excluded many options, for example digital and barrier options. Finite difference methods are easy to implement, but they have large mean square errors. The likelihood ratio method does not require any smoothness in the discounted payoff because it is based on probability density instead. But its application is limited, because it requires the explicit knowledge of the relevant probability densities and often has large variance. Estimating second derivative is even more difficult than estimating first derivative regardless of the method used. The pathwise method is generally inapplicable to second derivative of option prices since a kink in an option payoff becomes a discontinuity in the derivative of the payoff. Combination of pathwise method and likelihood ratio method generally produce better gamma estimates than likelihood ratio method alone. We will study a new method for Greek estimation known as the sequential Monte Carlo method combined with Malliavin calculus method. In Malliavin calculus, also known as the stochastic calculus of variations, the Malliavin derivative differentiates a random variable with respect to its underlying noise generating process, see Nualart (1995). The Malliavin calculus method for the estimation of Greeks developed in Fournie *et al.* (1999, 2001), Malliavin and Thalmier (2006) first differentiates, but implementation requires the simulation of a discrete time approximation, thus discretizes next. The question remains as to discretize the underlying asset price process first and then differentiate or differentiate first and then discretize the asset price process. The first route leads to *Malliavin estimators.* Chen and Glasserman (2007) interchange the order and produce the same Malliavin estimators, that is, they discretize first and then differentiate. For the discretization, Chen and Glasserman (2007) used Euler approximation which is a first order approximation. In this paper, our first step is to use Milstein approximation which a second order approximation to improve the discretization error and have faster rate of convergence. Bishwal (2011) developed many higher order discretization schemes for approximation of stochastic integrals. One can employ some of these schemes, especially the stochastic Boole's rule, for Greek estimation in order to further improve the accuracy of approximation. Finally, numerical experiments should be performed. We suppose that the underlying model dynamics, e.g, price of a stock, are given by the stochastic differential equation

$$dX_t = f(X_t)dt + g(X_t)dW_t, \ X_0 = x$$

where W_t is a standard Brownian motion and f and g are smooth functions satisfying the existence and uniqueness of the solution of the equation. With a discounted payoff function h and maturity T, option price is

$$u(x) = E(h(X_T))$$

and option delta is

$$u'(x) = E(h'(X_T))\frac{dX_T}{dx}$$

where

$$\frac{dX_t}{dx} = Y_t$$

is the pathwise derivative of X_t with respect to the initial state x. Thus

$$\xi_1 := h'(X_T)\frac{dX_T}{dx} = h'(X_T)Y_T$$

is an unbiased estimator of $u'(x)$. The dynamics of the pathwise derivative is given by the stochastic differential equation

$$dY_t = f'(X_t)Y_t dt + g'(X_t)Y_t dW_t, \quad Y_0 = 1,$$

that is,

$$Y_t = \exp\left\{(f'(X_s) - \frac{1}{2}(g'(X_s))^2]ds + \int_0^t g'(X_s)dW_s\right\}.$$

It can be shown that

$$u'(x) = E^x\left[h(X_T)\int_0^T a(t)\frac{Y_t}{g(X_t)}dW_t\right]$$

where $\int_0^T a(t)dt = 1$. The stochastic integral

$$\int_0^T a(t)\frac{Y_t}{g(X_t)}dW_t$$

is called the *Malliavin weight*. With $a(t) = \frac{1}{T}, \quad 0 \le t \le T$, the Malliavin estimator is given by

$$\xi_3 := h(X_T)\frac{1}{T}\int_0^T \frac{Y_t}{g(X_t)}dW_t,$$

see Fournie *et al.* (1999). For the Black-Scholes model

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

$$u'(x) = E^x\left[h(X_T)\frac{W_T}{x\sigma_0 T}\right].$$

Thus the Malliavin estimator of Δ is

$$h(X_T)\frac{W_T}{x\sigma T}.$$

With $p(x, X_T)$ being the transition density of X_T given x, the likelihood ratio method (LRM) estimator is given by

$$\xi_2 := h(X_T)\frac{d}{dx}\log p(x, X_T).$$

The likelihood ratio method (LRM) Black-Scholes call delta estimator is given by

$$e^{-rT}(X_T - K)^+\frac{\log(S_t/x) - (r - \frac{1}{2}\sigma^2)T}{x\sigma T}$$

which can be simulated as

$$e^{-rT}(X_T - K)^+\frac{Z}{x\sigma\sqrt{T}}.$$

Based on Euler discretization of the price SDE, this estimator can be seen as a limit of the average of the combinations of pathwise and LRM estimators. One could study higher order efficient Monte Carlo simulation methods for the derivation of the derivative (differentiation) of the derivative price (option price), that is estimate the Greeks or sensitivities without using Malliavin derivative but using elementary techniques. One could study the convergence of the Greek estimators when the discretization of the SDE is done using the Milstein scheme and other higher order schemes to the corresponding Malliavin estimators. Unbiasedness and the rate of convergence of the new estimators will be studied. Based on discrete observations, higher order approximation of the stochastic integral, studied in Bishwal (2008, 2011) could be used to obtain better Greek estimators. The performance of the new estimators could be numerically tested using computer programs, e.g., MATLAB.

Delta, Vega and Gamma of European Options

Weak regularity properties are assumed for the payoff F. The stochastic integration by parts formula allows removing the derivative of the payoff function, thus improving the numerical approximation. It is possible to prove the validity of the integration by parts formula for weakly differentiable (or even differentiable in a distributional sense) functions.

Examples: Consider the Black-Scholes model for the underlying asset under the physical measure

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

The Black-Scholes dynamics for the underlying asset of an option under the equivalent martingale measure is given by

$$S_T = x \exp\left\{\sigma W_T + \left(r - \frac{1}{2}\sigma^2\right) T\right\}.$$

Delta:

$$D_s S_T = \sigma S_T \quad \text{and} \quad \partial_x S_T = \frac{S_T}{x}.$$

In the SIBP if $u = 1$ and $Y = \partial_\alpha X$, we obtain

$$E[\partial_\alpha F(X)] = E\left[F(X)\int_0^T \frac{\partial_\alpha X}{\int_0^T D_s X ds} \diamond dW_t\right].$$

Using this we obtain

$$\begin{aligned}\Delta &= e^{-rT}\partial_x E[F(S_T)]\\ &= e^{-rT} E\left[F(S_T)\int_0^T \frac{\partial_x S_T}{\int_0^T D_s S_T ds} \diamond dW_t\right]\\ &= e^{-rT} E\left[F(S_T)\int_0^T \frac{1}{\sigma T x} dW_t\right] = \frac{e^{-rT}}{\sigma T x} E[F(S_T)W_T].\end{aligned}$$

Vega:

$$\partial_\sigma S_T = (W_T - 2\sigma T)S_T, \quad D_s S_T = \sigma S_T.$$

$$\mathcal{V} = e^{-rT}\partial_\sigma E[F(S_T)] = e^{-rT} E\left[F(S_T)\left(\frac{W_T - \sigma T}{\sigma T} W_T - \frac{1}{\sigma}\right)\right].$$

Gamma:

$$\begin{aligned}\Gamma = e^{-rT}\partial_{xx}E[F(S_T)] &= \frac{e^{-rT}}{\sigma T}E[\partial_x \frac{F(S_T)}{x}W_T] \\ &= \frac{e^{-rT}}{\sigma T x^2}E[F(S_T)W_T] + \frac{e^{-rT}}{\sigma T x}E[\partial_x F(S_T)W_T] \\ &= \frac{e^{-rT}}{\sigma T x^2}E[F(S_T)W_T] + \frac{e^{-rT}}{\sigma T x}E[F'(S_T)\partial_x S_T W_T] \\ &= \frac{e^{-rT}}{\sigma T x^2}E[F(S_T)W_T] + \frac{e^{-rT}}{\sigma T x}E[F(S_T)\int_0^T \frac{W_T}{\sigma T x}\diamond dW_t] \\ &= \frac{e^{-rT}}{\sigma T x^2}E[F(S_T)W_T] + \frac{e^{-rT}}{\sigma T x}\frac{1}{\sigma T x}E[F(S_T)(W_T^2 - T)] \\ &= \frac{e^{-rT}}{\sigma T x^2}E[F(S_T)(\frac{W_T^2 - T}{\sigma T} - W_T)].\end{aligned}$$

Delta of Asian option

Let the arithmetic average be defined as

$$X = \frac{1}{T}\int_0^T S_t dt.$$

Observe that

$$\partial_x X = \frac{X}{x},$$

$$\int_0^T D_s X ds = \int_0^T \int_0^T D_s S_t dt ds = \sigma \int_0^T \int_0^T S_t ds dt = \sigma \int_0^T t S_t dt.$$

The Delta of the asian option is given by

$$\begin{aligned}\Delta = e^{-rT}\partial_x E[F(X)] = \frac{e^{-rT}}{x}E[F'(X)X] &= \frac{e^{-rT}}{\sigma x}E\left[F(X)\int_0^T \frac{\int_0^T S_s ds}{\int_0^T s S_s ds}\diamond dW_t\right] \\ &= \frac{e^{-rT}}{x}E\left[F(X)\left(\frac{1}{I_1}\left(\frac{W_T}{\sigma} + \frac{I_2}{I_1}\right) - 1\right)\right]\end{aligned}$$

where

$$I_j = \frac{\int_0^T t^j S_t dt}{\int_0^T S_t dt}, \quad j = 1, 2.$$

Sensitivity with respect to correlation in Heston Model

Consider the Heston model

$$dS_t = \sqrt{V_t} S_t \left\{\sqrt{1-\rho^2} dW_t^1 + \rho dW_t^2\right\},$$

$$dV_t = \alpha(\beta - V_t)dt + \sigma\sqrt{V_t} dW_t^2$$

where W_t^1 and W_t^2 are two independent Brownian motions and the leverage $\rho \in (-1, 1)$. Observe that

$$S_T = S_0 + \int_0^T \sqrt{V_t} dW_t^1 + \int_0^T \sqrt{V_t} dW_t^2 - \frac{1}{2}\int_0^T v_t dt.$$

and

$$\partial_\rho S_T = S_T G$$

where

$$G := -\frac{\rho}{\sqrt{1-\rho^2}}\int_0^T \sqrt{V_t}dW_t^1 + \int_0^T \sqrt{V_t}dW_t^2.$$

Then

$$\partial_\rho E[F(S_T)] = \frac{1}{T\sqrt{1-\rho^2}}E\left[F(S_T)\left(G\int_0^T \frac{1}{\sqrt{V_t}}dW_t^1 + \frac{\rho T}{\sqrt{1-\rho^2}}\right)\right].$$

Similarly, other Greeks should be calculated.

4. Bootstrap and Volatility Estimation

In a series of papers, given in the reference, by Barndorff-Nielsen and Shephard (2001-2007) (henceforth BN-S) have developed an asymptotic theory for *realized volatility-like measures*. In particular, for a general stochastic volatility model, BN-S established a central limit theorem (CLT) for realized volatility over a fixed interval of time, for instance a day, as the number of intraday returns increases to infinity. They have also showed that a CLT applies to empirical measures based on powers of intra day returns (realized power variation) and products of powers of absolute returns (e.g., bipower variation). BN-S also provided a joint asymptotic distribution theory for the realized volatility and realized bipower variation, and showed how to use this distribution to test for the presence of jumps in asset prices.

For the first time, Goncalves and Meddahi (2004) proposed bootstrap methods for evaluating high frequency data such as realized volatility to improve upon the first order theory of asymptotic mixed normal approximations of BN-S. They studied two bootstrap methods for realized volatility: an i.i.d. (uniform) bootstrap and wild bootstrap. The i.i.d. bootstrap (cf. Efron (1979)) generates bootstrap pseudo intraday returns by resampling with replacement the original set of intraday returns. The wild bootstrap observations are generated by multiplying each original intraday return by an i.i.d. draw from a distribution that is completely independent of the original data. The wild bootstrap was introduced by Wu (1986), and further studied by Liu (1988) and Mammen (1993), in the context of cross-section linear regression models subject to unconditional heteroskedasticity in the error term. Zhang, Mykland and Ait-Sahalia (2004) and Zhang (2004) considered an application of subsampling method to realized volatility under stochastic volatility. In particular, they use subsampling plus averaging the bias correct the realized volatility measure when microstructure noise is present. The generalized bootstrap techniques could be used to estimate the distribution (as opposed to bias) of realized volatility.

A popular bootstrap for serially dependent data is *block bootstrap*. In our context, intraday returns are (conditionally on the volatility path) independent, and this implies that blocking is not necessary for asymptotic refinements of the bootstrap. The issue here is *heteroskedasticity* and not serial correlation.

Malliavin calculus is a powerful tool to study the asymptotic expansion of distribution of estimators in diffusion type models, see Yoshida (1997). The *nondegeneracy* of the Malliavin covariance plays the same role in diffusion case as the Cramer condition in the i.i.d. case. One should use Malliavin calculus for obtaining Edgeworth expansion of generalized bootstrap statistics.

Goncalves and Meddahi (2004) used Monte Carlo simulations and formal Edgeworth expansions to compare the accuracy of the bootstrap and normal approximations. They showed that i.i.d. provides an asymptotic refinement when volatility is constant. The absolute magnitude of the coefficients describing the i.i.d. bootstrap error is smaller than the coefficients entering in the first term Edgeworth expansion of the original statistic. The Edgeworth expansion for wild bootstrap shows that it provides an asymptotic refinement when the volatility is heterogeneous if one chooses the external random variable used to construct the wild bootstrap observations appropriately.

One could use generalized bootstrap schemes for higher order accuracy in estimation and testing in classical stochastic volatility models driven by standard Brownian motion. Then one could also consider stochastic volatility models with jumps (for instance, Poisson type jumps) and stochastic volatility models with long memory driven by fractional Brownian motion.

Hall and Presnell (1999) introduced a class of *weighted-bootstrap techniques*, called *biased-bootstrap* or *b-bootstrap* methods. It is motivated by the need to adjust more conventional, uniform-bootstrap methods in a surgical way, so as to alter some of their features while leaving others unchanged. In the b-bootstrap, resampling probabilities are chosen to minimize the distance of the weighted bootstrap distribution from the usual, uniform bootstrap distribution, conditional on the data and subject to constraints that are designed to improve the statistical performance. Empirical likelihood methods may be viewed as a particular case of the b-bootstrap as also many techniques suggested for refining the generalised method of moments. The b-bootstrap has connection to tilting methods. It has also application to dependent data in the context of empirical likelihood. Methods related to both empirical likelihood and the b-bootstrap include the bootstrap likelihood technology of implied likelihood method for making likelihood calculations from confidence intervals. Depending on the nature of the adjustment, the biased bootstrap can be used to reduce bias, or reduce variance, or render some characteristic equal to a predetermined quantity. More specifically, applications of b-bootstrap methods include hypothesis testing (b-bootstrap enables simulation under the null hypothesis even when the hypothesis is false), b-bootstrap competitor to *Tibshirani's variance stabilization method* both density estimation and nonparametric regression under constraints, robustification of general statistical procedures, sensitivity analysis, generalized method of moments, outlier trimming, skewness and kurtosis reduction, shrinkage, and many more.

Bootstrap confidence intervals could also be studied, together with error reduction techniques. Many ideas of linear regression problems could be extended to stochastic volatility models. Bose and Chatterjee (2005) introduced a generalized bootstrap technique for estimators obtained by solving estimating equations. Special cases are the classical bootstrap of Efron, the delete-d jackknife, and variations of the Bayesian bootstrap. Under fairly general conditions they investigate to establish (a) asymptotic normality of estimator and consistency of bootstrap when model dimension is fixed or increasing with data size; (b) asymptotic representation of resampling variance estimator and (c) higher order accuracy of the new generalized bootstrap estimator for the bias corrected, studentized estimator.

In the classical setting, for instance, consider estimating the variance of the least squares estimate in linear regressions. There are several resampling schemes available in the literature. By establishing representation results, Liu and Singh (1992) classified these into two groups:

Those that are *efficient* but not consistent under heteroscedasticity and those that are consistent under heteroscedasticity (*robust*) but not efficient.

Classes of generalized bootstrap are introduced and in some sense all of the above schemes are special cases of these bootstraps. By establishing higher order expansions, one can distinguish between the estimators within the robust and the efficient class. First order representation results are also established for high dimensional regression models where the number of parameters increases with the sample size.

For the related problem of estimating the entire distribution of the realized volatility, one should study consistency of the generalized bootstrap. It is known from the existing works that the paired bootstrap (which is robust) is not second order accurate for the ordinary least squares estimator. One could show that with proper bias correction and studentization, a (smooth) generalization of the paired bootstrap may be second order accurate. One could then extend these ideas to estimates obtained by solving martingale estimating equations. One could establish representation results for the bootstrap estimator and obtain some first and second order distribution results. Representation results could also be obtained for the bootstrap variance estimator.

One should also think to investigate how these ideas can be implemented in estimating the distribution of M estimators. Generalized bootstrap schemes in *martingale estimation function* should be introduced and their first and second order behavior should be studied. New as well as unifying results on resampling plans in a large class of M estimators could be obtained which includes all standard resampling schemes and a large class of common estimators used in statistics. The performance of the bootstrap for *nonregular models* when the standard conditions often assumed are violated, could be also be investigated. The following jackknife and bootstrap techniques could be studied for stochastic volatility models:

1) Delete-1 jackknife (Quenoville (1949)), 2) Weighted Jackknife (Hinkley (1977)), 3) Weighted Jackknife (Wu (1986)), 4) Weighted Jackknife (Liu and Singh (1992)), 5) Residual bootstrap (Efron (1979)), 6) External bootstrap (Wu (1986)), 7) Weighted bootstrap (Liu (1988)), 8) Uncorrelated weights bootstrap (Chatterjee and Bose (2005)).

Goncalves and Meddahi (2004) compared wild bootstrap and i.i.d. bootstrap for realized volatility. One could study generalized bootstrap methods for realized volatility. One could also study other estimation methods. One should also study bootstrapping the Fourier estimator and the Wavelet estimator of the realized volatility. One should employ several different generating mechanisms for the instantaneous volatility process, Ornstein-Uhlenbeck, long memory and jump processes. Market microstructure contamination could also be entertained using a model with bid-ask bounce. Fourier estimators are known to be superior to realized volatility and wavelet estimators when considering bid-ask bounce. One should also consider bootstrapping the weighted realized volatility. This class nests several estimators.

One should also study bootstrapping the realized range-based variance, a statistics that replaces every squared return of realized variance with a normalized squared range. Realized range-based variance is asymptotically mixed normal and *five times more efficient* than the realized variance. This is not surprising. In contrast to the estimator based on daily returns, the range utilizes the information from the entire sample path and one would expect it to be superior.

One should also study bootstrapping continuous time GARCH model which is called COG-ARCH model. In this context one should consider heavy-tailed distributions (in the sense that all moments are not finite) of the volatility.

Beyond the first order theory of asymptotic mixed normal distribution of the realized volatility, one should obtain the Berry-Esseen bounds. Then one should consider the properties of generalized bootstrap estimators of realized volatility and study their Edgeworth expansion.

One should also study the problem of testing of the parametric form of the volatility in a stochastic differential equation driven by fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. This class of models is important as it captures the long memory behavior of the log-share prices. One should then extend to fractional stochastic volatility model.

Testing about volatility is an important problem in derivative pricing. A misspecification of the volatility function could lead to misspecfied derivative prices. When the price process follows the classical diffusion, Ait-Sahalia (1996) studied the volatility testing problem using nonparametric density matching. His test is based on the comparison of parametric marginal density with its nonparametric estimator. However, the classical diffusion does not take in to account the long memory behavior of the price process. One could take the long memory behavior of the price in to account using the fractional diffusion and study the testing problem for volatility. One should study the asymptotic behavior of the test statistic.

A normalized fractional Brownian motion $\{W_t^H, t \geq 0\}$ with Hurst parameter $H \in (0,1)$ is a centered Gaussian process with continuous sample paths whose covariance kernel is given by

$$E(W_t^H W_s^H) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}), \quad s,t \geq 0.$$

The process is self similar (scale invariant) and it can be represented as a stochastic integral with respect to standard Brownian motion. For $H = \frac{1}{2}$, the process is a standard Brownian motion. For $H \neq \frac{1}{2}$, the fBm is not a semimartingale and not a Markov process, but a Dirichlet process. The increments of the fBm are negatively correlated for $H < \frac{1}{2}$ (anti-persistent) and positively correlated for for $H > \frac{1}{2}$ and in this case they display long-range dependence (persistence). The parameter H which is also called the self similarity parameter, measures the intensity of the long range dependence.

We assume that the price process of an underlying is a one dimensional fractional diffusion satisfying the fractional Itô SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t^H, \ 0 < t \leq T$$

where μ and σ are smooth functions such that a unique solution of the above SDE exists and $\{W_t^H, t \in [0,T]\}$ is a fractional Brownian motion with Hurst coefficient $H > 1/2$. Assume the σ^2 has continuous derivatives up to second order.

For the pricing of contingent claims with a payoff function that depends on the evolution of the underlying process X during the time period $[0,T]$, the derivative prices are calculated under the risk-neutral quasi-martingale measure, which is not influenced by the drift function, but the diffusion function σ captures the volatility of the underlying.

In the field of interest rate modeling, the specification of μ is also important. However, a first step in evaluating a particular parametric interest rate model could be test of parametric form of its volatility coefficient.

It is often assumed that the diffusion coefficient belongs to a set of parametric functions ,i.e, there exists an unknown parameter $\theta_0 \in \Theta$ such that $\sigma(x) = \sigma(\theta_0, x)$. The hypotheses to be tested are the null hypothesis
$\mathcal{H}_0$: There exists $\theta_0 \in \Theta$: for every $t \in [0,T]$: $\sigma(X_t) = \sigma(\theta_0, X_t)$ P-a.s.
versus the alternative hypothesis
$\mathcal{H}_1$: For all $\theta \in \Theta$: for every $t \in [0,T]$: $|\sigma(X_t) - \sigma(\theta, X_t)| \geq c_n\ \delta_n(X_t)$ P-a.s.

Here δ_n is the local shift in the alternative, a sequence of bounded functions and c_n is the order of difference between $\mathcal{H}_0$ and $\mathcal{H}_1$.
Assumptions:
(A1) The following holds for σ^2:

$$|\sigma^2(\theta, x) - \sigma^2(\theta_0, x)| \leq D(x)|\theta - \theta_0| \ \ \forall x \in I_x$$

where $D(x)$ is a constant depending on x and the set I_x is defined by

$$I_x := \{x : L_T(x) \geq \epsilon > 0\}$$

with an arbitrary number ϵ, where $L_T(x)$ denotes the local time of X at time T.
(A2) $\hat{\theta}$ is a $1-H$ root consistent parametric estimator of θ within the family of the parametric model, i.e., $|\hat{\theta} - \theta| = O_P(n^{-(1-H)})$.

We introduce the following test statistics

$$\tau_n(x) = \{nh_n L_n(x_i)\}^{1-H} \left(\frac{S_n(x)}{\widetilde{\sigma}^2(\hat{\theta}, x)} - 1 \right).$$

The proposed test statistic is asymptotically equivalent to the L_2 distance between $S_n(\cdot)$ and $\widetilde{\sigma}^2(\hat{\theta}, \cdot)$.

One should study the asymptotic behavior of the above test statistic. Corradi and Swanson (2003) studied bootstrap specification test using an i.i.d. bootstrap. One should study bootstrap specification test using the generalized bootstrap methods.

5. Jumps and Long-Memory

Recently long memory processes, i.e. processes with slowly decaying autocorrelation and processes with jumps have received attention in finance, engineering and physics. The simplest continuous time long memory process is the fractional Brownian motion discovered by Kolmogorov (1940) and later on studied by Levy (1948) and Mandelbrot and van Ness (1968). Continuous time long memory jump process is fractional Levy process. Hence fractional Levy process can also be called the *Kolmogorov-Levy process.*

A normalized fractional Brownian motion $\{W_t^H, t \geq 0\}$ with Hurst parameter $H \in (0,1)$ is a centered Gaussian process with continuous sample paths whose covariance kernel is given by

$$E(W_t^H W_s^H) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}), \quad s, t \geq 0.$$

The process is self similar (scale invariant) and it can be represented as a stochastic integral with respect to standard Brownian motion. For $H = \frac{1}{2}$, the process is a standard Brownian motion. For $H \neq \frac{1}{2}$, the fBm is not a semimartingale and not a Markov process, but a

Dirichlet process. The increments of the fBm are negatively correlated for $H < \frac{1}{2}$ and positively correlated for $H > \frac{1}{2}$ and in this case they display long-range dependence. The parameter H which is also called the self similarity parameter, measures the intensity of the long range dependence. The ARIMA(p, d, q) with autoregressive part of order p, moving average part of order q and fractional difference parameter $d \in (0,\ 0.5)$ process converge in Donsker sense to fBm. See Mishura (2008).

The following processes, namely, fractional Poisson process, fractional Levy process, and sub-fractional Levy processes could be used for the innovation driving the stochastic volatility process.

Fractional Poisson process

A fractional Poisson process $\{\overline{W}_H(t), t > 0\}$ with Hurst parameter $H \in (1/2, 1)$ is defined as

$$\overline{W}_H(t) = \frac{1}{\Gamma(H - \frac{1}{2})} \int_0^t u^{\frac{1}{2}-H} \left(\int_0^u \tau^{H-\frac{1}{2}} (\tau - u)^{H-\frac{3}{2}} d\tau \right) dR(u)$$

where

$$R(u) = \frac{N(u)}{\sqrt{\lambda}} - \sqrt{\lambda}u, \lambda > 0$$

and $N(u)$ is a Poisson process.

The process is self-similar in the wide sense, has wide sense stationary increments, has fat-tailed non-Gaussian distribution, and exhibits long range dependence. The process converges to fractional Brownian motion in distribution. The process is self similar in the asymptotic sense.

Strict sense, wide sense and asymptotic sense self-similarity are equivalent for fractional Brownian motion.Stock returns are far from being self-similar in strict sense. The stochastic volatility model we consider is

$$dS_t = \sqrt{V}_t dB(t),$$

$$dV_t = \mu V_t dt + \sigma V_t d\overline{W}_H(t),\ t \geq 0$$

The advantage of fractional Poisson process is that whereas fractional Brownian motion allows for arbitrage when used as noise in the stock price process, the shot noise process itself can be chosen arbitrage-free.

Hawkes processes are an efficient generalization of the Poisson processes to model a sequence of arrivals over time of some types of events, that present self-exciting feature, in the sense that each arrival increases the rate of future arrivals for some period of time. This class of counting processes allows one to capture self-exciting phenomena in a more accurate way compared to inhomogeneous Poisson processes or Cox processes. In finance, they are accurate to model for example credit risk contagion, order book or microstructure noises's feature of financial markets.

A Hawkes process is a counting process A_t with stochastic intensity λ_t given by

$$\lambda_t = \mu + \int_0^t \Phi(t - s) dA_s$$

where $\mu > 0$ and $\Phi : \mathbb{R} \to \mathbb{R}_+$ are two parameters. The parameter $\mu > 0$ is called the *background intensity* and the function Φ is called the *excitation function*. When $\Phi = 0$, this a homogeneous Poisson process.

A fractional Hawkes process $\{A_H(t), t > 0\}$ with Hurst parameter $H \in (1/2, 1)$ is defined as

$$A_H(t) = \frac{1}{\Gamma(H - \frac{1}{2})} \int_0^t u^{\frac{1}{2}-H} \left(\int_u^t \tau^{H-\frac{1}{2}} (\tau - u)^{H-\frac{3}{2}} d\tau \right) d\widetilde{R}(u)$$

where

$$\widetilde{R}(u) = \frac{A(u)}{\sqrt{\lambda_t}} - \sqrt{\lambda_t} u$$

and $A(u)$ is a Hawkes process with stochastic intensity λ_t.

Fractional Levy Process

Levy driven processes of Ornstein-Uhlenbeck type have been extensively studied over the last few years and widely used in finance, see Barndorff-Nielsen and Shephard (2001). FLOU process generalizes fOU process to include jumps. The fractional Levy Ornstein-Uhlenbeck (fOU) process, is an extension of fractional Ornstein-Uhlenbeck process with fractional Levy motion (fLM) driving term. In finance, it could be useful as a generalization of fractional Vasicek model, as one-factor short-term interest rate model which could take into account the long memory effect and jump of the interest rate. This process was introduced by Marquardt (2006) who also introduced fractional Levy process in Marquardt (2006). Using suitable transformation of the process, one can obtain a nonlinear stationary process satisfying a fractional SDE, see Buchmann and Kluppelberg (2005). Brent (2003) used the process as temperature and obtained weather derivative arbitrage free pricing formulas for European and average type options. Cheridito *et al.* obtained the fOU process as a Lamperti transformation of the fBM. The model parameter is usually unknown and must be estimated from data.

Fractional Levy Process is defined as

$$M_{H,t} = \frac{1}{\Gamma(H + \frac{1}{2})} \int_{\mathbb{R}} [(t-s)_+^{H-1/2} - (-s)_+^{H-1/2}] dM_s, \quad t \in \mathbb{R}$$

where $M_t, t \in \mathbb{R}$ is a Levy process on $\mathbb{R}$ with $E(M_1) = 0,\ E(M_1^2) < \infty$ and without Brownian component. FLP has the following properties:
1) The covariance of the process is given by

$$cov(M_{H,t}, M_{H,s}) = \frac{E(L(1)^2}{2\Gamma(2H+1)\sin(\pi H)} [|t|^{2H} + |s|^{2H} - |t-s|^{2H}].$$

2) M_H is not a martingale. For a large class of Levy processes, M_d is neither a semimartingale.
3)M_H is Hölder continuous of any order β less than $H - \frac{1}{2}$. 4) M_H has stationary increments.
5) M_H is symmetric. 6) M_H is self similar. 7) M_H has infinite total variation on compacts.
8) The FIMA (fractionally integrated moving average) process is defined as

$$Y_H(t) = \int_{-\infty}^t g_H(t-u) L(du),\ t \in \mathbb{R}$$

where

$$g_H(t) = \frac{1}{\Gamma(H - \frac{1}{2})} \int_0^t g(t-s) s^{H-\frac{3}{2}} ds,\ t \in \mathbb{R}$$

and the kernel g is the kernel of a short memory moving average process.

The process $Y_H(t)$ can be written as

$$Y_H(t) = \int_{-\infty}^{t} g(t-u)dM_{H,u},\ t \in \mathbb{R}.$$

Assuming that the kernel $g : \mathbb{R} \to \mathbb{R}$ satisfies $g(t) = 0$ for all $t < 0$ (causality) and $|g(t)| \leq Ce^{-ct}$ for some constants $C > 0$ and $c > 0$ (has short memory), the FIMA process is stationary and is infinite divisible.

Consider the kernel

$$g(t-s) = \sigma e^{\theta(t-s)} I_{(0,\infty)}(t-s)$$

then

$$g_H(t) = \frac{\sigma}{\Gamma(H-\frac{1}{2})} \int_0^{\infty} e^{\theta(t-s)} I_{(0,\infty)}(t-s) s^{H-\frac{3}{2}} ds,\ t \in \mathbb{R}$$

Note that

$$U_t^{H,\theta,\sigma} = \int_{\mathbb{R}} g_H(t-u)L(du),\ t \in \mathbb{R}$$

is the fractional Levy Ornstein-Uhlenbeck (FLOU) process satisfying the fractional Langevin equation

$$dU_t = \theta U_t dt + \sigma dM_{H,t},\ t \in \mathbb{R}.$$

The process has long memory.

Consider the Ornstein-Uhlenbeck process X_t satisfying the Itô stochastic differential equation

$$dX_t = \theta X_t dt + dM_t^H,\quad t \geq 0$$

where $\{M_t^H\}$ is a fractional Levy motion with $H > 1/2$ with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and $\theta < 0$ is the unknown parameter to be estimated on the basis of continuous observation of the process $\{X_t\}$ on the time interval $[0, T]$.

$$X_t = \int_{-\infty}^{t} e^{\theta(t-s)} dM_s.$$

This process is stationary and is a process with long memory.

It can be shown that X_{t_i} is a stationary discrete time AR(1) process with autoregression coefficient $\phi \in (0,1)$ with the following representation

$$X_{t_i} = \phi X_{t_{i-1}} + \epsilon_{t_{i-1}}$$

where

$$\phi = e^{-\theta h},\quad \epsilon_{t_{i-1}} = \int_{t_{i-1}}^{t_i} e^{-\theta(t_i-u)} dM_u.$$

For example M is a gamma process or an inverse-Gaussian process. The fractional Levy process (FLP) is a generalization of FBM. The FLP is not a martingale and not even a semimartingale in general. FLP is a natural counterpart to FBM. It is Holder continuous. It has stationary increments. It is symmetric. It is self-similar. Fractional stable process is a special case of FLP.

The fractional Levy Ornstein-Uhlenbeck (FLOU) process X was introduced in Marquardt (2006). FLOU process is a LOU process which includes long memory.

We indroduce the ratio estimator of the drift. The ratio estimator of θ is defined as

$$\hat{\theta}_n := -\log\left[\min_{1\leq i\leq n} \frac{X_{i\Delta}}{X_{(i-1)\Delta}}\right].$$

This estimator is motivated by the extreme value theory for the correlation parameter of an AR(1) process whose innovation distribution is positive. See Davis and McCormick (1989). In the case of exponential AR(1) process, it coincides with the maximum likelihood estimator. See Nielsen and Shephard (2003).

The weak consistency of the ratio estimator in the LOU process was studied in Jongbloed *et al.* (2006). The strong consistency and asymptotic Weibullness was studied in Brockwell, Davis and Yang (2007) in the case of Gamma innovations. One can obtain strong consistency and asymptotic Weibullness of the ratio estimator for the fLOU process.

Sub-fractional Levy Process

As a generalization of fractional Brownian motion we get the Hermite process of order k with Hurst parameter $H \in (\frac{1}{2}, 1)$ which is defined as a multiple Wiener-Itô integral of order k with respect to standard Brownian motion $(B(t))_{t\in\mathbb{R}}$

$$Z_t^{H,k} := c(H,k)\int_{\mathbb{R}}\int_0^t \prod_{j=1}^{k}(s-y_i)_+^{-(\frac{1}{2}+\frac{H-1}{2})}ds\ dB(y_1)dB(y_2)\cdots dB(y_k)$$

where $x_+ = \max(x,0)$ and the constant $c(H,k)$ is a normalizing constant that ensures $E(Z_t^{H,k})^2 = 1$.

For $k=1$ the process is fractional Brownian motion W_t^H with Hurst parameter $H \in (0,1)$. For $k=2$ the process is Rosenblatt process. For $k \geq 2$, the process is non-Gaussian.

The Rosenblatt process is not a semimartingale and for $H > 1/2$, the quadratic variation is 0. The distribution of the process is infinitely divisible. It is unknown yet whether the process is Markov or not.

The covariance kernel $R(t,s)$ is given by

$$R(t,s) := E[Z_t^{H,k}Z_s^{H,k}] = c(H,k)^2\int_0^t\int_0^s\left[(u-s)_+^{-(\frac{1}{2}+\frac{H-1}{2})}ds(v-y)_+^{-(\frac{1}{2}+\frac{H-1}{2})}dy\right]^k dudv.$$

Let

$$\beta(p,q) := \int_0^1 z^{p-1}(1-z)^{q-1}dz,\ \ p,q>0$$

be the beta function.

Using the identity

$$\int_0^1\int_{\mathbb{R}}(u-s)_+^{a-1}ds(v-y)_+^{a-1}dy = \beta(a,2a-1)|u-v|^{2a-1},$$

we have

$$\begin{aligned}
R(t,s) &= c(H,k)^2\beta\left(\frac{1}{2}-\frac{1-H}{k},\frac{2H-2}{k}\right)^k\int_0^t\int_0^s\left(|u-v|^{\frac{2H-2}{k}}\right)^k dvdu\\
&= c(H,k)^2\frac{\beta(\frac{1}{2}-\frac{1-H}{k},\frac{2H-2}{k})^k}{H(2H-1)}\frac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H}).
\end{aligned}$$

In order to obtain $E(Z_t^{(H,k)})^2 = 1$, choose

$$c(H,k)^2 = \left(\frac{\beta(\frac{1}{2} - \frac{1-H}{k}, \frac{2H-2}{k})^k}{H(2H-1)} \right)^{-1}$$

and we have

$$R(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

Thus the covariance structure of the Hermite process and fractional Brownian motion are the same. The process $Z_t^{(H,k)}$ is H-self similar with stationary increments and all moments are finite.

For any $p \geq 1$,

$$E|Z_t^{(H,k)} - Z_s^{(H,k)}|^p \leq c(p,H,k)|t-s|^{pH}.$$

Thus the Hermite process has Hölder continuous paths of order $\delta < H$.

A weighted fBm (wfBm) ξ_t has the covariance function

$$q(s,t) = \int_0^{s \wedge t} u^a[(t-u)^b + (s-u)^b]du, \quad s,t \geq 0$$

where $a > -1, \ -1 < b \leq 1, \ |b| \leq 1 + a$. When $a = 0$, it is the usual fBm with Hurst parameter $(b+1)/2$ up to a multiplicative constant. For $b = 0$ it is a time-inhomogeneous Bm.

The function u^a is called the weight function of wfBm. For $a = 0$, this process is usual fBm with Hurst parameter $(b+1)/2$. For the case $b = 1$, this process has the covariance of the process $\int_0^t W_{r^a} dr$ where W is standard Brownian motion. For $b = 0$, this process is time-inhomogeneous Bm. The finite dimensional distributions of the process $(T^{-a/2}(\xi_{t+T} - \xi_T)), t \geq 0$ converge as $T \to \infty$ to those of fBm with Hurst parameter $(1+b)/2$ multiplied by $(2/(1+b)))^{1/2}$. The process has asymptotically stationary increments for long time intervals, but not for short time intervals. For $b \neq 0$, the process is neither a semimartingale nor a Markov process.

This process occurs as the limit of occupation time fluctuations of a particle system of independent particles moving in $\mathbb{R}^d$ with symmetric α-stable Levy process, $0 < \alpha \leq 2$, started from an inhomogeneous Poisson configuration with intensity measure $dx/(1+|x|^\gamma), 0 < \gamma \leq d = 1 < \alpha, a = -\gamma/\alpha, \ b = 1 - 1/\alpha, -1 < a < 0, 0 < b \leq 1 + a$. The homogeneous case $\gamma = 0$ gives fBm.

A bi-fractional Brownian motion (bfBm) has covariance

$$\frac{1}{2}(s^{2H} + t^{2H})^k - |t-s|^{2Hk}), \quad s,t \geq 0, \quad 0 < k \leq 1.$$

For $k = 1$, it reduces to fBm. For $H = 1/2$, bfBm can be extended for $1 < k < 2$.

Consider the Gaussian process with the covariance function

$$K_H(s,t) = (2-2H)\left(s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |s-t|^{2H}\right]\right), \quad s,t > 0$$

for $1 < H \leq 2$. The case $H = 1/2$ corresponds to Bm.

This process occurs as the limit of occupation time fluctuations of a particle system undergoing a critical branching, i.e., each particle independently, at an exponentially distributed lifetime, disappears with probability $1/2$ or is replaced with two particles at the same site with probability $1/2$. For $\alpha = 2$, one reaches super processes.

Recently, sub-fractional Brownian (sub-FBM) motion which is a centered Gaussian process with covariance function

$$C_H(s,t) = s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |s-t|^{2H}\right], \quad s,t > 0$$

for $0 < H < 1$ introduced by Bojdecki, Gorostiza and Talarczyk (2004) has received some attention recently. The interesting feature of this process is that this process has some of the main properties of FBM, but the increments of the process are nonstationary, more weakly correlated on non-overlapping time intervals than that of FBM, and its covariance decays polynomially at a higher rate as the distance between the intervals tends to infinity. It would be interesting to see extension of this paper to sub-FBM case. We generalize sub-fBM to sub-FLP.

Sub-fractional Levy process is defined as

$$S_{H,t} = \frac{1}{\Gamma(H+\frac{1}{2})}\int_{\mathbb{R}}[(t-s)_+^{H-1/2} - (-s)_+^{H-1/2}]dM_s, \quad t \in \mathbb{R}$$

where $M_t, t \in \mathbb{R}$ is a Levy process on $\mathbb{R}$ with $E(M_1) = 0$, $E(M_1^2) < \infty$ and without Brownian component. SFLP has the following properties:
1) The covariance of the process is given by

$$cov(S_{H,t}, S_{H,s}) = s^{2H} + t^{2H} + \frac{E(L(1)^2}{2\Gamma(2H+1)\sin(\pi H)}[|t|^{2H} + |s|^{2H} - |t-s|^{2H}].$$

2) S_H is not a martingale. For a large class of Levy processes, S_H is neither a semimartingale. 3) S_H is Hölder continuous of any order β less than $H - \frac{1}{2}$. 4) S_H has nonstationary increments. 5) S_H is symmetric. 6) S_H is self similar. 7) S_H has infinite total variation on compacts.

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