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ON THE NEW ROBUST LYAPUNOV UNIFORM STABILITY APPROACH FOR NONLINEAR IMPULSIVE CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

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Abstract. This study investigates the uniform stability of the trivial solution for nonlinear impulsive Caputo fractional differential equations (ICFrDEs), leveraging on the powerful framework of the vector Lyapunov functions. By employing a novel class of piecewise continuous Lyapunov functions - an extension of traditional Lyapunov functions, and integrating comparison results, we derive comprehensive sufficient conditions for the uniform stability of the trivial solution of the system. To illustrate the applicability and advancements of these findings, a detailed example is provided, showcasing improvements over existing results and highlighting the broader potential of this approach.

1. Introduction

For the past thirty years, the theory of fractional differential equations (FrDEs) which is seen as an extension or the generalization of the traditional concept of differential equations, have been used to model various real life problems and phenomena (see [1,7,8]).

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The earliest study of the concept began about the 19th century during a mathematical discourse that transpired between Leibnitz and L'hopital when the former introduced the notation for derivative to be d^n/dx^n. This notation, however generated lots of questions among mathematicians - the question of whether the validity of the order n of the derivative will hold if it is extended to non-integer. In any case, as a direct question to Leibnitz, L'hopital in his letter asked: "what will be the result if $n = 1/2$?" In his response, Leibnitz averred, "it could be an apparent paradox from which one day useful consequences would be drawn." Since then, research interest in the study of fractional order derivatives has sparked of. With this development, it has been observed that the behavior of several systems, including phenomena with memory and hereditary characteristics can be modelled by fractional dynamical systems (see [3, 4, 10]). As an interesting developing area of research in calculus during the last couple of decades, fractional calculus offers an enormous robust characteristics with strong relevance in contemporary applications. For a more exhaustive discourse on the subject, we would refer interested readers to monographs [19, 32, 33].

Again, in the qualitative theory of FrDEs, one of the properties of research interest is the stability of solutions. As argued in [5, 11, 12], stability property enables us to analyse the behavior of solutions starting at varied points. Fundamental results on the stability properties of solutions of FrDEs using the scalar Lyapunov function were examined in [11, 15, 24, 33], and sufficient conditions for the stability as well as the uniform stability properties of FrDEs using the vector Lyapunov function were examined in [6–9].

Moreso, in the analysis of the stability properties of solutions of fractional order systems, one of the viable tool that is often employed is the Lyapunov's second method, also called the Lyapunov's direct method. This method has been argued among researchers to be more versatile in examining the stability properties of solutions compared to other approaches like the monotone iteration method, Laplace Adomian decomposition method, Laplace transform method, the Razumikhin technique, the use of matrix inequality, variational homotopy method, modified predictor-corrector method, Elzaki transform method, etc. This fact is infact premised on the ideal that the method allows us to examine the stability of solutions of differential systems without first solving the given systems. The approach involves seeking an appropriately continuous Lyapunov functions that is positive definite whose time derivative along the trajectory curve or solution path is negative semidefinite.

Furthermore, rapidly evolving alongside the theory of FrDEs is the mathematical theory of impulsive differential equations (IDEs) which are also considered as very relevant models for describing the true state of several real life processes and phenomena. The theory of IDEs is much more robust and veritable in the modelling of real life situations in engineering, physics, economics, computer science, finance, etc., compared to the corresponding theory of differential equations [21].

Now, many evolution processes are characterized by the fact that at certain moments of time they experience an abrupt change of state. These processes of perturbations are so abrupt that their duration is often times considered to be negligible in comparison with the overall duration of the process. The efficient applications of impulsive differential systems require the finding of criteria for stability of their solutions [34]. However, suffice to state here that, the use of Lyapunov's second method has a restriction in its application to IDEs, notwithstanding its versatility in the analysis of the stability properties of solutions of differential systems. As observed in [35], the application of classical (continuous) Lyapunov functions significantly limits the potential of the method, but since the solutions of the systems under assessment are piecewise continuous, it becomes necessary to use Lyapunov functions which are analogous with discontinuities of the first kind. Qualitative results on the stability properties of impulsive differential systems have been examined in [12,21,36]. Also, [7,9] examined the eventual stability properties of impulsive fractional order systems using the vector Lyapunov functions.

In this paper, the uniform stability of ICFrDEs is examined using the vector Lyapunov functions which is generalized by a class of piecewise continuous Lyapunov functions. Together with the comparison results, sufficient conditions for the uniform stability of the system is established with an illustrated example.

2. Preliminaries, Definitions and Notations

Let $\mathbb{R}^n$ be an n-dimensional Euclidean space with norm $\|.\|$, and let Ω be a domain in $\mathbb{R}^n$ containing the origin; $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R} = (-\infty, \infty)$, $t_0 \in \mathbb{R}_+, t > 0$.
Let $J \subset \mathbb{R}_+$ and define the following class of functions $PC^q[J, \Omega] = \alpha : J \to \Omega, \alpha(t)$ as piecewise continuous mapping with order q from the domain J into the range Ω with points of discontinuity $t_k \in J$ at which $\alpha(t_k^+)$ exists.
Fractional calculus being the generalization of the classical calculus to non integer order allows for the extension of the traditional concepts of derivative and integral to functions with fractional orders. It allows for functions with non integer orders which makes it much more flexible in describing real world systems. See [2, 20, 25].
There are several definitions of fractional derivatives and fractional integrals

General case. Let the number $n - 1 < q < n, q > 0$ be given, where n is a natural number and $\Gamma(.)$ denotes the gamma function.

1 The Riemann-Liouville fractional derivative of order q of $x(t)$ is given by (see [8]

$$ {}_{t_0}^{RL}D_t^q x(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_{t_0}^{t} (t-s)^{n-q-1} x(s) ds, t \geq t_0 $$

2 The Caputo fractional derivative of order q of $x(t)$ is defined by (see [7])

$$ {}_{t_0}^{C}D^q x(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{n-q-1} x^{(n)}(s) ds, t \geq t_0 $$

The Caputo derivatives has many properties that are similar to those of the standard derivatives, which makes them easier to understand and apply. The initial conditions of fractional differential equations using the Caputo derivative are also easier to interpret in physical context, which is another reason why it is often used in applications of fractional calculus.

3 The Grunwald-Letnikov fractional derivative of order q of $x(t)$ is given by (see [9])

$$ D_0^q x(t) = \lim_{h \to 0^+} \frac{1}{h^q} \sum_{r=0}^{[\frac{(t-t_0)}{h}]} (-1)^r ({}^qC_r) x(t - rh), t \geq t_0 $$

and the Grunwald-Letnikov fractional Dini derivative of order q of $x(t)$ is given by (see [11])

$$D_0^q x(t) = \limsup_{h\to 0^+} \frac{1}{h^q} \sum_{r=0}^{[\frac{(t-t_0)}{h}]} (-1)^r ({}^qC_r) x(t-rh), t \geq t_0$$

where ${}^qC_r$ are the binomial coefficients and $[\frac{(t-t_0)}{h}]$ denotes the integer part of $\frac{(t-t_0)}{h}$.

Particular case. (when n=1). In most applications, the order of q is often less than 1, so that $q \in (0,1)$. For simplicity of notation, we will use ${}^CD^q$ instead of ${}^C_{t_0}D^q$ and the Caputo fractional derivative of order q of the function $x(t)$ is

$$(1)\qquad {}^CD^q x = \frac{1}{\Gamma(1-q)} \int_{t_0}^t (t-s)^{-q} x' ds, t \geq t_0$$

3. Impulses in Fractional Order Systems

Consider the initial value problem (IVP) for the system of fractional differential equations (FrDE) with a Caputo derivative for $0 < q < 1$,

$$(2)\qquad {}^CD^q x = f(t,x), t \geq t_0, x(t_0) = x_0,$$

where $x \in \mathbb{R}^N$, $f \in C[\mathbb{R}_+ \times \mathbb{R}^N, \mathbb{R}^N]$, $f(t,0) \equiv 0$ and $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^N$.
Some sufficient conditions for the existence of the global solutions to (4) are considered in [30, 38, 39]. The IVP for FrDE (4) is equivalent to the following Volterra integral equation (See [12]),

$$(3)\qquad x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s, x(s)) ds, t \geq t_0$$

Consider the initial value problem for the system of impulsive fractional differential equations (IFrDE) with a Caputo derivative for $0 < q < 1$,

$$(4)\qquad \begin{aligned} {}^CD^q x &= f(t,x), t \geq t_0, t \neq t_k, k = 1,2,... \\ \Delta x &= I_k(x(t_k)), k \in N, t = t_k \\ x(t_0) &= x_0, \end{aligned}$$

where $x, x_0 \in \mathbb{R}^N$, $f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$, and $t_0 \in \mathbb{R}_+$, $I_k : \mathbb{R}^N \to \mathbb{R}^N$, $k = 1,2,...$
under the following assumptions: (A_0)

(i) $0 < t_1 < t_2 < ... < t_k < ...$, and $t_k \to \infty$ as $k \to \infty$;

(ii) $f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$, is continuous in $(t_{k-1}, t_k]$ and for each $x \in \mathbb{R}^N$, $k = 1,2,...$, $\lim_{(t,y)\to(t_k^+,x)} f(t,y) = f(t_k^+, x)$ exists;

(iii) $I_k : \mathbb{R}^N \to \mathbb{R}^N$

In this paper, we assume that $f(t,0) \equiv 0$, $I_k(0) = 0$ for all k, so that we have the trivial solution for (4), and the points $t_k, k = 1,2,...$ are fixed such that $t_1 < t_2 < ...$ and $\lim_{k\to\infty} t_k = \infty$. The system (4) with initial condition $x(t_0) = x_0$ is assumed to have a solution $x(t; t_0, x_0) \in PC^q([t_0, \infty), \mathbb{R}^N)$.

Remark 3.1. *The second equation in (4) is called the impulsive condition, and the function $I_k(x(t_k))$ gives the amount of jump of the solution at the point t_k.*

The function V is continuous in $(t_{k-1}, t_k] \times \mathbb{R}^N$ and for each $x \in \mathbb{R}^N, k = 1, 2, ...,$ $\lim\limits_{(t,y)\to(t_k^+,x)} V(t, y) = V(t_k^+, x), V$ is locally Lipschitzian in x and $V(t, 0) \equiv 0$.
Now, for any function $V(t, x) \in PC([t_0, \infty) \times \xi, \mathbb{R}_+^N)$ we define the Caputo fractional Dini derivative as:
(5)

$$^cD_+^q V(t,x) = \limsup_{h\to 0^+} \frac{1}{h^q}\{V(t,x) - V(t_0,x_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}({}^qC_r)[V(t-rh, x-h^q f(t,x)) - V(t_0,x_0)]\}$$

$t \geq t_0$ where $t \in [t_0, \infty), x, x_0 \in \xi$, $\xi \subset \mathbb{R}^N$ and there exists $h > 0$ such that $t - rh \in [t_0, T)$

Definition 3.1. *A function $g \in PC[\mathbb{R}^n, \mathbb{R}^n]$ is said to be quasi-monotone non-decreasing in x, if $x \leq y$ and $x_i = y_i$ for $1 \leq i \leq n$ implies $g_i(x) \leq g_i(y), \forall i$.*

Definition 3.2. *The zero solution of (4) is said to be:*
$(S1)$ stable if for every $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$ there exist $\delta = \delta(\epsilon, t_0) > 0$, continuous in t_0 such that for any $x_0 \in \mathbb{R}^N$, $\|x_0\| \leq \delta$ implies $\|x(t; t_0, x_0)\| < \epsilon$ for $t \geq t_0$.
$(S2)$ uniformly stable if for every $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$ there exist $\delta = \delta(\epsilon) > 0$, continuous in t_0 such that for any $x_0 \in \mathbb{R}^N$, $\|x_0\| \leq \delta$ implies $\|x(t; t_0, x_0)\| < \epsilon$ for $t \geq t_0$.

Definition 3.3. *A function $a(r)$ is said to belong to the class $\mathcal{K}$ if $a \in PC([0, \psi), \mathbb{R}_+], a(0) = 0$, and $a(r)$ is strictly monotone increasing in r.*

In this paper, we define the following sets:

$$\bar{S}_\psi = \{x \in \mathbb{R}^N : \|x\| \leq \psi\}$$
$$S_\psi = \{x \in \mathbb{R}^N : \|x\| < \psi\}$$

It suffices to say that the inequalities between vectors are understood to be component-wise inequalities.

We will use the comparison results for the impulsive Caputo fractional differential equation of the type

$$\begin{aligned} {}_{t_0}^{c}D^q u = g(t, u), t \geq t_0, t \neq t_k, k = 1, 2, ... \\ \Delta u = \psi_k(u(t_k)), k \in N, t = t_k \\ u(t_0^+) = u_0, \end{aligned} \tag{6}$$

existing for $t \geq t_0$, where $u \in \mathbb{R}^n, \mathbb{R}_+ = [t_0, \infty), g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n, g(t, 0) \equiv 0$,
where g is the continuous mapping of $\mathbb{R}_+ \times \mathbb{R}^n$ into $\mathbb{R}^n$. The function $g \in PC[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ is such that for any initial data $(t_0, u_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, the system (6) with initial condition $u(t_0) = u_0$ is assumed to have a solution $u(t; t_0, u_0) \in PC^q([t_0, \infty), \mathbb{R}^n)$.

Lemma 3.1. *Assume $m \in PC([t_0, T] \times \bar{S}_\psi, \mathbb{R}^N)$ and suppose there exists $t^* \in (t_0, T]$ such that for $\alpha_1 < \alpha_2$, $m(t^*, \alpha_1) = m(t^*, \alpha_2)$ and $m(t, \alpha_1) < m(t, \alpha_2)$ for $t_0 \leq t < t^*$. Then if the Caputo fractional Dini derivative of m exists at t^*, then the inequality ${}^CD_+^q m(t^*, \alpha_1) - {}^CD_+^q m(t^*, \alpha_2) > 0$ holds.*

Proof. Let $V(t,x) = m(t,\alpha_1) - m(t,\alpha_2)$.
Applying (5), we have

$$\begin{aligned} {}^{C}D_{+}^{q}(m(t^*,\alpha_1) - m(t^*,\alpha_2)) &= \limsup_{h\to 0^+} \frac{1}{h^q}\{[m(t^*,\alpha_1) - m(t^*,\alpha_2)] - [m(t_0,\alpha_1) - m(t_0,\alpha_2)] \\ &- \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} qC_r[m(t^*-rh,\alpha_1) - m(t^*-rh,\alpha_2)] \\ &-[m(t_0,\alpha_1) - m(t_0,\alpha_2)]\} \end{aligned}$$

When $m(t^*,\alpha_1) = m(t^*,\alpha_2)$, we have

$$\begin{aligned} {}^{C}D_{+}^{q}(m(t^*,\alpha_1) - m(t^*,\alpha_2)) &= \limsup_{h\to 0^+} \frac{1}{h^q}\{-[m(t_0,\alpha_1) - m(t_0,\alpha_2)] - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} qC_r \\ &[m(t^*-rh,\alpha_1) - m(t^*-rh,\alpha_2)] - [m(t_0,\alpha_1) - m(t_0,\alpha_2)]\} \\ &= -\limsup_{h\to 0^+} \frac{1}{h^q}\{[m(t_0,\alpha_1) - m(t_0,\alpha_2)] \\ &+ \limsup_{h\to 0^+} \frac{1}{h^q} \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r} qC_r[m(t^*-rh,\alpha_1) - m(t^*-rh,\alpha_2)] \\ &- \limsup_{h\to 0^+} \frac{1}{h^q} \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r} qC_r[m(t_0,\alpha_1) - m(t_0,\alpha_2)] \\ &= -\limsup_{h\to 0^+} \frac{1}{h^q}\{[m(t_0,\alpha_1) - m(t_0,\alpha_2)] \\ &- \limsup_{h\to 0^+} \frac{1}{h^q} \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r} qC_r[m(t_0,\alpha_1) - m(t_0,\alpha_2)] \\ &= -\limsup_{h\to 0^+} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^{r} qC_r[m(t_0,\alpha_1) - m(t_0,\alpha_2)] \end{aligned}$$

Applying equation 3.8 in [11], we have

$$ {}^{c}D_{+}^{q}(m(t^*,\alpha_1) - m(t^*,\alpha_2)) = -\frac{(t-t_0)^{-q}}{\Gamma(1-q)}[m(t_0,\alpha_1) - m(t_0,\alpha_2)] $$

By the lemma, we have that

$$ m(t,\alpha_1) - m(t,\alpha_2) < 0, \text{for } t_0 \le t < t^* $$

And so it follows that

$$ {}^{c}D_{+}^{q}(m(t^*,\alpha_1) - m(t^*,\alpha_2)) > 0 $$

□

Remark 3.2. *Lemma* 3.2 *extends Lemma* 1 *in [11], where the vectors* $m(t,\alpha_1)$ *and* $m(t,\alpha_2)$ *are compared component-wise.*

4. Fractional Differential Inequalities and Comparison results for Impulsive vector Fractional Differential Equations

In this section, we assume that $0 < q < 1$.

Theorem 4.1. *Assume that*

(i) $g \in PC[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ *and is continuous in* $(t_{k-1}, t_k]$, $k = 1, 2, ...$ *and* $g(t, u)$ *is quasimonotone nondecreasing in* u *for each* $u \in \mathbb{R}^n$ *and* $\lim_{(t,y)\to(t_k^+,u)} g(t, u) = g(t_k^+, u)$ *exists;*

(ii) $V \in PC[\mathbb{R}_+ \times \mathbb{R}^N, \mathbb{R}_+^N]$ *be locally Lipschitzian in* x *such that*

$$^{C}D_+^q V(t, x) \le g(t, V(t, x)), t \ne t_k, (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N$$

and

$$V(t, x + I_k(x(t_k))) \le \rho_k(V(t, x)), t = t_k, x \in S_\psi$$

and the function $\rho_k : \mathbb{R}_+^N \to \mathbb{R}_+^N$ *is nondecreasing for* $k = 1, 2, ...$

(iii) $r(t) = r(t; t_0, u_0) \in PC^q([t_0, T], \mathbb{R}^n)$ *is the maximal solution of the the IVP for the IFrDE* (6).

Then,

$$V(t, x(t)) \le r(t), t \ge t_0 \tag{7}$$

where $x(t) = x(t; t_0, x_0) \in PC^q([t_0, T], \mathbb{R}^N)$ *is any solution of* (4) *existing on* $[t_0, \infty)$, *provided that*

$$V(t_0^+, x_0) \le u_0. \tag{8}$$

Proof. Let $\eta \in \bar{S}_\psi =: \{\eta \in \mathbb{R}^n : \|\eta\| \le \psi\}$ be a small enough arbitrary vector and consider the initial value problem for the following system of fractional differential equations.

$$\begin{aligned} {}^{c}D^q u = g(t, u) + \eta, \Delta u = \psi_k(u(t_k)), t = t_k, k = 1, 2, ... \\ u(t_0^+) = u_0 + \eta \end{aligned} \tag{9}$$

for $t \in [t_0, \infty)$.

The function $u_\eta(t, \alpha)$ is a solution of (9), where $\alpha > 0$, if and only if it satisfies the Volterra Integral equation

$$u_\eta(t, \alpha) = u_0 + \eta + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1}(g(s, u_\eta(s, \alpha)) + \eta)ds, \; t \in [t_0, \infty) \tag{10}$$

Let the function $m(t, \alpha) \in PC([t_0, T] \times \bar{S}_\psi, \mathbb{R}^N)$ be defined as $m(t, \alpha) = V(t, x^*(t))$.

We now prove that

$$m(t, \alpha) < u_\eta(t, \alpha), \quad \text{for} \quad t \in [t_0, \infty) \tag{11}$$

Observe that the inequality (11) holds for $t = t_0$ i.e

$$m(t_0, \alpha) = V(t_0, x_0) \le u_0 < u_\eta(t_0, \alpha)$$

Assume that the inequality (11) is not true, then there exist a point $t_1 > t_0$ such that

$$m(t_1, \alpha) = u_\eta(t_1, \alpha) \quad \text{and} \quad m(t, \alpha) < u_\eta(t, \alpha) \quad \text{for} \quad t \in [t_0, t_1)$$

It follows from Lemma (3.1) that

$$^{C}D_{+}^{q}m(t_1,\alpha) -^{C} D_{+}^{q}u_{\eta}(t_1,\alpha) > 0$$

So that

$$^{C}D_{+}^{q}(V(t_1,x(t_1))) >^{C} D_{+}^{q}(u_{\eta}(t_1,\alpha))$$

and using (9) we arrive at

$$^{C}D_{+}^{q}(V(t_1,x(t_1))) > g(t_1,u_{\eta}(t_1,\alpha)+\eta) > g(t_1,u(t_1,\alpha))$$

Therefore,

$$^{C}D_{+}^{q}(m(t_1,\alpha)) > g(t_1,u(t_1,\alpha)) \tag{12}$$

For $t \in [t_0,T]$, we maintain that $x^*(t)$ satisfies (4) and the equality,

$$\limsup_{h\to 0^+} \frac{1}{h^q}[x^*(t) - x_0 - S(x^*(t),h)] = f(t,x^*(t)) \tag{13}$$

holds, where $x^*(t)$ is any other solution of (4).

$$S(x^*(t),h) = \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}(^{q}C_r)[x^*(t-rh)-x_0)] \tag{14}$$

is the Grunwald Letnikov fractional derivative and $[\frac{t-t_0}{h}]$ is the integer part of $\frac{t-t_0}{h}$.

Multiply (13) through by h^q we have,

$$\begin{aligned}
\limsup_{h\to 0^+}[x^*(t) - x_0 - S(x^*(t),h)] &= h^q f(t,x^*(t))\\
x^*(t) - x_0 - \limsup_{h\to 0^+}[S(x^*(t),h)] &= h^q f(t,x^*(t))\\
x^*(t) - x_0 - [S(x^*(t),h)+\rho(h^q)] &= h^q f(t,x^*(t))
\end{aligned}$$

$$x^*(t) - h^q f(t,x^*(t)) = [S(x^*(t),h) + x_0 + \rho(h^q)] \tag{15}$$

For $t \in [t_0,T]$, we have

$$m(t,\alpha) - m(t_0,\alpha) - \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}(^{q}C_r)[m(t-rh,\alpha)-m(t_0,\alpha)]$$

$$= V(t,x^*(t)) - V(t_0,x_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1q}C_r[V(t-rh,x^*(t)-h^q f(t,x^*(t)) - V(t_0,x_0)]$$

$$\begin{aligned}
= \;& V(t,x^*(t)) - V(t_0,x_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1q}C_r[V(t-rh,x^*(t)-h^q f(t,x^*(t)) - V(t_0,x_0)]\\
+ \;& \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}(^{q}C_r)\{[V(t-rh,S(x^*(t),h)+x_0+\rho(h^q) - V(t_0,x_0)]\\
& -[V(t-rh,x^*(t-rh) - V(t_0),x_0)]\}
\end{aligned} \tag{16}$$

Since $V(t,x)$ is locally Lipschtzian in the second variable, we have

$$\leq L|(-1)^{r+1}|\|\sum_{r=1}^{[\frac{t-t_0}{h}]}({}^qC_r)[S(x^*(t),h)+x_0+\rho(h^q)-x^*(t-rh)]\|$$

where $L>0$ is the Lipschitz constant.

$$\leq L\|\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^qC_r[S(x^*(t),h)+\rho(h^q)-(x^*(t-rh)-x_0)]\| \tag{17}$$

Using equations (14), equation (17) becomes,

$$\leq L\|\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^qC_r(\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1q}C_r[x^*(t-rh)-x_0)]+\rho(h^q)-(x^*(t-rh)-x_0)]\|$$

$$\begin{aligned}\leq\ & L\|\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^qC_r(-1)^{r+1}(\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^qC_r[x^*(t-rh)-x_0)]+\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^qC_r\rho(h^q)\\ &-\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^qC_r(x^*(t-rh)-x_0)]\|\end{aligned}$$

$$\leq L(-1)^{r+1}\|\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^qC_r(x^*(t-rh)-x_0)[\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^qC_r-1]+\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^qC_r\rho(h^q)\| \tag{18}$$

Substituting equation (4.12) into (4.10) yields,

$$=V(t,x^*(t))-V(t_0,x_0)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1q}C_r[V(t-rh,x^*(t)-h^qf(t,x^*(t))-V(t_0,x_0)]$$

$$+L\|\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1q}C_r(x^*(t-rh)-x_0)[\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1q}C_r-1]+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1q}C_r\rho(h^q)\|$$

$$=V(t,x^*(t))-V(t_0,x_0)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1q}C_r[V(t-rh,x^*(t)-h^qf(t,x^*(t))-V(t_0,x_0)]$$

$$+L\|\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1q}C_r(x^*(t-rh)-x_0)[-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{rq}C_r-1]+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1q}C_r\rho(h^q)\|$$

Dividing through by $h^q>0$ and taking the *limsup* as $h\to 0^+$ we have,

$$\begin{aligned}{}^CD_+^qm(t,\alpha)\ =\ & \limsup_{h\to0^+}\frac{1}{h^q}[V(t,x^*(t))-V(t_0,x_0)\\ &-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1q}C_r[V(t-rh,x^*(t)-h^qf(t,x^*(t))-V(t_0,x_0)]]\\ &+\limsup_{h\to0^+}\frac{1}{h^q}L\|\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1q}C_r(x^*(t-rh)-x_0)[-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{rq}C_r-1]\end{aligned}$$

$$+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1q}C_r\rho(h^q)\|$$

Recall

$$\lim_{h\to 0^+}\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{rq}C_r=-1 \; and \; \limsup_{h\to 0^+}\frac{1}{h^q}\rho(h^q)=0$$

from equations (3.6) and (3.7) in [11] we have,

$$^C D_+^q m(t,\alpha) = ^C D_+^q V(t,x^*(t)) + L\|\sum_{r=1}^{[\frac{t-t_0}{h}]}(x^*(t-rh)-x_0)[-(-1)-1]+0\|$$

$$^C D_+^q m(t,\alpha) = ^C D_+^q V(t,x^*(t)) + 0$$

Using condition (ii) of Theorem 4.1 we have

$$^C D_+^q m(t,\alpha) \le g(t,V(t,x^*(t))) = g(t,m(t,\alpha)) \tag{19}$$

Also,

$$m(t_0^+,\alpha)\le u_0 \; and \; m(t_k^+,\alpha) = V(t_k^+,x(t_k)+I_k(x(t_k))\le \rho_k(m(t_k)) \tag{20}$$

Now, equation (19) with $t=t_1$ contradicts (12), hence (11) holds.

□

For $t\in[t_0,T]$, we now show that whenever $\eta_1<\eta_2$, then

$$u_{\eta_1}(t,\alpha) < u_{\eta_2}(t,\alpha) \tag{21}$$

It is obvious that (21) holds for $t=t_0$. Assume the inequality (21) is not true. Then there exist a point $t_1>t_0$ such that $u_{\eta_1}(t_1,\alpha)=u_{\eta_2}(t_1,\alpha)$ and $u_{\eta_1}(t,\alpha)<u_{\eta_2}(t,\alpha)$ for $t\in[t_0,t_1)$. By lemma (3.1), we have that

$$^C D_+^q(u_{\eta_1}(t_1,\alpha)-u_{\eta_2}(t_1,\alpha))>0$$

However,

$$\begin{aligned} ^C D_+^q(u_{\eta_1}(t_1,\alpha)-u_{\eta_2}(t_1,\alpha_)) &= {}^C D_+^q u_{\eta_1}(t_1,\alpha) - ^C D_+^q u_{\eta_2}(t_1,\alpha) \\ &= g(t_1,u(t_1,\alpha)+\eta_1)) - [g(t_1,u(t_1,\alpha)+\eta_2))] \\ &= \eta_1-\eta_2<0 \end{aligned}$$

which is a contradiction and so (4.15) is true. Thus, equations (4.5) and (4.15) guarantee that the family of solutions $\{u_\eta(t,\alpha)\}$, $t\in[t_0,T]$ of (4.3) is uniformly bounded, i.e. there exists $P>0$ with $|u_\eta(t,\alpha)|\le P$, with bound P on $[t_0,T]$.
We now show that the family $\{u_\eta(t,\alpha)\}$ is equicontinuous on $[t_0,T]$. Assume $K=sup\{g(t,x):(t,x)\in[t_0,T]\times[-P,P]\}$. Also, fix a decreasing sequence $\{\eta_i\}_{i=1}^{\infty}(t)$, such that $\lim_{i\to\infty}\eta_i=0$ and

consider a sequence of functions $u_{\eta_i}(t,\alpha)$. Again let $t_1, t_2 \in [t_0, T]$ with $t_1 < t_2$, then we have the following estimate

$$\begin{aligned}
\|u_{\eta_i}(t_2,\alpha) - u_{\eta_i}(t_1,\alpha)\| &= \|u_0 + \eta_i + \frac{1}{\Gamma(q)}\int_{t_0}^{t_2}(t_2-s)^{q-1}(g(s,u_{\eta_i}(s,\alpha)) + \eta_i) \\
&- (u_0 + \eta_i + \frac{1}{\Gamma(q)}\int_{t_0}^{t_1}(t_1-s)^{q-1}(g(s,u_{\eta_i}(s,\alpha)) + \eta_i))\| \\
&= \frac{1}{\Gamma(q)}\|\int_{t_0}^{t_2}(t_2-s)^{q-1}(g(s,u_{\eta_i}(s,\alpha)))ds \\
&- \int_{t_0}^{t_1}(t_1-s)^{q-1}(g(s,u_{\eta_i}(s,\alpha)))ds\| \\
&\leq \frac{k}{\Gamma(q)}\left|\int_{t_0}^{t_2}(t_2-s)^{q-1} - \int_{t_0}^{t_1}(t_1-s)^{q-1}\right|ds \\
&= \frac{k}{\Gamma(q)}\left|-(\int_{t_0}^{t_1}(t_1-s)^{q-1} - \int_{t_0}^{t_2}(t_2-s)^{q-1})\right|ds \\
&= \frac{k}{\Gamma(q)}\left|\int_{t_0}^{t_1}(t_1-s)^{q-1} - (\int_{t_0}^{t_1}(t_2-s)^{q-1} + \int_{t_1}^{t_2}(t_2-s)^{q-1})\right|ds \\
&= \frac{k}{\Gamma(q)}\left|\int_{t_0}^{t_1}(t_1-s)^{q-1} - \int_{t_0}^{t_1}(t_2-s)^{q-1} - \int_{t_1}^{t_2}(t_2-s)^{q-1}\right|ds \\
&\leq \frac{k}{\Gamma(q)}\left|\int_{t_0}^{t_1}(t_1-s)^{q-1} - \int_{t_0}^{t_1}(t_2-s)^{q-1}\right|ds + \left|\int_{t_1}^{t_2}(t_2-s)^{q-1}\right|ds \\
&= \frac{k}{\Gamma(q)}\left|\frac{(t_1-t_0)^q}{q} + \frac{(t_2-t_1)^q}{q} - \frac{(t_2-t_0)^q}{q}\right| + |\frac{(t_2-t_1)^q}{q}| \\
&\leq \frac{k}{\Gamma(q+1)}(t_1-t_0)^q + (t_2-t_1)^q - (t_2-t_0)^q + (t_2-t_1)^q \\
&= \frac{k}{\Gamma(q+1)}(t_1-t_0)^q - (t_2-t_0)^q + 2(t_2-t_1)^q \\
&\leq \frac{2k}{\Gamma(q+1)}(t_2-t_1)^q < \epsilon
\end{aligned}$$

provided $\|t_2 - t_1\| < \delta = (\frac{\epsilon\Gamma(q+1)}{2k})^{\frac{1}{q}}$, proving that the family of solutions $\{u_{\eta_i}(t;\alpha)\}$ is equicontinuous. By the Arzela-Ascoli theorem, $\{u_{\eta_i}(t;\alpha)\}$has a subsequence $\{u_{\eta_{i_j}}(t;\alpha)\}$ which converges uniformly to a function $r(t)$ on $[t_0, T]$. We then show that $r(t)$ is a solution of (10). Equation (10) becomes

$$u_{\eta_{i_j}}(t,\alpha) = u_{0_{i_j}} + \eta_{i_j} + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}(g_{i_j}(s,u_{i_j}(s,\eta_{i_j})) + \eta_{i_j})ds \tag{22}$$

Taking the limit as $i_j \to \infty$ in (22), yields

$$r(t) = u_0 + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}(g(s,r(t)))ds \tag{23}$$

Thus, $r(t)$ is a solution of (6) on $[t_0, T]$. We claim that $r(t)$ is the maximal solution of (6). Then from (11), we have that $m(t,\alpha) < u_\eta(t,\alpha) \leq r(t)$ on $[t_0, T]$.

5. Main Results

In this section, we will obtain sufficient conditions for the uniform stability of the system (4).

Theorem 5.1 (Uniform Stability). *Assume the following*

(i) $g \in PC[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ *satisfies* $(A_0)(ii)$ *and* $g(t,u)$ *is quasi-monotone non-decreasing in* u *with* $g(t,0) \equiv 0$.

(ii) $V : \mathbb{R}_+ \times S_\psi \to \mathbb{R}^N_+$, $V \in \mathbb{L}$ *is locally Lipschitzian in* x *with* $V(t,0) \equiv 0$ *such that*

$$ {}^C D^q_+ V(t,x) \le g(t, V(t,x)), t \ne t_k, (t,x) \in \mathbb{R}_+ \times S_\psi \tag{24}$$

holds for all $(t,x) \in \mathbb{R}_+ \times S_\psi$.

(iii) *there exists a* $\psi_0 > 0$ *such that* $x_0 \in S_\psi$ *implies that*

$$x + I_k(x) \in S_\psi \text{ and } V(t, x + I_k(x)) \le \psi_k(V(t,x)), t = t_k, x \in S_\psi$$

and the function $\psi_k : \mathbb{R}^N_+ \to \mathbb{R}^N_+$ *is nondecreasing for* $k = 1, 2, ...$

(iv) $b(\|x\|) \le V_0(t,x) \le a(\|x\|)$, *where* $a, b \in \mathcal{K}$ *and* $V_0(t,x) = \sum_{i=1}^N V_i(t,x)$

Then the uniform stability of the trivial solution $u = 0$ *of the IFrDE* (6) *implies the uniform stability of the trivial solution* $x = 0$ *of* (4).

Proof. Let $0 < \epsilon < \psi$ and $t_0 \in R_+$ be given.
Assume that the solution $u = 0$ of (6) is uniformly stable. Then given each $b(\epsilon) > 0$, and $t_0 \in R_+$, there exist a positive function $\delta_1 = \delta_1(\epsilon) > 0$ such that whenever

$$u_0 = \sum_{i=1}^n u_{i0} < \delta, \text{ we have } \sum_{i=1}^n u_i(t; t_0, u_0) \le b(\epsilon), t \ge t_0 \tag{25}$$

where $u(t; t_0, u_0)$ is any solution of (6).
let us choose $V(t_0^+, x_0) \le u_0$ and

$$\sum_{i=1}^n u_{i0} = a(t_0, \|x_0\|)$$

Since $a(t, \mathbb{K})$ and $a \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$ we can find a positive function $\delta = \delta(t_0, \epsilon) > 0$ such that

$$a(t_0, \|x_0\|) < \delta_1 \text{ and } \|x_0\| < \delta \tag{26}$$

hold simultaneously. We claim that if

$$\|x_0\| \le \delta, \text{ then } \|x(t, t_0, x_0)\| \le \epsilon, t \ge t_0.$$

Suppose that this claim is not true. Then there would exists a point $t_1 > t_0$ and a solution $x(t)$ with $\|x_0\| < \delta$ such that

$$\|x(t_1)\| = \epsilon \text{ and } \|x(t)\| < \epsilon, \text{ for } t \in [t_0, t_1). \tag{27}$$

This implies that $x(t) + I_k(x) \in S_\psi$ for $t \in [t_0, t_1)$.
From (7) we have that

$$V_0(t, x(t)) \le r_0(t, t_0, u_0) \text{ for } t \in [t_0, t_1). \tag{28}$$

Combining condition (iv) and (28) we have

$$b(\epsilon) \le \sum_{i=1}^n V_i(t_1, x(t_1)) \le \sum_{i=1}^n r_i(t; t_0, u_0) \tag{29}$$

Using equations (25), (27) and (29) we have,

$$b(\epsilon) \leq \sum_{i=1}^{n} V_i(t_1, x(t_1)) \leq \sum_{i=1}^{n} r_i(t; t_0, u_0) < b(\epsilon)$$

which leads to an absurdity that $b(\epsilon) < b(\epsilon)$.
Hence, the uniform stability of the trivial solution $u = 0$ of (6) implies the uniform stability of the trivial solution $x = 0$ of (4). □

6. Application

Let the points $t_k, t_k < t_{k+1}$, $lim_{k\to\infty} t_k \to \infty$ be fixed. Consider the vector impulsive Caputo fractional differential equations

$$
\begin{aligned}
{}^{C}D^{q}x_1(t) &= x_1 sinx_1 + \frac{x_2^2 cosecx_2}{x_1} - 5x_1, t \neq t_k \\
{}^{C}D^{q}x_2(t) &= 3x_2 cosecx_2 + 4x_2 sinx_1 - \frac{x_1^2 secx_2}{x_2}, t \neq t_k \\
\Delta x_1 &= s_k(x(t_k)), \Delta x_2 = n_k(x(t_k)), t = t_k, k = 1, 2, ...
\end{aligned} \tag{30}
$$

for $t \geq t_0$, with initial conditions

$$x_1(t_0^+) = x_{10} \quad \text{and} \quad x_2(t_0^+) = x_{20}$$

Consider a vector $V = (V_1, V_2)^T$, where
$V_1(t, x_1, x_2) = x_1^2$ and $V_2(t, x_1, x_2) = x_2^2$, with $x = (x_1, x_2) \in \mathbb{R}^2$, and its associated norm defined by $\|x\| = \sqrt{x_1^2 + x_2^2}$.
Now

$$V_0(t, x) = \sum_{i=1}^{2} V_i(t, x_1, x_2) = x_1^2 + x_2^2$$

and so $b(\|x\|) \leq V_0(t, x) \leq a(\|x\|)$ with $b(r) = r$ and $a(r) = r^2$, implying that $a, b \in \mathcal{K}$. From (5), we compute the Caputo fractional Dini derivative for $V_1(t, x_1, x_2) = x_1^2$ for $t > 0, t \neq t_k$ as follows:

$$
\begin{aligned}
{}^{C}D_{+}^{q}V_1(t; x_1, x_2) &= \limsup_{h\to 0^+} \frac{1}{h^q}\Big\{V(t,x) - V(t_0, x_0) \\
&\quad + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)[V(t - rh, x - h^q f(t,x)) - V(t_0, x_0)]\Big\} \\
&= \limsup_{h\to 0^+} \frac{1}{h^q}\big\{x_1^2 - x_{10}^2 + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)[(x_1 - h^q f_1(t; x_1, x_2)^2 - x_{10}^2]\big\} \\
&\leq \limsup_{h\to 0^+} \frac{1}{h^q}\big\{x_1^2 - x_{10}^2| + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)[x_1^2 - 2x_1 h^q f_1(t; x_1, x_2) \\
&\quad + h^{2q} f_1(t, x_1) - x_{10}^2]\big\} \\
&= \limsup_{h\to 0^+} \frac{1}{h^q}\big\{x_1^2 - x_{10}^2 + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) x_1^2
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) 2x_1 h^q f_1(t;x_1,x_2) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) h^{2q} f_1(t;x_1,x_2)\}\\
= \;& \limsup_{h\to 0^+} \frac{1}{h^q}\{x_1^2 + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) x_1^2 - x_{10}^2 - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) x_{10}^2\\
&-2x_1 \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) h^q f_1(t;x_1,x_2)|\}\\
= \;& \limsup_{h\to 0^+} \frac{1}{h^q}\{ \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) x_1^2 - \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) x_{10}^2\\
&-2x_1 \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) h^q f_1(t;x_1,x_2)\}
\end{aligned}
$$

Recall that from equations (3.7) and (3.8) in [11], we have

$$
\limsup_{h\to 0^+} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) \;=\; \frac{x_1^2}{t^q\Gamma(1-q)},
$$

and

$$
\lim_{h\to 0^+} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r {}^qC_r = -1
$$

Substituting we obtain

$$
\begin{aligned}
{}^C D_+^q V_1(t;x_1,x_2) &\le \frac{x_1^2}{t^q\Gamma(1-q)} - \frac{x_{10}^2}{t^q\Gamma(1-q)} + 2x_1 f_1(t;x_1,x_2)\\
{}^C D_+^q V_1(t;x_1,x_2) &\le \frac{x_1^2}{t^q\Gamma(1-q)} + 2x_1 f_1(t;x_1,x_2)\\
{}^C D_+^q V_1(t;x_1,x_2) &\le \frac{x_1^2}{t^q\Gamma(1-q)} + 2x_1(x_1 sinx_1 + \frac{x_2^2 cosecx_2}{x_1} - 5x_1)\\
&\le \frac{x_1^2}{t^q\Gamma(1-q)} + 2x_1^2 sinx_1 + 2x_2^2 cosecx_2 - 10x_1^2
\end{aligned}
$$

As $t\to\infty$, $\frac{x_1^2}{t^q\Gamma(1-q)} \to 0$, so that we have

$$
\begin{aligned}
{}^C D_+^q V_1(t;x_1,x_2) &\le +2x_1^2 sinx_1 + 2x_2^2 cosecx_2 - 10x_1^2\\
&\le x_1^2(-10 + 2|sinx_1|) + x_2^2(2\frac{1}{|sinx_2|})\\
&\le x_1^2(-8) + x_2^2(2)\\
&\le -8V_1 + 2V_2
\end{aligned}
$$

Therefore,

$$
{}^C D_+^q V_1(t;x_1,x_2) \le -8V_1 + 2V_2 \tag{31}
$$

Also, for $x_0 \in S_\psi$, for $t = t_k, k = 1, 2, ..,$ we have

$$V(t, x(t) + s_k) = |s_k + x(t)| \leq V(t, x(t))$$

Similarly, using (5), we compute the Caputo fractional Dini derivative for $V_2(t, x_1, x_2) = x_2^2$ as follows:

$$\begin{aligned}
{}^C D_+^q V_2(t; x_1, x_2) &= \limsup_{h\to 0^+} \frac{1}{h^q}\{V(t,x) - V(t_0, x_0) \\
&\quad + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)[V(t - rh, x - h^q f(t,x)) - V(t_0, x_0)]\} \\
&= \limsup_{h\to 0^+} \frac{1}{h^q}\{x_2^2 - x_{20}^2 + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)[(x_2 - h^q f_2(t; x_1, x_2)^2 - x_{20}^2]\} \\
&\leq \limsup_{h\to 0^+} \frac{1}{h^q}\{x_2^2 - x_{20}^2| + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)[x_2^2 - 2x_2 h^q f_2(t; x_1, x_2) \\
&\quad + h^{2q} f_2(t, x_2) - x_{20}^2]\} \\
&= \limsup_{h\to 0^+} \frac{1}{h^q}\{x_2^2 - x_{20}^2 + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)x_2^2 - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)2x_2 h^q f_2(t; x_1, x_2) \\
&\quad - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) h^{2q} f_2(t; x_1, x_2)\} \\
&= \limsup_{h\to 0^+} \frac{1}{h^q}\{x_2^2 + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)x_2^2 - x_{20}^2 - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)x_{20}^2 \\
&\quad - 2x_2 \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) h^q f_2(t; x_1, x_2)|\} \\
&= \limsup_{h\to 0^+} \frac{1}{h^q}\{ \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)x_2^2 - \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)x_{20}^2 \\
&\quad - 2x_2 \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) h^q f_2(t; x_1, x_2)\} \\
{}^C D_+^q V_2(t; x_1, x_2) &\leq \frac{x_2^2}{t^q\Gamma(1-q)} - \frac{x_{20}^2}{t^q\Gamma(1-q)} + 2x_2 f_2(t; x_1, x_2) \\
{}^C D_+^q V_2(t; x_1, x_2) &\leq \frac{x_2^2}{t^q\Gamma(1-q)} + 2x_2 f_2(t; x_1, x_2) \\
{}^C D_+^q V_2(t; x_1, x_2) &\leq \frac{x_2^2}{t^q\Gamma(1-q)} + 2x_2(-3x_2 cosecx_2 + 4x_2 sinx_1 - \frac{x_1^2 secx_2}{x_2}) \\
&\leq \frac{x_1^2}{t^q\Gamma(1-q)} - 6x_2^2 cosecx_2 + 4x_2^2 sinx_1 + 2x_1^2 secx_2 \\
&\leq x_2^2(-6cosecx_2 + 4sinx_1) + x_1^2(2secx_2) \\
&\leq x_2^2(-6\frac{1}{|sinx_2|} + 4|sinx_1|) + x_1^2(\frac{2}{|cosx_2|}) \\
&\leq x_2^2(-6+4) + x_1^2(2) \\
&\leq x_2^2(-2) + x_1^2(2)
\end{aligned}$$

$$\leq \quad -2V_2 + 2V_1$$

Therefore,

$$ {}^{C}D_{+}^{q}V_1(t;x_1,x_2) \leq 2V_1 - 2V_2 \tag{32}$$

Also, for $x_0 \in S_\psi$, for $t = t_k, k = 1, 2, ..,$ we have

$$V(t, x(t) + n_k) = |n_k + x(t)| \leq V(t, x(t))$$

Combining (32) and (31), we have

$$ {}^{C}D_{+}V \leq \begin{pmatrix} -8 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = g(t, V) \tag{33}$$

Now consider the comparison system

$$ {}^{C}D^{q}u = g(t, u) = Au \tag{34}$$

where $A = \begin{pmatrix} -8 & 2 \\ 2 & -2 \end{pmatrix}$.

The vectorial inequality (33) and all other conditions of Theorem 5.1 are satisfied since the eigenvalues of A are all negative real parts. Hence, the system (34) is uniformly stable . Therefore, the trivial solution $x_0 = 0$ of the system of IFrDE (30) is uniformly stable.

7. Conclusion

In this study, we have explored the uniform stability of the trivial solution for nonlinear ICFrDEs using the framework of vector Lyapunov functions. By introducing a generalized class of piecewise continuous Lyapunov functions and employing comparison results, we derived the uniform stability of the trivial solution of the fractional dynamical systems. The provided example not only validates the theoretical results but also extends and refines existing findings, demonstrating the utility and versatility of the proposed approach. This work contributes a significant step forward in the stability analysis of fractional impulsive systems and open avenues for further research into more complex systems and applications.

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