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INTERACTION DYNAMICS FOR (3+1)-DIMENSIONAL COMBINED PKP-BKP EQUATION

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Abstract. This study explores the (3+1)-dimensional combined pKP-BKP equation and its variants, specifically when certain parameters vanish, using the Hirota bilinear method and symbolic computation. We have successfully derived various solutions, including breather waves, two-wave interactions, lump-periodic solutions, and other novel interaction forms. All the presented solutions have been verified to satisfy the original equations through back-substitution, aided by the Wolfram Mathematica package. To illustrate the nature of these solutions, their visual characteristics are graphically depicted. The results contribute valuable insights into fundamental nonlinear fluid dynamics and enhance our understanding of computational physics and engineering science within complex nonlinear, higher-dimensional wave fields.

1. Introduction

Nonlinear evolution equations arising in fluid dynamics, optical fibers, plasma dynamics, and related fields have been extensively studied in the literature. Integrable equations are known to admit exact multi-soliton solutions. Solitons are stable, localized waves whose shapes, amplitudes, and velocities remain unchanged during propagation. Their interactions are considered elastic if these properties remain invariant before and after the interaction, as exemplified by the Korteweg-de Vries (KdV) equation [1–12]. However, inelastic interactions can also occur, leading to phenomena such as fission and fusion of waves, as seen in the Burgers equation. Furthermore, both elastic and inelastic interactions in various continuous integrable equations have been widely analyzed and discussed.
Among the fundamental methods for obtaining soliton solutions are the inverse scattering transform [1-10], the Riemann-Hilbert technique [11–20], the Darboux transformation [16–25], and the Hirota direct method [6–36]. Notable solutions in mathematical physics, such as breather, complexiton, lump, and rogue wave solutions can be seen as particular reductions of soliton solutions under various conditions. Numerous effective approaches [12–24] have been employed to investigate the complete integrability of nonlinear evolution equations, addressing both elastic and inelastic interactions to produce new results in scientific research.

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Various prominent solutions, including breathers, lumps, and rogue waves, have been derived using a range of sophisticated techniques, such as the algebraicgeometric method, the inverse scattering method, the Backlund transformation, Painleve analysis, Lax integrability, the Darboux transformation [15–22], and the Hirota bilinear method [1–5, 22–36], among others. For example, in [34], solutions for a (2+1)-dimensional coupled partially nonlocal nonlinear Schrodinger equation were obtained via a coupled relation using the Darboux method. Similarly, in [35, 36], N-soliton solutions, breathers, and lumps for the (2+1)-dimensional Hirota-Satsuma-Ito equation were derived using the Hirota bilinear method, the complex conjugate parameter, and the long-wave limit method. The Hirota bilinear method is recognized as an efficient, convenient, and powerful technique for exploring integrable nonlinear models in various contexts, including Bose-Einstein condensates, plasma physics, ferromagnetic chains, water waves, and nonlinear optical fibers. Its effectiveness has attracted widespread attention among researchers. Integrable equations can be reformulated into Hirota bilinear forms through suitable dependent variable transformations. Moreover, computer algebra systems such as Maple and Mathematica are invaluable for managing the otherwise labor-intensive computations involved.
The potential Kadomtsev-Petviashvili (pKP) equation read as

$$V_{xt} + 6V_xV_{xx} + V_{xxxx} + aV_{yy} = 0. \tag{1.1}$$

obtained by V in the KdV equation with V_x, and integrating with respect to x once.
Moreover, the (2+1)-dimensional integrable B-type Kadomtsev-Petviashvili (BKP) equation is given by

$$(15(V_x)^3 + 15V_xV_{xxx} + V_{xxxxx})_x + 5(V_{xxxy} + 3(V_xV_y)_x) + V_{xt} - V_{yy} = 0, \tag{1.2}$$

that describes the interactions between exponentially localized structures, and has been used as a model for the shallow water wave in fluids and the electrostatic wave potential in plasmas [1–18]. Recently, Ma [1] proposed a combined form of the pKP eq. (1.1) and the BKP eq. (1.2), known as the pKP-BKP equation, expressed as

$$a_1(15(V_x)^3+15V_xV_{xxx}+V_{xxxxx})_x+a_2(6V_xV_{xx}+V_{xxxx})+a_3(V_{xxxy}+3(V_xV_y)_x)+a_4V_{xx}+a_5V_{xt}+a_6V_{yy} = 0, \tag{1.3}$$

where a_i are arbitrary constants and $a_5 \neq 0$. This newly proposed equation went under thorough studies with useful findings in [1–5]. In [1], the Hirota conditions for N-soliton solutions are studied and analyzed. Variety of soliton molecules were formed through solitons such as the kink, lump or breather for the physical quantity $V(x, y, t)$ in [2–5]. Resonant multi-soliton, M-breather, M-lump and hybrid solutions of a combined pKP-BKP equation were formally derived in [2–5]. Interactions such as collisions between lump waves and periodic waves, as well as collisions between lumps and single or double kink soliton solutions, were also explored in [6–36]. The combined pKP-BKP eq. (1.3), as a novel development, has been investigated in depth, producing many significant results.
In [37], this equation was further extended to a new (3+1)-dimensional pKP-BKP model, leading to additional advancements beyond the findings reported in [1–5]. The newly proposed (3+1)-dimensional pKP-BKP equation is given by:

$$V_{xt}+\alpha(15(V_x)^3+15V_xV_{xxx}+V_{xxxxx})_x+\beta(6V_xV_{xx}+V_{xxxx})+\gamma(V_{xxxy}+3(V_xV_y)_x)+aV_{xx}+bV_{xt}+cV_{xz}+\mu V_{yy} = 0, \tag{1.4}$$

where some additional terms, namely bV_{xy}, and cV_{xz} were added to the pKP-BKP eq. (1.3), a_5 is set equal to 1, and $v = V(x, y, z, t)$. The newly proposed pKP-BKP eq. (1.4) is slightly different from Eq. (1.3) because of the newly added terms and the extension to (3+1)-dimensions. The physical background of the governing eq. (1.4) represents the pioneer work of combining nonlinear evolution equations to reveal nonlinear integrable equations as discussed in [1]. Our main aim from this work is use the Hirota's method to to transform the extended (3+1)-dimensional combined pKP-BKP equation and its special cases into a bilinear form and then obtain breather wave, two-waves, lump-kink and other new interactions for each of the model equation and its special case.

2. The (3+1)-Dimensional Combined pKP-BKP Equation

The (3+1)-Dimensional Combined pKP-BKP Equation [37] is given by eq. (1.4).

2.1. The breather solution.

To obtain the breather solution for eq(1.4) we first use the transformation

$$V(x, y, z, t) = 2\frac{\partial \log(f(x, y, z, t))}{\partial x}, \tag{2.1}$$

to convert eq.(1.4) into a Hirota bilinear form as:

$$\begin{aligned} &f\left(af_{\text{xx}} + \text{b}f_{\text{xy}} + cf_{\text{xz}} + f_{\text{xt}} + \beta f_{\text{xxxx}} + \gamma f_{\text{xxxy}} + \mu f_{\text{yy}} + \alpha \text{f}_{\text{xxxxxx}}\right) - af_x^2 - f_y\left(bf_x + \gamma f_{\text{xxx}}\right) - cf_xf_z \\ &-f_tf_x - 4\beta f_xf_{\text{xxx}} - 6f_{\text{xxxxx}}\alpha f_x - 3\gamma f_xf_{\text{xxy}} + 15f_{\text{xxxx}}\alpha \text{f}_{\text{xx}} + 3\gamma f_{\text{xx}}f_{\text{xy}} - 10\alpha f_{\text{xxx}}^2 - \mu f_y^2 + 3\beta \text{f}_{\text{xx}}^2 = 0. \end{aligned} \tag{2.2}$$

To obtain the breather solution, we use the test function [38]:

$$f = q_2 \exp\left(p_1\chi_1\right) + \exp\left(-p_1\chi_1\right) + q_1 \cos\left(p_0\chi_2\right), \tag{2.3}$$

where $\chi_1 = \eta_1 t + x + y + \eta_2 z,\ \chi_2 = \eta_3 t + x + y + \eta_4 z$.

Substituting eq.(2.3) into eq.(2.2) then and performing the necessary manipulations, we obtain a polynomial in powers of exponential and trigonometric functions. Collecting the coefficients of the same power of trig and exponential functions and equating each to zero, yields an algebraic system of equations. Solving this system of equation, we obtain the values of the parameter that are involved.

Case One

When

$p_0 = -ip_1,\ q_1 = -2\sqrt{q_2},\ \eta_3 = -2a - 2b - c\eta_2 - c\eta_4 - \eta_1 - 2\mu - 32\alpha p_0^4 + 8\beta p_0^2.$

$$f = -2\sqrt{q_2}\cosh\left(p_1\left(k + x + y + \eta_4 z\right)\right) + q_2 e^{p_1(\eta_1 t + x + y + \eta_2 z)} + e^{-p_1(\eta_1 t + x + y + \eta_2 z)}. \tag{2.4}$$

Substitute eq.(2.4) into eq.(2.1), we obtain the exact breather solution as

$$V_1(x, y, z, t) = \frac{2\left(-2p_1\sqrt{q_2}\sinh\left(p_1\left(k + x + y + \eta_4 z\right)\right) + p_1 q_2 e^{p_1(\eta_1 t + x + y + \eta_2 z)} + p_1\left(-e^{-p_1(\eta_1 t + x + y + \eta_2 z)}\right)\right)}{-2\sqrt{q_2}\cosh\left(p_1\left(k + x + y + \eta_4 z\right)\right) + q_2 e^{p_1(\eta_1 t + x + y + \eta_2 z)} + e^{-p_1(\eta_1 t + x + y + \eta_2 z)}}, \tag{2.5}$$

where $k = t\left(-2a - 2b - c\eta_2 - c\eta_4 - \eta_1 - 2\mu - 32\alpha p_1^4 - 8\beta p_1^2\right)$.

2.2. **The two-waves solution.** To obtain the two-waves solution, we use the test function [38]:

$$f=\delta_3\sin\left(\chi_2\right)+\delta_4\sinh\left(\chi_3\right)+\delta_1\exp\left(\chi_1\right)+\delta_2\exp\left(-\chi_1\right), \tag{2.6}$$

where $\chi_1=\eta_1 t+x+y+\eta_2 z,\ \chi_2=\eta_3 t+x+y+\eta_4 z,\ \chi_3=\eta_5 t+x+y+\eta_6 z.$

Substituting eq.(2.6) into eq.(2.2) and performing the necessary manipulations, we obtain a polynomial in powers of exponential with trigonometric and exponential with hyperbolic functions. Collecting the coefficients of the same power of exponential with trigonometric and exponential with hyperbolic function and equating each to zero, yields an algebraic system of equations. Solving this system of equation, we obtain the values of the parameter that are involved.

Case One

When
$\delta_2=-\frac{\delta_4^2}{4\delta_1},\ \delta_3=0,\ \eta_5=-2a-32\alpha-2b-8\beta-c\eta_2-c\eta_6-8\gamma-\eta_1-2\mu$, we obtain

$$f=\delta_4\sinh\left(k+x+y+\eta_6 z\right)-\frac{\delta_4^2 e^{-\eta_1 t-x-y-\eta_2 z}}{4\delta_1}+\delta_1 e^{\eta_1 t+x+y+\eta_2 z}. \tag{2.7}$$

Substitute eq.(2.7) into eq.(2.1), we obtain the exact two-waves solution as

$$V_2(x,y,z,t)=\frac{2\left(\delta_4\cosh\left(k+x+y+\eta_6 z\right)+\frac{\delta_4^2 e^{-\eta_1 t-x-y-\eta_2 z}}{4\delta_1}+\delta_1 e^{\eta_1 t+x+y+\eta_2 z}\right)}{\delta_4\sinh\left(k+x+y+\eta_6 z\right)-\frac{\delta_4^2 e^{-\eta_1 t-x-y-\eta_2 z}}{4\delta_1}+\delta_1 e^{\eta_1 t+x+y+\eta_2 z}}, \tag{2.8}$$

where $k=t\left(-2a-32\alpha-2b-8\beta-c\eta_2-c\eta_6-8\gamma-\eta_1-2\mu\right).$

Case Two

When
$\delta_2=\frac{\delta_3^2(10\alpha-3\beta-3\gamma)}{4\delta_1(10\alpha+3\beta+3\gamma)},\ \delta_4=0;\eta_1=-a+4\alpha-b+2\beta-c\eta_2+2\gamma-\mu,\ \eta_3=-a+4\alpha-b-2\beta-c\eta_4-2\gamma-\mu,$
we obtain

$$f=\delta_3\sin(k)+\frac{\delta_3^2(10\alpha-3\beta-3\gamma)e^{-\mathrm{k1}-x-y-\eta_2 z}}{4\delta_1(10\alpha+3\beta+3\gamma)}+\delta_1 e^{\mathrm{k1}+x+y+\eta_2 z}. \tag{2.9}$$

Substitute eq.(2.9) into eq.(2.1), we obtain another exact two-waves solution as

$$V_3(x,y,z,t)=\frac{2\left(\delta_3\cos(k)-\frac{\delta_3^2(10\alpha-3\beta-3\gamma)e^{-\mathrm{k1}-x-y-\eta_2 z}}{4\delta_1(10\alpha+3\beta+3\gamma)}+\delta_1 e^{\mathrm{k1}+x+y+\eta_2 z}\right)}{\delta_3\sin(k)+\frac{\delta_3^2(10\alpha-3\beta-3\gamma)e^{-\mathrm{k1}-x-y-\eta_2 z}}{4\delta_1(10\alpha+3\beta+3\gamma)}+\delta_1 e^{\mathrm{k1}+x+y+\eta_2 z}}, \tag{2.10}$$

where $k=t\left(-a+4\alpha-b-2\beta-c\eta_4-2\gamma-\mu\right)+x+y+\eta_4 z,\quad \mathrm{k1}=t\left(-a+4\alpha-b+2\beta-c\eta_2+2\gamma-\mu\right).$

2.3. **The lump-periodic solution.** To obtain the lump-periodic solution, we use the test function [38]

$$f=q_2\cos\left(\chi_2\right)+q_1\cosh\left(\chi_1\right)+q_3\cosh\left(\chi_3\right). \tag{2.11}$$

Substituting eq.(2.11) into eq.(2.2) and performing the necessary manipulations, we obtain a polynomial in powers of hyperbolic with trigonometric functions. Collecting the coefficients of the same power and equating each to zero, yields an algebraic system of equations. Solving this system of equation, we obtain the values of the parameter that are involved.

Case One

When

$q_1 = q_3,\ q_2 = 0,\ \eta_5 = -2a - 32\alpha - 2b - 8\beta - c\eta_2 - c\eta_6 - 8\gamma - \eta_1 - 2\mu.$

we have,

$$f = q_3 \cosh(k + x + y + \eta_6 z) + q_3 \cosh(\eta_1 t + x + y + \eta_2 z)\,. \tag{2.12}$$

Substitute eq.(2.12) into eq.(2.1), we obtain the exact lump-periodic solution as

$$V_4(x,y,z,t) = \frac{2\left(q_3 \sinh(k + x + y + \eta_6 z) + q_3 \sinh(\eta_1 t + x + y + \eta_2 z)\right)}{q_3 \cosh(k + x + y + \eta_6 z) + q_3 \cosh(\eta_1 t + x + y + \eta_2 z)}, \tag{2.13}$$

where $k = t\left(-2a - 32\alpha - 2b - 8\beta - c\eta_2 - c\eta_6 - 8\gamma - \eta_1 - 2\mu\right).$

2.4. **The new interaction solution.** For the new interaction solution, we use the test function [38],

$$f = q_2 \exp(p_1\chi_1) + q_3 \exp(-p_1\chi_1) + q_1 \sin(p_0\chi_2) + q_4 \sinh(p_2\chi_3)\,, \tag{2.14}$$

where $\chi_1 = \eta_1 t + x + y + \eta_2 z,\ \chi_2 = \eta_3 t + x + y + \eta_4 z,\ \chi_3 = \eta_5 t + x + y + \eta_6 z.$

Substituting eq.(2.14) into eq.(2.2) and performing the necessary manipulations, we obtain a polynomial in powers of hyperbolic with trigonometric functions. Collecting the coefficients of the same power and equating each to zero, yields an algebraic system of equations. Solving this system of equation, we obtain the values of the parameter that are involved.

Case One

When

$p_1 = -ip_0,\ q_1 = -2\sqrt{q_2}\sqrt{q_3},\ q_4 = 0; \eta_3 = -2a - 2b - c\eta_2 - c\eta_4 - \eta_1 - 2\mu - 32\alpha p_0^4 + 8\beta p_0^2 + 8\gamma p_0^2.$

we have

$$f = -2\sqrt{q_2}\sqrt{q_3}\sin\left(p_0\left(k + x + y + \eta_4 z\right)\right) + q_2 e^{-ip_0(\eta_1 t + x + y + \eta_2 z)} + q_3 e^{ip_0(\eta_1 t + x + y + \eta_2 z)}. \tag{2.15}$$

Substitute eq.(2.15) into eq.(2.1), we obtain exact new interaction solution as

$$V_5(x,y,z,t) = \frac{2\left(-2p_0\sqrt{q_2}\sqrt{q_3}\cos\left(p_0\left(k + x + y + \eta_4 z\right)\right) + (-i)p_0 q_2 e^{-ip_0(\eta_1 t + x + y + \eta_2 z)} + ip_0 q_3 e^{ip_0(\eta_1 t + x + y + \eta_2 z)}\right)}{-2\sqrt{q_2}\sqrt{q_3}\sin\left(p_0\left(k + x + y + \eta_4 z\right)\right) + q_2 e^{-ip_0(\eta_1 t + x + y + \eta_2 z)} + q_3 e^{ip_0(\eta_1 t + x + y + \eta_2 z)}}, \tag{2.16}$$

where $k = t\left(-2a - 2b - c\eta_2 - c\eta_4 - \eta_1 - 2\mu - 32\alpha p_0^4 + 8\beta p_0^2 + 8\gamma p_0^2\right).$

Case Two

When

$p_1 = -p_2,\ q_1 = 0,\ q_4 = \frac{2\sqrt{q_2}\sqrt{q_3}\sqrt{a+b+c\eta_6+\eta_5+\mu+16\alpha p_2^4+4\beta p_2^2+4\gamma p_2^2}}{\sqrt{-a-b-c\eta_6-\eta_5-\mu-16\alpha p_2^4-4\beta p_2^2-4\gamma p_2^2}},$

$\eta_2 = \frac{-2a-2b-c\eta_6-\eta_1-\eta_5-2\mu-32\alpha p_2^4-8\beta p_2^2-8\gamma p_2^2}{c},$

we have

$$f = q_2 e^{-p_2\left(\frac{kz}{c} + \eta_1 t + x + y\right)} + q_3 e^{p_2\left(\frac{kz}{c} + \eta_1 t + x + y\right)} + \frac{2\sqrt{k}\sqrt{q_2}\sqrt{q_3}\sinh\left(p_2\left(\eta_5 t + x + y + \eta_6 z\right)\right)}{\sqrt{-k}}. \tag{2.17}$$

Substitute eq.(2.17) into eq.(2.1), we obtain another new interaction solution as

(2.18)

$$V_6(x,y,z,t)=\frac{2\left(-p_2q_2e^{-p_2\left(\frac{kz}{c}+\eta_1t+x+y\right)}+p_2q_3e^{p_2\left(\frac{kz}{c}+\eta_1t+x+y\right)}+\frac{2\sqrt{\mathrm{k1}}p_2\sqrt{q_2}\sqrt{q_3}\cosh(p_2(\eta_5t+x+y+\eta_6z))}{\sqrt{-\mathrm{k1}}}\right)}{q_2e^{-p_2\left(\frac{kz}{c}+\eta_1t+x+y\right)}+q_3e^{p_2\left(\frac{kz}{c}+\eta_1t+x+y\right)}+\frac{2\sqrt{\mathrm{k1}}\sqrt{q_2}\sqrt{q_3}\sinh(p_2(\eta_5t+x+y+\eta_6z))}{\sqrt{-\mathrm{k1}}}},$$

where $k=-2a-2b-c\eta_6-\eta_1-\eta_5-2\mu-32\alpha p_2^4-8\beta p_2^2-8\gamma p_2^2,\ \mathrm{k1}=a+b+c\eta_6+\eta_5+\mu+16\alpha p_2^4+4\beta p_2^2+4\gamma p_2^2$.

3. The (3+1)-Dimensional Combined pKP-BKP Equation with $\beta=0$

For $\beta=0$ the new (3+1)-Dimensional pKP-BKP Equation [37] is given as:

$$V_{xt}+\alpha(15(V_x)^3+15V_xV_{xxx}+V_{xxxxx})_x+\gamma(V_{xxxy}+3(V_xV_y)_x)+aV_{xx}+bV_{xt}+cV_{xz}+\mu V_{yy}=0. \tag{3.1}$$

3.1. **The breather solution.** To obtain the breather solution for eq.(3.1), we first use the transformation given in eq.(2.1) and convert eq.(3.1) into a Hirota bilinear form as:

(3.2)

$$f\left(af_{\mathrm{xx}}+\mathrm{bf}_{\mathrm{xy}}+cf_{\mathrm{xz}}+f_{\mathrm{xt}}+\gamma f_{\mathrm{xxxy}}+\mu f_{\mathrm{yy}}+\alpha\mathrm{f}_{\mathrm{xxxxxx}}\right)-af_x^2-f_y\left(bf_x+\gamma f_{\mathrm{xxx}}\right)-cf_xf_z-f_tf_x$$

$$-6f_{\mathrm{xxxxx}}\alpha\mathrm{f}_x-3\gamma f_xf_{\mathrm{xxy}}+15f_{\mathrm{xxxx}}\alpha\mathrm{f}_{\mathrm{xx}}+3\gamma f_{\mathrm{xx}}f_{\mathrm{xy}}-10\alpha f_{\mathrm{xxx}}^2-\mu f_y^2=0.$$

Substituting eq.(2.3) into eq.(3.2) then and performing the necessary manipulations, we obtain a polynomial in powers of exponential and trigonometric functions. Collecting the coefficients of the same powers of these functions and equating each to zero, yields an algebraic system of equations. Solving this system of equation, we obtain the values of the parameter that are involved.

Case One

When

$p_1=ip_0,\ q_1=-2\sqrt{q_2},\ \eta_3=-2a-2b-c\eta_2-c\eta_4-\eta_1-2\mu-32\alpha p_0^4+8\gamma p_0^2$, we have

$$f=-2\sqrt{q_2}\cos\left(p_0\left(k+x+y+\eta_4z\right)\right)+q_2e^{ip_0(\eta_1t+x+y+\eta_2z)}+e^{-ip_0(\eta_1t+x+y+\eta_2z)}. \tag{3.3}$$

Substitute eq.(3.3) into eq.(2.1), we obtain the breather solution to eq.(3.1) as:

(3.4)

$$V_1(x,y,z,t)=\frac{2\left(2p_0\sqrt{q_2}\sin\left(p_0\left(k+x+y+\eta_4z\right)\right)+ip_0q_2e^{ip_0(\eta_1t+x+y+\eta_2z)}+(-i)p_0e^{-ip_0(\eta_1t+x+y+\eta_2z)}\right)}{-2\sqrt{q_2}\cos\left(p_0\left(k+x+y+\eta_4z\right)\right)+q_2e^{ip_0(\eta_1t+x+y+\eta_2z)}+e^{-ip_0(\eta_1t+x+y+\eta_2z)}}.$$

where $k=t\left(-2a-2b-c\eta_2-c\eta_4-\eta_1-2\mu-32\alpha p_0^4+8\gamma p_0^2\right)$.

3.2. **The two-waves solution.** To obtain the two-waves solution, we use the test function given in eq.(2.5): Substituting eq.(2.6) into eq.(3.2) and performing the necessary manipulations, we obtain a polynomial in powers of exponential with trigonometric and exponential with hyperbolic functions. Collecting the coefficients of the same powers of these functions and equating each to zero, yields an algebraic system of equations. Solving this system of equation, we obtain the values of the parameter that are involved.

Case One

When

$\delta_2 = -\frac{\delta_4^2}{4\delta_1},\ \delta_3 = 0,\ \eta_5 = -2a - 32\alpha - 2b - c\eta_2 - c\eta_6 - 8\gamma - \eta_1 - 2\mu,$
we obtain

$$f = \delta_4 \sinh(k + x + y + \eta_6 z) - \frac{\delta_4^2 e^{-\eta_1 t - x - y - \eta_2 z}}{4\delta_1} + \delta_1 e^{\eta_1 t + x + y + \eta_2 z}. \tag{3.5}$$

Substitute eq.(3.5) into eq.(2.1), we obtain the exact two-waves solution to eq.(3.1) as

$$V_2(x,y,z,t) = \frac{2\left(\delta_4 \cosh(k + x + y + \eta_6 z) + \frac{\delta_4^2 e^{-\eta_1 t - x - y - \eta_2 z}}{4\delta_1} + \delta_1 e^{\eta_1 t + x + y + \eta_2 z}\right)}{\delta_4 \sinh(k + x + y + \eta_6 z) - \frac{\delta_4^2 e^{-\eta_1 t - x - y - \eta_2 z}}{4\delta_1} + \delta_1 e^{\eta_1 t + x + y + \eta_2 z}}, \tag{3.6}$$

where $k = t(-2a - 32\alpha - 2b - c\eta_2 - c\eta_6 - 8\gamma - \eta_1 - 2\mu)$.

Case Two

Wnen
$\delta_2 = \frac{\delta_3^2(10\alpha - 3\gamma)}{4\delta_1(10\alpha + 3\gamma)},\ \delta_4 = 0,\ \eta_1 = -a + 4\alpha - b - c\eta_2 + 2\gamma - \mu,\ \eta_3 = -a + 4\alpha - b - c\eta_4 - 2\gamma - \mu,$ we have

$$f = \delta_3 \sin(k) + \frac{\delta_3^2(10\alpha - 3\gamma)e^{-\mathrm{k1}t - x - y - \eta_2 z}}{4\delta_1(10\alpha + 3\gamma)} + \delta_1 e^{\mathrm{k1}t + x + y + \eta_2 z}. \tag{3.7}$$

Substitute eq.(3.7) into eq.(2.1), we obtain the another exact two-waves solution to eq.(3.1) as
$V_3(x,y,z,t) =$
(3.8)

$$\frac{2\left(-\frac{\delta_3^2(10\alpha-3\gamma)\exp(-t(-a+4\alpha-b-c\eta_2+2\gamma-\mu)-x-y-\eta_2 z)}{4\delta_1(10\alpha+3\gamma)} + \delta_1 \exp(t(-a+4\alpha-b-c\eta_2+2\gamma-\mu)+x+y+\eta_2 z) + \delta_3 \cos(k)\right)}{\frac{\delta_3^2(10\alpha-3\gamma)\exp(-t(-a+4\alpha-b-c\eta_2+2\gamma-\mu)-x-y-\eta_2 z)}{4\delta_1(10\alpha+3\gamma)} + \delta_1 \exp(t(-a+4\alpha-b-c\eta_2+2\gamma-\mu)+x+y+\eta_2 z) + \delta_3 \sin(k)},$$

where $k = t(-a + 4\alpha - b - c\eta_4 - 2\gamma - \mu) + x + y + \eta_4 z,\ \mathrm{k1} = -a + 4\alpha - b - c\eta_2 + 2\gamma - \mu.$

3.3. **The lump-periodic solution.** To obtain the lump-periodic solution, we use the test function eq.(2.11). Substituting eq.(2.11) into eq.(3.2) and performing the necessary manipulations, we obtain a polynomial in powers of hyperbolic with trigonometric functions. Collecting the coefficients of the same powers of thses functions and equating each to zero, yields an algebraic system of equations. Solving this system of equation, we obtain the values of the parameter that are involved.

Case One

When
$q_1 = -q_3,\ q_2 = 0,\ \eta_5 = -2a - 32\alpha - 2b - c\eta_2 - c\eta_6 - 8\gamma - \eta_1 - 2\mu,$ we obtain
(3.9)

$$f = q_3 \cosh(t(-2a - 32\alpha - 2b - c\eta_2 - c\eta_6 - 8\gamma - \eta_1 - 2\mu) + x + y + \eta_6 z) - q_3 \cosh(\eta_1 t + x + y + \eta_2 z).$$

Substitute eq.(3.9) into eq.(2.1), we obtain exact lump-periodic solution to eq.(3.1) as
(3.10)

$$V_4(x,y,z,t) = \frac{2(q_3 \sinh(t(-2a - 32\alpha - 2b - c\eta_2 - c\eta_6 - 8\gamma - \eta_1 - 2\mu) + x + y + \eta_6 z) - q_3 \sinh(\eta_1 t + x + y + \eta_2 z))}{q_3 \cosh(t(-2a - 32\alpha - 2b - c\eta_2 - c\eta_6 - 8\gamma - \eta_1 - 2\mu) + x + y + \eta_6 z) - q_3 \cosh(\eta_1 t + x + y + \eta_2 z)}.$$

3.4. **The new interaction solution.** For the new interaction solution, we use the test function given eq.(2.14). Substituting eq.(2.14) into eq.(3.2) and performing the necessary manipulations, we obtain a polynomial in powers of hyperbolic with trigonometric functions. Collecting the coefficients of the same powers of these functions and equating each to zero, yields an algebraic system of equations. Solving this system of equation, we obtain the values of the parameter that are involved.

Case One

When

$p_1 = -ip_0,\ q_1 = 2\sqrt{q_2}\sqrt{q_3},\ q_4 = 0,\ \eta_3 = -2a - 2b - c\eta_2 - c\eta_4 - \eta_1 - 2\mu - 32\alpha p_0^4 + 8\gamma p_0^2,$

we obtain

$$f = 2\sqrt{q_2}\sqrt{q_3}\sin(k) + q_2 e^{-ip_0(\eta_1 t + x + y + \eta_2 z)} + q_3 e^{ip_0(\eta_1 t + x + y + \eta_2 z)}. \tag{3.11}$$

Substitute eq.(3.11) into eq.(2.1), we obtain exact new interaction solution to eq.(3.1) as

$$V_5(x,y,z,t) = \frac{2\left(2p_0\sqrt{q_2}\sqrt{q_3}\cos(k) + (-i)p_0 q_2 e^{-ip_0(\eta_1 t + x + y + \eta_2 z)} + ip_0 q_3 e^{ip_0(\eta_1 t + x + y + \eta_2 z)}\right)}{2\sqrt{q_2}\sqrt{q_3}\sin(k) + q_2 e^{-ip_0(\eta_1 t + x + y + \eta_2 z)} + q_3 e^{ip_0(\eta_1 t + x + y + \eta_2 z)}}, \tag{3.12}$$

where $k = p_0\left(t\left(-2a - 2b - c\eta_2 - c\eta_4 - \eta_1 - 2\mu - 32\alpha p_0^4 + 8\gamma p_0^2\right) + x + y + \eta_4 z\right).$

Case-Two

When

$p_1 = p_2,\ q_1 = 0,\ q_4 = \frac{2\sqrt{q_2}\sqrt{q_3}\sqrt{a+b+c\eta_6+\eta_5+\mu+16\alpha p_2^4+4\gamma p_2^2}}{\sqrt{-a-b-c\eta_6-\eta_5-\mu-16\alpha p_2^4-4\gamma p_2^2}},\ \eta_2 = \frac{-2a-2b-c\eta_6-\eta_1-\eta_5-2\mu-32\alpha p_2^4-8\gamma p_2^2}{c},$

we obtain

$$f = q_2 e^{p_2\left(\frac{kz}{c} + \eta_1 t + x + y\right)} + q_3 e^{-p_2\left(\frac{kz}{c} + \eta_1 t + x + y\right)} + \frac{2\sqrt{\mathrm{k1}}\sqrt{q_2}\sqrt{q_3}\sinh\left(p_2\left(\eta_5 t + x + y + \eta_6 z\right)\right)}{\sqrt{-\mathrm{k1}}}. \tag{3.13}$$

Substitute eq.(3.13) into eq.(2.1), we obtain another exact new interaction solution to eq.(3.1) as

$$V_6(x,y,z,t) = \frac{2\left(p_2 q_2 e^{p_2\left(\frac{kz}{c} + \eta_1 t + x + y\right)} - p_2 q_3 e^{-p_2\left(\frac{kz}{c} + \eta_1 t + x + y\right)} + \frac{2\sqrt{\mathrm{k1}}p_2\sqrt{q_2}\sqrt{q_3}\cosh(p_2(\eta_5 t + x + y + \eta_6 z))}{\sqrt{-\mathrm{k1}}}\right)}{q_2 e^{p_2\left(\frac{kz}{c} + \eta_1 t + x + y\right)} + q_3 e^{-p_2\left(\frac{kz}{c} + \eta_1 t + x + y\right)} + \frac{2\sqrt{\mathrm{k1}}\sqrt{q_2}\sqrt{q_3}\sinh(p_2(\eta_5 t + x + y + \eta_6 z))}{\sqrt{-\mathrm{k1}}}}, \tag{3.14}$$

where $k = -2a - 2b - c\eta_6 - \eta_1 - \eta_5 - 2\mu - 32\alpha p_2^4 - 8\gamma p_2^2,\ \mathrm{k1} = a + b + c\eta_6 + \eta_5 + \mu + 16\alpha p_2^4 + 4\gamma p_2^2.$

4. The (3+1)-Dimensional combined pKP-BKP Equation with $\gamma = 0$

For $\gamma = 0$, the new (3+1)-Dimensional pKP-BKP Equation is given as [37]:

$$V_{xt} + \alpha(15(V_x)^3 + 15V_xV_{xxx} + V_{xxxxx})_x + \beta(6V_xV_{xx} + V_{xxxx}) + aV_{xx} + bV_{xt} + cV_{xz} + \mu V_{yy} = 0, \tag{4.1}$$

4.1. **The breather solution.** To obtain the breather solution for eq.(4.1) we first use the transformation given in eq.(2.1), to convert eq.(4.1) into a Hirota bilinear form as:

$$\begin{aligned} &f\left(af_{\mathrm{xx}} + \mathrm{b}f_{\mathrm{xy}} + cf_{\mathrm{xz}} + f_{\mathrm{xt}} + \beta f_{\mathrm{xxxx}} + \mu f_{\mathrm{yy}} + \alpha f_{\mathrm{xxxxxx}}\right) - af_x^2 - bf_xf_y - cf_xf_z - f_tf_x - 4\beta f_xf_{xxx} \\ &-6f_{\mathrm{xxxxx}}\alpha f_x + 15f_{\mathrm{xxxx}}\alpha f_{\mathrm{xx}} - 10\alpha f_{\mathrm{xxx}}^2 - \mu f_y^2 + 3\beta f_{\mathrm{xx}}^2 = 0. \end{aligned} \tag{4.2}$$

Substituting eq.(2.3) into eq.(4.2) then and performing the necessary manipulations, we obtain a polynomial in powers of exponential and trigonometric functions. Collecting the coefficients of the same powers of these functions and equating each to zero, yields an algebraic system of equations. Solving this system of equation, we obtain the values of the parameter that are involved.

Case One

When

$p_1 = ip_0,\ q_1 = -2\sqrt{q_2},\ \eta_3 = -2a - 2b - c\eta_2 - c\eta_4 - \eta_1 - 2\mu - 32\alpha p_0^4 + 8\beta p_0^2,$

we obtain

$$f = -2\sqrt{q_2}\cos(k) + q_2 e^{ip_0(\eta_1 t + x + y + \eta_2 z)} + e^{-ip_0(\eta_1 t + x + y + \eta_2 z)}. \tag{4.3}$$

Substitute eq.(4.3) into eq.(2.1), we obtain exact breather solution to eq.(4.1) as

$$V_1(x,y,z,t) = \frac{2\left(2p_0\sqrt{q_2}\sin(k) + ip_0 q_2 e^{ip_0(\eta_1 t + x + y + \eta_2 z)} + (-i)p_0 e^{-ip_0(\eta_1 t + x + y + \eta_2 z)}\right)}{-2\sqrt{q_2}\cos(k) + q_2 e^{ip_0(\eta_1 t + x + y + \eta_2 z)} + e^{-ip_0(\eta_1 t + x + y + \eta_2 z)}}, \tag{4.4}$$

where $k = p_0\left(t\left(-2a - 2b - c\eta_2 - c\eta_4 - \eta_1 - 2\mu - 32\alpha p_0^4 + 8\beta p_0^2\right) + x + y + \eta_4 z\right).$

4.2. **The Two-Waves Solution.** To obtain the two-waves solution, we use the test function given in eq.(2.6). Substituting eq.(2.6) into eq.(4.2) and performing the necessary manipulations, we obtain a polynomial in powers of exponential with trigonometric and exponential with hyperbolic functions. Collecting the coefficients of the same powers of these functions and equating each to zero, yields an algebraic system of equations. Solving this system of equation, we obtain the values of the parameter that are involved.

Case One

When

$\delta_2 = -\frac{\delta_4^2}{4\delta_1},\ \delta_3 = 0,\ \eta_5 = -2a - 32\alpha - 2b - 8\beta - c\eta_2 - c\eta_6 - \eta_1 - 2\mu,$

we obtain

$$f = \delta_4 \sinh(k) - \frac{\delta_4^2 e^{-\eta_1 t - x - y - \eta_2 z}}{4\delta_1} + \delta_1 e^{\eta_1 t + x + y + \eta_2 z}. \tag{4.5}$$

Substitute eq.(4.541) into eq.(2.1), we obtain exact two-waves solution to eq.(4.1) as

$$V_2(x,y,z,t) = \frac{2\left(\delta_4\cosh(k) + \frac{\delta_4^2 e^{-\eta_1 t - x - y - \eta_2 z}}{4\delta_1} + \delta_1 e^{\eta_1 t + x + y + \eta_2 z}\right)}{\delta_4\sinh(k) - \frac{\delta_4^2 e^{-\eta_1 t - x - y - \eta_2 z}}{4\delta_1} + \delta_1 e^{\eta_1 t + x + y + \eta_2 z}}, \tag{4.6}$$

where $k = t\left(-2a - 32\alpha - 2b - 8\beta - c\eta_2 - c\eta_6 - \eta_1 - 2\mu\right) + x + y + \eta_6 z.$

Case Two

When

$\delta_2 = \frac{\delta_3^2(10\alpha - 3\beta)}{4\delta_1(10\alpha + 3\beta)},\ \delta_4 = 0; \eta_1 = -a + 4\alpha - b + 2\beta - c\eta_2 - \mu,\ \eta_3 = -a + 4\alpha - b - 2\beta - c\eta_4 - \mu,$

we have

$$f = \frac{\delta_3^2(10\alpha - 3\beta)\exp(-t(-a + 4\alpha - b + 2\beta - c\eta_2 - \mu) - x - y - \eta_2 z)}{4\delta_1(10\alpha + 3\beta)} +$$

$$\delta_1 \exp\left(t\left(-a + 4\alpha - b + 2\beta - c\eta_2 - \mu\right) + x + y + \eta_2 z\right) + \delta_3 \sin(k). \tag{4.7}$$

Substitute eq.(4.7) into eq.(2.1), we obtain another exact two-waves solution to eq.(4.1) as

$$V_3(x,y,z,t) = \frac{2\left(-\frac{\delta_3^2(10\alpha-3\beta)\exp(-t(-a+4\alpha-b+2\beta-c\eta_2-\mu)-x-y-\eta_2 z)}{4\delta_1(10\alpha+3\beta)} + \delta_1\exp\left(t\left(-a+4\alpha-b+2\beta-c\eta_2-\mu\right)+x+y+\eta_2 z\right)+\delta_3\cos(k)\right)}{\frac{\delta_3^2(10\alpha-3\beta)\exp(-t(-a+4\alpha-b+2\beta-c\eta_2-\mu)-x-y-\eta_2 z)}{4\delta_1(10\alpha+3\beta)} + \delta_1\exp\left(t\left(-a+4\alpha-b+2\beta-c\eta_2-\mu\right)+x+y+\eta_2 z\right)+\delta_3\sin(k)}, \tag{4.8}$$

where $k = t\left(-a+4\alpha-b-2\beta-c\eta_4-\mu\right)+x+y+\eta_4 z$.

4.3. **The lump-periodic solution.** To obtain the lump-periodic solution, we use the test function eq.(2.11). Substituting eq.(2.11) into eq.(4.2) and performing the necessary manipulations, we obtain a polynomial in powers of hyperbolic with trigonometric functions. Collecting the coefficients of the same powers of these and equating each to zero, yields an algebraic system of equations. Solving this system of equation, we obtain the values of the parameter that are involved.

Case One When

$q_1 = -q_3,\ q_2 = 0,\ \eta_5 = -2a-32\alpha-2b-8\beta-c\eta_2-c\eta_6-\eta_1-2\mu,$

we obtain

$$f = q_3\cosh(k) - q_3\cosh\left(\eta_1 t+x+y+\eta_2 z\right). \tag{4.9}$$

Substitute eq.(4.9) into eq.(2.1), we obtain exact lump-periodic solution to eq.(4.1) as

$$V_4(x,y,z,t) = \frac{2\left(q_3\sinh(k) - q_3\sinh\left(\eta_1 t+x+y+\eta_2 z\right)\right)}{q_3\cosh(k) - q_3\cosh\left(\eta_1 t+x+y+\eta_2 z\right)}, \tag{4.10}$$

where $k = t\left(-2a-32\alpha-2b-8\beta-c\eta_2-c\eta_6-\eta_1-2\mu\right)+x+y+\eta_6 z$.

4.4. **The new interaction solution.** For the new interaction solution, we use the test function given in eq.(2.14). Substituting eq.(2.14) into eq.(4.2) and performing the necessary manipulations, we obtain a polynomial in powers of hyperbolic with trigonometric functions. Collecting the coefficients of the same powers of thses functions and equating each to zero, yields an algebraic system of equations. Solving this system of equation, we obtain the values of the parameter that are involved.

Case One

when

$p_1 = \sqrt{-p_0^2},\ q_1 = 2\sqrt{q_2}\sqrt{q_3},\ q_4 = 0; \eta_3 = -2a-2b-c\eta_2-c\eta_4-\eta_1-2\mu-32\alpha p_0^4+8\beta p_0^2$, we obtain

$$f = 2\sqrt{q_2}\sqrt{q_3}\sin(k) + q_2 e^{\sqrt{-p_0^2}(\eta_1 t+x+y+\eta_2 z)} + q_3 e^{-\sqrt{-p_0^2}(\eta_1 t+x+y+\eta_2 z)}. \tag{4.11}$$

Substitute eq.(4.11) into eq.(2.15), we obtain exact new interaction solution to eq.(4.1) as

$$V_5(x,y,z,t) = \frac{2\left(2p_0\sqrt{q_2}\sqrt{q_3}\cos(k) + \sqrt{-p_0^2}q_2 e^{\sqrt{-p_0^2}(\eta_1 t+x+y+\eta_2 z)} - \sqrt{-p_0^2}q_3 e^{-\sqrt{-p_0^2}(\eta_1 t+x+y+\eta_2 z)}\right)}{2\sqrt{q_2}\sqrt{q_3}\sin(k) + q_2 e^{\sqrt{-p_0^2}(\eta_1 t+x+y+\eta_2 z)} + q_3 e^{-\sqrt{-p_0^2}(\eta_1 t+x+y+\eta_2 z)}}, \tag{4.12}$$

where $k = p_0\left(t\left(-2a-2b-c\eta_2-c\eta_4-\eta_1-2\mu-32\alpha p_0^4+8\beta p_0^2\right)+x+y+\eta_4 z\right)$.

Case Two

when

$$p_1 = p_2,\ q_1 = 0,\ q_4 = \frac{2\sqrt{q_2}\sqrt{q_3}\sqrt{a+b+c\eta_6+\eta_5+\mu+16\alpha p_2^4+4\beta p_2^2}}{\sqrt{-a-b-c\eta_6-\eta_5-\mu-16\alpha p_2^4-4\beta p_2^2}},\ \eta_2 = \frac{-2a-2b-c\eta_6-\eta_1-\eta_5-2\mu-32\alpha p_2^4-8\beta p_2^2}{c},$$

we obtain

$$(4.13)\quad f = q_2 e^{p_2\left(\frac{kz}{c}+\eta_1 t+x+y\right)} + q_3 e^{-p_2\left(\frac{kz}{c}+\eta_1 t+x+y\right)} + \frac{2\sqrt{\mathrm{k1}}\sqrt{q_2}\sqrt{q_3}\sinh\left(p_2\left(\eta_5 t+x+y+\eta_6 z\right)\right)}{\sqrt{-\mathrm{k1}}}.$$

Substitute eq.(4.13) into eq.(2.1), we obtain another exact new interaction solution to eq.(4.1) as

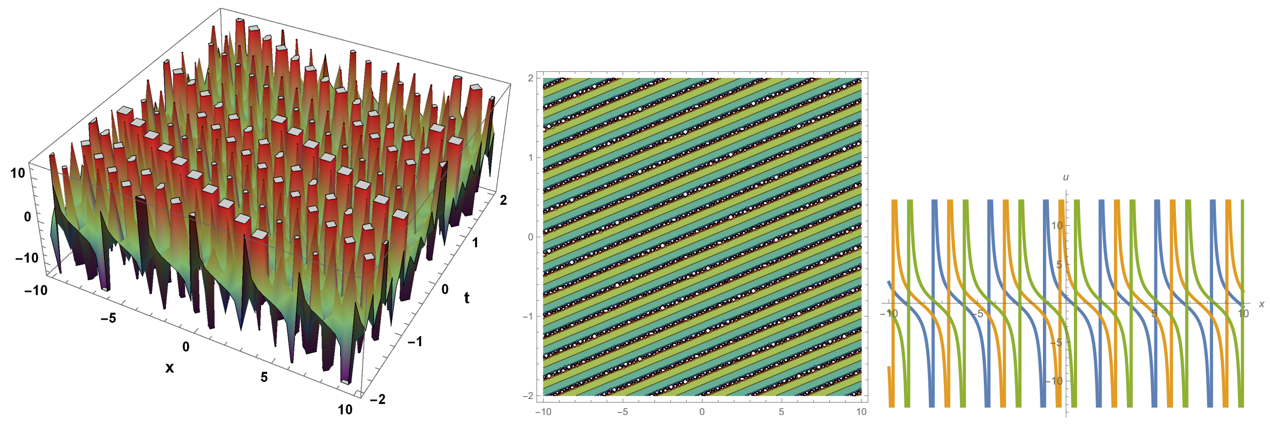
(4.14)

$$V_6(x,y,z,t) = \frac{2\left(p_2 q_2 e^{p_2\left(\frac{kz}{c}+\eta_1 t+x+y\right)} - p_2 q_3 e^{-p_2\left(\frac{kz}{c}+\eta_1 t+x+y\right)} + \frac{2\sqrt{\mathrm{k1}}\,p_2\sqrt{q_2}\sqrt{q_3}\cosh(p_2(\eta_5 t+x+y+\eta_6 z))}{\sqrt{-\mathrm{k1}}}\right)}{q_2 e^{p_2\left(\frac{kz}{c}+\eta_1 t+x+y\right)} + q_3 e^{-p_2\left(\frac{kz}{c}+\eta_1 t+x+y\right)} + \frac{2\sqrt{\mathrm{k1}}\sqrt{q_2}\sqrt{q_3}\sinh(p_2(\eta_5 t+x+y+\eta_6 z))}{\sqrt{-\mathrm{k1}}}}.$$

where $k = -2a-2b-c\eta_6-\eta_1-\eta_5-2\mu-32\alpha p_2^4-8\beta p_2^2,\ \mathrm{k1} = a+b+c\eta_6+\eta_5+\mu+16\alpha p_2^4+4\beta p_2^2$.

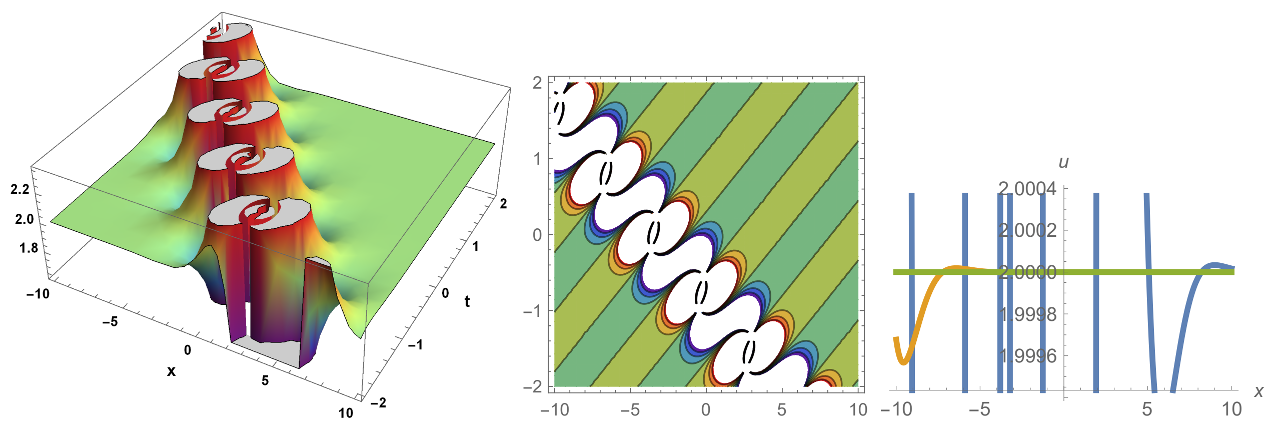
5. Graphical Representation

In this section, we shall present the graphical representation for some of the obtained results by choosing suitable values for the parameters that are involved.



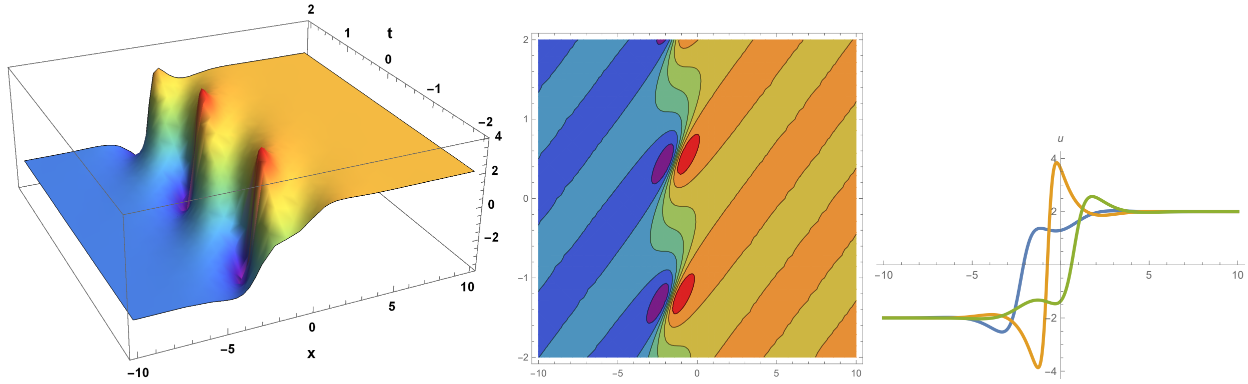
(A) 3D Plot (B) Contour Plot (C) 2D Plot

FIGURE 1. 3D, contour and 2D of $V_1(x,y,z,t)$ plot given by eq. (2.5)



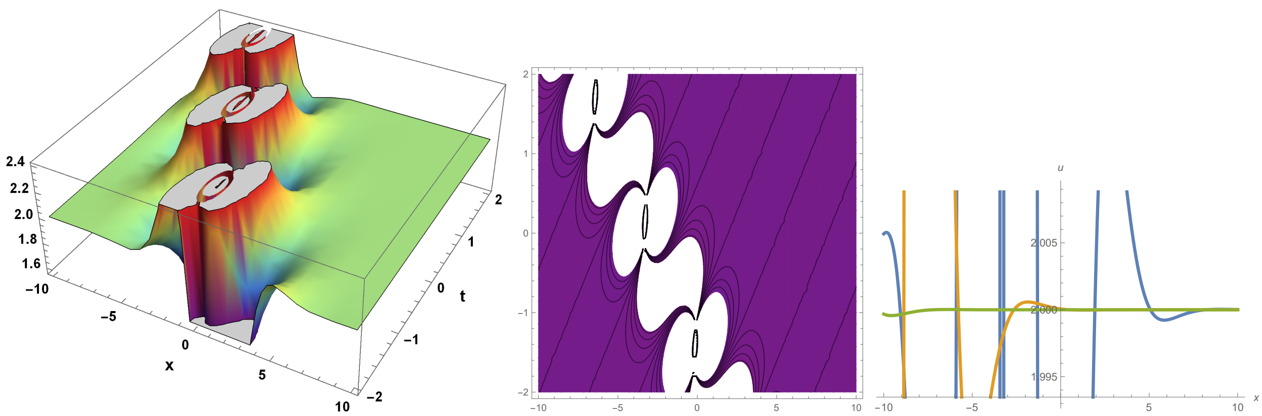
(A) 3D Plot (B) Contour Plot (C) 2D Plot

FIGURE 2. 3D, contour and 2D of $V_3(x,y,z,t)$ plot given by eq. (2.10)



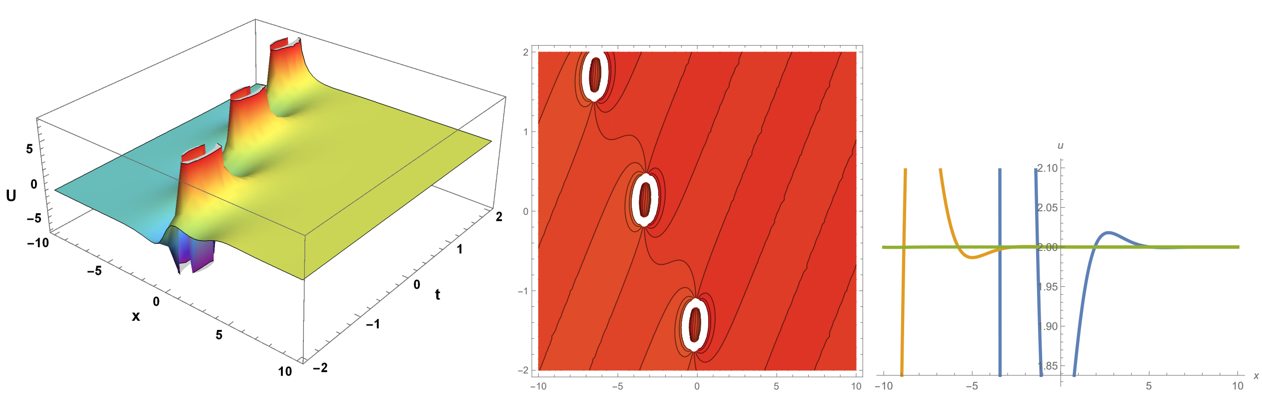
(A) 3D Plot (B) Contour Plot (C) 2D Plot

FIGURE 3. 3D, contour and 2D of $Re[V_3(x,y,z,t)]$ plot given by eq. (2.10)



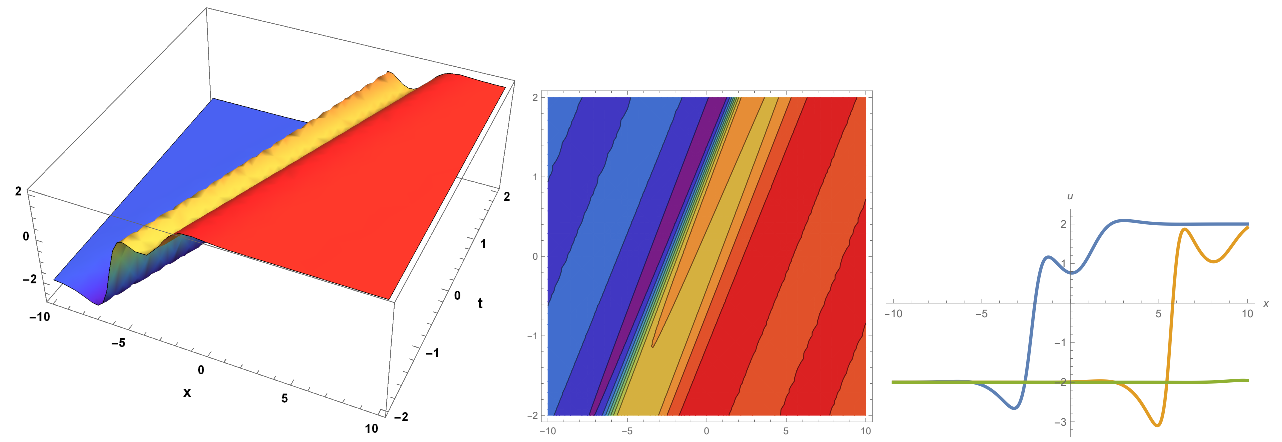
(A) 3D Plot (B) Contour Plot (C) 2D Plot

FIGURE 4. 3D, contour and 2D of $V_3(x,y,z,t)$ plot given by eq. (3.8)



(A) 3D Plot (B) Contour Plot (C) 2D Plot

FIGURE 5. 3D, contour and 2D of $V_3(x,y,z,t)$ plot given by eq. (4.8)



(A) 3D Plot (B) Contour Plot (C) 2D Plot

FIGURE 6. 3D, contour and 2D of $V_3(x, y, z, t)$ plot given by eq. (4.8)

In figure 1, we depict surface profile of the breather solution $V_1(x, y, z, t)$ given by eq.(2.5) in 3D, contour, and 2D by taking the following parameter values $p_0 = q_2 = -1,\ a = b = c = 1; \mu = \alpha = \beta = \gamma = \eta_1 = \eta_2 = \eta_4 = y = z = 1$.

In figure 2, we depict absolute plot of surface profile of the two-waves solution $V_3(x, y, z, t)$ given by eq.(2.10) in 3D, contour, and 2D by taking the following parameter values $\delta_1 = \delta_2 = \delta_3 = a = b = c = \mu = \alpha = \beta = \gamma = \eta_1 = \eta_2 = \eta_4 = y = z = 1$.

In figure 3, we depict real plot of surface profile of the two-waves solution $V_3(x, y, z, t)$ given by eq.(2.10) in 3D, contour, and 2D by taking the following parameter values $\delta_1 = 0.8,\ \delta_3 = 0.31,\ \eta_2 = 1.2,\ \eta_4 = -0.87,\ a = 1.46,\ b = -0.13,\ \mu = -1.39,\ c = -0.8,\ \gamma = 0.87,\ \alpha = -0.54,\ \beta = -0.39,\ z = -0.76,\ y = 1$.

In figure 4, we depict absolute plot of surface profile of the two-waves solution $V_3(x, y, z, t)$ given by eq.(3.8) in 3D, contour, and 2D by taking the following parameter values $\delta_1 = \delta_3 = 1; a = b = c = 1; \mu = \alpha = \beta = \gamma = \eta_1 = \eta_2 = \eta_4 = y = z = 1$.

In figure 5, we depict absolute plot of surface profile of the two-waves solution $V_3(x, y, z, t)$ given by eq.(4.8) in 3D, contour, and 2D by taking the following parameter values $\delta_1 = \delta_3 = 1; a = b = c = 1; \mu = \alpha = \beta = \gamma = \eta_1 = \eta_2 = \eta_4 = y = z = 1$.

In figure 6, we depict real plot of surface profile of the two-waves solution $V_3(x, y, z, t)$ given by eq.(4.8) in 3D, contour, and 2D by taking the following parameter values $\delta_1 = 0.8,\ p_0 = 1.3,\ p_2 = 1.3,\ q_2 = 0.56,\ q_3 = 0.56,\ \delta_3 = 0.8,\ \delta_4 = 0.31,\ \eta_1 = 1.2,\ \eta_2 = 1.2,\ \eta_4 = -0.87,\ \eta_5 = -0.87,\ \eta_6 = 1.2,\ a = 1.46,\ b = -0.13,\ \mu = -1.39,\ c = -0.8,\ \gamma = 0.87,\ \alpha = -0.54,\ \beta = -0.39,\ z = -0.76,\ y = 1$.

6. CONCLUSION

In this study, the (3+1)-dimensional combined pKP-BKP equation and its special cases were analyzed using the Hirota bilinear method alongside symbolic computation techniques. A variety of exact solutions, including breather waves, two-wave interactions, lump-kink structures, lump-periodic solutions, and other novel interaction forms were successfully derived. All obtained solutions were rigorously verified by substituting them back into the original equation with the aid of the Mathematica 12 software package. To further illustrate their physical relevance, the solutions' characteristics were graphically presented for selected parameter values.

The results of this work provide valuable insights into the behavior of nonlinear wave interactions in multi-dimensional settings. They have potential application plications in advancing our understanding of nonlinear fluid dynamics, plasma waves, and wave propagation in nonlinear optical fibers. Moreover, these findings contribute to the modeling and analysis of complex phenomena in computational physics, applied mathematics, and engineering disciplines where multidimensional nonlinear wave processes play a significant role, such as oceanography, atmospheric dynamics, and energy transport in advanced material systems.

Author Contributions

A.D.: Formal analysis, Software, Methodology, Validation, A.T: Investigation, Writing–review ,Supervision & editing.

Conflict of Interest

The authors declare that they have no conflict of interest.

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