

ON INTERPOLATIVE BOYD-WONG AND MATKOWSKI TYPE CONTRACTIONS IN 2-BANACH SPACES

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ABSTRACT. In this paper, we demonstrate a fixed point theorem for generalized weakly contractive mappings in 2-Banach spaces which is a generalization of a Banach contraction mapping principle. On the other hand, we introduce interpolative Boyd-Wong and Matkowski type contractions on 2-Banach spaces. In particular, we prove the existence and uniqueness of a fixed point of such mappings in 2-Banach spaces.

1. INTRODUCTION

The Banach contraction mapping principle is one of interesting results in functional analysis. Moreover, it was applied in many branches of mathematics which is given by

Theorem 1.1. *Let $(\mathcal{F}, d)$ be a complete metric space and let $S : \mathcal{F} \longrightarrow \mathcal{F}$ be a mapping with*

$$(1) \quad d(Sz, Sf) \leq kd(z, f)$$

where $k \in (0, 1)$ and $z, f \in \mathcal{F}$, hence S has a unique fixed point.

There are several generalizations of this principle, see, e.g [1, 2, 21, 23].

Recently, Karapinar, Aydi and Mitrović [16] introduced interpolative Boyd-Wong and Matkowski type contractions and they proved the related fixed point theorems in the class of metric spaces.

The fixed point theory played a crucial role in functional analysis. Moreover, it used in many branches of science such as biology, chemistry, economics, engineering and computer science.

In a 2-Banach space framework, Gähler [7] initiated the study of 2-normed spaces. Recently, White [24] initiated and studied the concept of 2-Banach spaces.

Recently, the authors [10] demonstrated that a contraction mapping has a one fixed point in bounded and closed subsets of a 2-normed space. However, Kir and Kiziltunc [18] demonstrated several results on fixed points in 2-Banach spaces.

In this paper, we demonstrate a fixed point theorem for weakly contractive mappings in 2-Banach spaces which is a generalization of a Banach contraction mapping principle. On the other hand, we introduce interpolative Boyd-Wong and Matkowski type contractions on 2-Banach spaces. In particular, we prove the existence and uniqueness of a fixed point of such mappings in 2-Banach spaces.

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2. PRELIMINARIES

We begin with preliminaries:

Definition 2.1. [7] Let $\mathcal{X}$ be a real vector space with $\dim \mathcal{X} \geq 2$. A 2-norm on $\mathcal{X}$ is a function $\|\cdot\| : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_+$ such that

- (i) $\|x, y\| = 0$ if and only if x and y are linearly dependent;
- (ii) For all $x, y \in \mathcal{X}$, $\|x, y\| = \|y, x\|$;
- (iii) For any $x, y \in \mathcal{X}$ and $\lambda \in \mathbb{R}$, $\|\lambda x, y\| = |\lambda| \|x, y\|$;
- (iv) For each $x, y, z \in \mathcal{X}$, $\|x + y, z\| \leq \|x, z\| + \|y, z\|$.

The pair $(\mathcal{X}, \|\cdot, \cdot\|)$ is called a 2-normed space.

Definition 2.2. [6] Let $\mathcal{X}$ be a 2-normed space. A sequence $(x_n)_{n \in \mathbb{N}} \subset \mathcal{X}$ is said to be a Cauchy sequence if $\lim_{m, n \rightarrow \infty} \|x_n - x_m, y\| = 0$ for any $y \in \mathcal{X}$.

Definition 2.3. [24] Let $\mathcal{X}$ be a 2-normed space. A sequence $(x_n)_{n \in \mathbb{N}} \subset \mathcal{X}$ converges in $\mathcal{X}$ if there exists an element $x \in \mathcal{X}$ such that for all $y \in \mathcal{X}$, $\lim_{n \rightarrow \infty} \|x_n - x, y\| = 0$. If $(x_n)_{n \in \mathbb{N}}$ converges to x , we write $x_n \rightarrow x$ as $n \rightarrow \infty$.

Definition 2.4. [24] A 2-normed space in which every Cauchy sequence converges will be called a 2-Banach space.

Definition 2.5. [10] Let $\mathcal{X}$ be a 2-normed space then the mapping $S : \mathcal{X} \rightarrow \mathcal{X}$ is said to be a contraction if there exists $\lambda \in (0, 1)$ such that

$$(2) \quad \|Sx - Sy, z\| \leq \lambda \|x - y, z\|$$

for all $x, y, z \in \mathcal{X}$.

Lemma 2.1. [10] Let $\mathcal{X}$ be a 2-normed space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. Let $S : \mathcal{F} \rightarrow \mathcal{F}$ be a contraction, then S has a unique fixed point.

Similarly to the proof of [10, Lemma 3.10], we obtain:

Lemma 2.2. [10] Let $\mathcal{X}$ be a 2-Banach space. Let $S : \mathcal{X} \rightarrow \mathcal{X}$ be a contraction, then S has a unique fixed point.

Definition 2.6. [4] A mapping S on a 2-normed space $\mathcal{X}$ is called a Meir-Keeler contraction if given $\varepsilon > 0$, there exists $\delta > 0$ such that for each $x, y, z \in \mathcal{X}$,

$$(3) \quad \varepsilon \leq \|x - y, z\| < \varepsilon + \delta \Rightarrow \|Sx - Sy, z\| < \varepsilon.$$

Definition 2.7. [4] A mapping S on a 2-normed space $\mathcal{X}$ is called a Ćirić contraction if given $\varepsilon > 0$, there exists $\delta > 0$ such that for each $x, y, z \in \mathcal{X}$,

$$(4) \quad \varepsilon < \|x - y, z\| < \varepsilon + \delta \Rightarrow \|Sx - Sy, z\| \leq \varepsilon.$$

Theorem 2.3. [4] Let $\mathcal{X}$ be a 2-Banach space and let S be a Meir-Keeler contraction on $\mathcal{X}$, then S has a unique fixed point in $\mathcal{X}$.

Theorem 2.4. [4] Let $\mathcal{X}$ be a 2-Banach space and let S be a Ćirić contraction on $\mathcal{X}$, then S has a unique fixed point in $\mathcal{X}$.

Definition 2.8. [4] A mapping S on a 2-normed space $\mathcal{X}$ is called a Hardy-Rogers contraction if S is a self-mapping on $\mathcal{X}$ satisfying for each $x, y, z \in \mathcal{X}$,

$$(5) \quad \begin{aligned} \|Sx - Sy, z\| &\leq a\|x - Sx, z\| + b\|y - Sy, z\| + c\|x - Sy, z\| \\ &\quad + e\|y - Sx, z\| + f\|x - y, z\| \end{aligned}$$

where a, b, c, e, f are nonnegative and we put $\beta = a + b + c + e + f$.

Consider the following condition

$$(6) \quad \begin{aligned} x \neq y \implies \|Sx - Sy, z\| &< a\|x - Sx, z\| + b\|y - Sy, z\| \\ &\quad + c\|x - Sy, z\| + e\|y - Sx, z\| + f\|x - y, z\|. \end{aligned}$$

Theorem 2.5. [4] Let $\mathcal{X}$ be a 2-normed space and S a self-mapping on $\mathcal{X}$ satisfying for each $x, y, z \in \mathcal{X}$,

$$(7) \quad \begin{aligned} \|Sx - Sy, z\| &\leq a\|x - Sx, z\| + b\|y - Sy, z\| \\ &\quad + c\|x - Sy, z\| + e\|y - Sx, z\| + f\|x - y, z\| \end{aligned}$$

where a, b, c, e, f are nonnegative and we put $\beta = a + b + c + e + f$.

(i) If $\mathcal{X}$ is complete and $\beta < 1$, then S has a unique fixed point.

(ii) If (5) is modified to the condition (6) and in this case $\mathcal{X}$ is compact, S is continuous and $\beta = 1$, then S has a unique fixed point.

Theorem 2.6. [4] Let $\mathcal{X}$ be a 2-Banach space, a, b, c, e, f be monotonically decreasing functions from $[0, \infty)$ to $[0, 1)$ and let the sum of these five functions be less than 1. Assume that $S : \mathcal{X} \rightarrow \mathcal{X}$ satisfies condition (5) with $a = a(\|x - y, z\|), \dots, f = f(\|x - y, z\|)$ for each $x, y, z \in \mathcal{X}$. Then S has a unique fixed point.

Theorem 2.7. [4] Let $\mathcal{X}$ be a 2-Banach space and $S_n : \mathcal{X} \rightarrow \mathcal{X}$, $n = 1, 2, \dots$ satisfy the conditions of Theorem 2.6 with the coefficients a, b, c, e, f . Let $S_n x_n = x_n$ and assume that $S_n \rightarrow S$ pointwise on $\mathcal{X}$. Then $x = \lim_{n \rightarrow \infty} x_n$ is the unique fixed point of S .

Theorem 2.8. [4] Let $\mathcal{X}$ be a 2-Banach space and $S_n : \mathcal{X} \rightarrow \mathcal{X}$, $n = 1, 2, \dots$ be functions with at least one fixed point $x_n, n = 1, 2, \dots$. Let S satisfy the hypothesis of Theorem 2.6 and $S_n \rightarrow S$ uniformly on $\mathcal{X}$. Then $x = \lim_{n \rightarrow \infty} x_n$ is the unique fixed point of S .

Ettayb [4] established a generalization of Banach's principle of contraction mappings in 2-Banach space as follows.

Theorem 2.9. [4] Let $\mathcal{X}$ be a 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $S : \mathcal{F} \rightarrow \mathcal{F}$ is a continuous mapping such that S^k is a contraction for some $k \in \mathbb{N}^*$, then S has a unique fixed point.

3. MAIN RESULTS

In this section, we start with the following definition.

Definition 3.1. A mapping S on a 2-normed space $\mathcal{X}$ is called a generalized weak contractive if

$$(8) \quad \xi(\|Sx - Sy, z\|) \leq \xi(\|x - y, z\|) - \varphi(\|x - y, z\|)$$

where $x, y, z \in \mathcal{X}$ and $\varphi, \xi : [0, \infty) \longrightarrow [0, \infty)$ are continuous and nondecreasing functions such that $\varphi(t) = \xi(t) = 0$ if and only if $t = 0$.

Now, we present our main result.

Theorem 3.1. *Let $\mathcal{X}$ be a 2-Banach space. If S is a generalized weak contractive mapping on $\mathcal{X}$, then S has a unique fixed point.*

Proof. Let $x_0 \in \mathcal{X}$ and $\{x_n\}_{n=0}^\infty \in \mathcal{X}$ be given by $x_{n+1} = Sx_n$ for each $n \in \mathbb{N}$. Hence for each $z \in \mathcal{X}$,

$$(9) \quad \xi(\|x_n - x_{n+1}, z\|) = \xi(\|Sx_{n-1} - Sx_n, z\|)$$

$$(10) \quad \leq \xi(\|x_{n-1} - x_n, z\|) - \varphi(\|x_{n-1} - x_n, z\|)$$

$$(11) \quad \leq \xi(\|x_{n-1} - x_n, z\|)$$

hence for each $z \in \mathcal{X}$ and $n \in \mathbb{N}$,

$$(12) \quad \|x_n - x_{n+1}, z\| \leq \|x_{n-1} - x_n, z\|$$

then the sequence $(\|x_n - x_{n+1}, z\|)_n$ is decreasing, hence there exists $d \in [0, \infty)$ such that for each $z \in \mathcal{X}$, $\|x_n - x_{n+1}, z\| \longrightarrow d$ as $n \longrightarrow \infty$. Take $x = x_{n-1}$, $y = y_n$ in (8) and letting $n \longrightarrow \infty$, we get

$$(13) \quad \xi(d) \leq \xi(d) - \varphi(d).$$

Hence $\varphi(d) = 0$ thus $d = 0$. So for any $z \in \mathcal{X}$,

$$(14) \quad \lim_{n \rightarrow \infty} \|x_n - x_{n+1}, z\| = 0.$$

Next, we demonstrate that $(x_n)_n$ is a Cauchy sequence. Suppose that $(x_n)_n$ is not Cauchy, hence there exists $\varepsilon > 0$ for which we can find subsequences $(x_{m(k)})_k$ and $(x_{n(k)})_k$ of $(x_n)_n$ with $n(k) > m(k) > k$ such that for any $z \in \mathcal{X}$,

$$(15) \quad \|x_{m(k)} - x_{n(k)}, z\| \geq \varepsilon.$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (15). Hence

$$(16) \quad \|x_{m(k)} - x_{n(k)-1}, z\| < \varepsilon.$$

Thus we get for each $z \in \mathcal{X}$,

$$(17) \quad \varepsilon \leq \|x_{m(k)} - x_{n(k)}, z\| \leq \|x_{m(k)} - x_{n(k)-1}, z\| + \|x_{n(k)-1} - x_{n(k)}, z\| < \varepsilon + \|x_{n(k)-1} - x_{n(k)}, z\|.$$

Utilizing (14) and letting $n \longrightarrow \infty$, we obtain

$$(18) \quad \lim_{k \rightarrow \infty} \|x_{m(k)} - x_{n(k)}, z\| = \varepsilon.$$

Also,

$$(19) \quad \|x_{n(k)} - x_{m(k)}, z\| \leq \|x_{n(k)} - x_{n(k)-1}, z\| + \|x_{n(k)-1} - x_{m(k)-1}, z\| + \|x_{m(k)-1} - x_{m(k)}, z\|$$

and

$$(20) \quad \|x_{n(k)-1} - x_{m(k)-1}, z\| \leq \|x_{n(k)-1} - x_{n(k)}, z\| + \|x_{n(k)} - x_{m(k)}, z\| + \|x_{m(k)-1} - x_{m(k)}, z\|.$$

Taking $k \rightarrow \infty$ in the above two inequalities and utilizing (15), (18), we have

$$(21) \quad \lim_{k \rightarrow \infty} \|x_{n(k)-1} - x_{m(k)-1}, z\| = \varepsilon.$$

Putting $x = x_{m(k)-1}$ and $y = x_{n(k)-1}$ in (8) and utilizing (15), we get

$$(22) \quad \xi(\varepsilon) \leq \xi(\varepsilon) - \varphi(\varepsilon)$$

which is a contradiction with $\varepsilon > 0$. This proves that $(x_n)_n$ is a Cauchy sequence. Since $\mathcal{X}$ is complete, we get it is convergent to $w \in \mathcal{X}$ as $n \rightarrow \infty$. Substituting $x = x_{n-1}$ and $y = w$ in (8), we have

$$(23) \quad \xi(\|x_n - Sw, z\|) \leq \xi(\|x_{n-1} - w, z\|) - \varphi(\|x_{n-1} - w, z\|).$$

Letting $n \rightarrow \infty$, we get for each $z \in \mathcal{X}$,

$$(24) \quad \xi(\|w - Sw, z\|) \leq \xi(\|w - w, z\|) - \varphi(\|w - w, z\|) = 0 - \varphi(0) = 0.$$

Then $Sw = w$. To demonstrate the uniqueness of the fixed point, let us assume that w_1 and w_2 are two fixed points of S . Setting $x = w_1$ and $x = w_2$ in (8), we get for any $z \in \mathcal{X}$,

$$(25) \quad \xi(\|w_1 - w_2, z\|) = \xi(\|Sw_1 - Sw_2, z\|)$$

$$(26) \quad \leq \xi(\|w_1 - w_2, z\|) - \varphi(\|w_1 - w_2, z\|),$$

thus $\varphi(\|w_1 - w_2, z\|) = 0$ for each $z \in \mathcal{X}$. Hence $w_1 = w_2$. $\square$

In particular, if $\varphi(t) = kt, \xi(t) = t$ where $k \in (0, 1)$ we obtain Lemma 2.2. Let Ψ be the set of functions such that

- (i) $\varphi(0) = 0$;
- (ii) $\varphi(t) < t$ for all $t > 0$;
- (iii) φ is upper semi-continuous from the right.

For more details, see [2]. Now, we introduce an interpolative Boyd-Wong type contraction in a 2-normed space as follows.

Definition 3.2. A self-mapping S on a 2-normed space $\mathcal{X}$ is called an interpolative Boyd-Wong type contraction, if there exist $\lambda \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$ and a nondecreasing function $\varphi \in \Psi$ such that

$$(27) \quad \begin{aligned} \|Sx - Sy, z\| \leq & \varphi(\|x - y, z\|^\beta \cdot \|x - Sx, z\|^\alpha \cdot \|y - Sy, z\|^\gamma) \\ & \cdot \left[\frac{1}{2}(\|x - Sy, z\| + \|y - Sx, z\|)\right]^{1-\alpha-\beta-\gamma} \end{aligned}$$

for each $x, y \in \mathcal{X} \setminus \text{Fix}(S)$ and $z \in \mathcal{X}$ where $\text{Fix}(S) = \{u \in \mathcal{X} : Su = u\}$.

Theorem 3.2. Let $\mathcal{X}$ be a 2-Banach space and let S be an interpolative Boyd-Wong type contraction on $\mathcal{X}$, then S has a fixed point in $\mathcal{X}$.

Proof. Let $x_0 \in \mathcal{X}$. We will set a constructive sequence $\{x_n\}_{n=0}^\infty \in \mathcal{X}$ by $x_{n+1} = Sx_n$ for each $n \in \mathbb{N}$. If there exist a nonnegative integer n_0 such that $x_{n_0} = x_{n_0+1} = Sx_{n_0}$, then x_{n_0} forms

a fixed point. The proof is complete. Henceforwards, suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Putting $x = x_n$ and $y = x_{n-1}$ in (27), we have for all $z \in \mathcal{X}$,

$$\begin{aligned}
 \|x_n - x_{n+1}, z\| &= \|Sx_n - Sx_{n-1}, z\| \\
 &\leq \varphi(\|x_n - x_{n-1}, z\|^\beta \cdot \|x_n - Sx_n, z\|^\alpha \cdot \|x_{n-1} - Sx_{n-1}, z\|^\gamma \\
 &\quad \cdot [\frac{1}{2}(\|x_n - x_n, z\| + \|x_{n-1} - x_{n+1}, z\|)]^{1-\alpha-\beta-\gamma}) \\
 &= \varphi(\|x_n - x_{n-1}, z\|^\beta \cdot \|x_n - x_{n+1}, z\|^\alpha \cdot \|x_{n-1} - x_n, z\|^\gamma \\
 &\quad \cdot [\frac{1}{2}(\|x_{n-1} - x_{n+1}, z\|)]^{1-\alpha-\beta-\gamma}).
 \end{aligned}
 \tag{28}$$

If $\|x_{n-1} - x_n, z\| \leq \|x_n - x_{n+1}, z\|$, then from (28), we obtain

$$\begin{aligned}
 \|x_n - x_{n+1}, z\| &\leq \varphi(\|x_n - x_{n-1}, z\|^{\beta+\gamma} \cdot \|x_n - x_{n+1}, z\|^{1-\beta-\gamma}) \\
 &\leq \|x_n - x_{n-1}, z\|^{\beta+\gamma} \cdot \|x_n - x_{n+1}, z\|^{1-\beta-\gamma}.
 \end{aligned}
 \tag{29}$$

Then

$$\|x_n - x_{n+1}, z\|^{\beta+\gamma} \leq \|x_{n-1} - x_n, z\|^{\beta+\gamma}
 \tag{30}$$

which is a contradiction. Then $(\|x_n - x_{n+1}, z\|)_n$ is decreasing. As a result, there exists $M \in \mathbb{R}_+$ such that $\lim_{n \rightarrow \infty} \|x_n - x_{n+1}, z\| = M$ for all $z \in \mathcal{X}$. We claim that $M = 0$. Indeed, by (28), we derive that

$$\|x_n - x_{n+1}, z\| \leq \varphi(\|x_{n-1} - x_n, z\|).
 \tag{31}$$

Since φ is upper semi-continuous from the right, we have

$$M = \lim_{n \rightarrow \infty} \|x_n - x_{n+1}, z\| \leq \lim_{n \rightarrow \infty} \sup \varphi(\|x_{n-1} - x_n, z\|) \leq \varphi(M) < M
 \tag{32}$$

which is a contradiction, so we get $M = 0$. Now, we demonstrate that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. We argue by contradiction, that is, $(x_n)_{n \in \mathbb{N}}$ is not a Cauchy sequence. This means that there exists $\varepsilon > 0$ for which we can find subsequences of integers $(m_k)_k$ and $(n_k)_k$ with $n_k > m_k > k$ such that

$$\|x_{m_k} - x_{n_k}, z\| \geq \varepsilon.
 \tag{33}$$

Further, corresponding to m_k , we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ and satisfying (33). Then

$$\|x_{m_k} - x_{n_k-1}, z\| < \varepsilon.
 \tag{34}$$

Letting $x = x_{n_k-1}$ and $y = x_{m_k-1}$ in (27), we obtain for any $z \in \mathcal{X}$,

$$\begin{aligned}
 \|x_{n_k} - x_{m_k}, z\| &= \|Sx_{n_k-1} - Sx_{m_k-1}, z\| \\
 &\leq \varphi(\|x_{n_k-1} - x_{m_k-1}, z\|^\beta \cdot \|x_{n_k-1} - Sx_{n_k-1}, z\|^\alpha \cdot \|x_{m_k-1} - Sx_{m_k-1}, z\|^\gamma \\
 &\quad \cdot [\frac{1}{2}(\|x_{n_k-1} - x_{m_k}, z\| + \|x_{m_k-1} - Sx_{n_k-1}, z\|)]^{1-\alpha-\beta-\gamma}) \\
 &= \varphi(\|x_{n_k-1} - x_{m_k-1}, z\|^\beta \cdot \|x_{n_k-1} - x_{n_k}, z\|^\alpha \cdot \|x_{m_k-1} - x_{m_k}, z\|^\gamma \\
 &\quad \cdot [\frac{1}{2}(\|x_{n_k-1} - x_{m_k}, z\| + \|x_{m_k-1} - x_{n_k}, z\|)]^{1-\alpha-\beta-\gamma}).
 \end{aligned}$$

By the upper semi-continuity of φ , $M = 0$ and letting $k \rightarrow \infty$, we obtain that

$$\varepsilon \leq \varphi(0) = 0$$

which is a contradiction. Therefore $(x_n)_n \in \mathcal{X}$ is Cauchy. Then $(x_n)_n$ converges to certain $x \in \mathcal{X}$. Now, we demonstrate that $x \in \mathcal{X}$ is a fixed point of S . Hence for each $z \in \mathcal{X}$,

$$\begin{aligned} \|x_{n+1} - Sx, z\| &= \|Sx_n - Sx, z\| \\ (35) \quad &\leq \varphi(\|x_n - x, z\|^\beta \cdot \|x_n - Sx_n, z\|^\alpha \cdot \|x - Sx, z\|^\gamma) \\ &\quad \cdot \left[\frac{1}{2}(\|x_n - Sx, z\| + \|x - Sx_n, z\|)\right]^{1-\alpha-\beta-\gamma}. \end{aligned}$$

Letting $n \rightarrow \infty$ in (35) and using the upper semi-continuity of φ , we obtain

$$\|Sx - x, z\| \leq \varphi(0) = 0$$

for all $z \in \mathcal{X}$. Then $Sx = x$ thus x is a fixed point of S . $\square$

Proceeding as the proof of Theorem 3.2, we state the following result.

Corollary 3.3. *Let $\mathcal{X}$ be a 2-Banach space. Let $S : \mathcal{X} \rightarrow \mathcal{X}$ be a self-mapping. Suppose that there exist $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$ and a nondecreasing function $\varphi \in \Psi$ such that*

$$(36) \quad \|Sx - Sy, z\| \leq \varphi(\|x - y, z\|^\beta \cdot \|x - Sx, z\|^\alpha \cdot \|y - Sy, z\|^{1-\alpha-\beta})$$

for each $x, y \in \mathcal{X} \setminus \text{Fix}(S)$ and $z \in \mathcal{X}$ where $\text{Fix}(S) = \{u \in \mathcal{X} : Su = u\}$. Then S has a fixed point in $\mathcal{X}$.

Taking $\varphi(t) = \lambda t$ (where $\lambda \in [0, 1)$) in Theorem 3.2, we have:

Corollary 3.4. *Let $\mathcal{X}$ be a 2-Banach space. Let $S : \mathcal{X} \rightarrow \mathcal{X}$ be a self-mapping. Suppose that there exist $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$ and $\lambda \in [0, \infty)$ such that*

$$\begin{aligned} (37) \quad \|Sx - Sy, z\| &\leq \lambda(\|x - y, z\|^\beta \cdot \|x - Sx, z\|^\alpha \cdot \|y - Sy, z\|^\gamma) \\ &\quad \cdot \left[\frac{1}{2}(\|x - Sy, z\| + \|y - Sx, z\|)\right]^{1-\alpha-\beta-\gamma} \end{aligned}$$

for each $x, y \in \mathcal{X} \setminus \text{Fix}(S)$ and $z \in \mathcal{X}$ where $\text{Fix}(S) = \{u \in \mathcal{X} : Su = u\}$. Then S has a fixed point in $\mathcal{X}$.

Let Φ be the set of functions $\phi : [0, \infty) \rightarrow [0, \infty)$ such that

- (i) ϕ is nondecreasing,
- (ii) $\lim_{n \rightarrow \infty} \phi^n(t) = 0$ for all $t > 0$.

In order to state our next theorem we shall need the following well-known and easy, but useful, observation (by Matkowski [19]).

Lemma 3.5. [12] *Let $\phi \in \Phi$. Then $\phi(t) < t$ for all $t > 0$ and $\phi(0) = 0$.*

Now, we introduce an interpolative Matkowski type contraction in a 2-normed space as follows.

Definition 3.3. A self-mapping S on a 2-normed space $\mathcal{X}$ is called an interpolative Matkowski type contraction, if there exist $\lambda \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$ and a nondecreasing function $\phi \in \Phi$ such that

$$\begin{aligned} (38) \quad \|Sx - Sy, z\| &\leq \phi(\|x - y, z\|^\beta \cdot \|x - Sx, z\|^\alpha \cdot \|y - Sy, z\|^\gamma) \\ &\quad \cdot \left[\frac{1}{2}(\|x - Sy, z\| + \|y - Sx, z\|)\right]^{1-\alpha-\beta-\gamma} \end{aligned}$$

for each $x, y \in \mathcal{X} \setminus \text{Fix}(S)$ and $z \in \mathcal{X}$ where $\text{Fix}(S) = \{u \in \mathcal{X} : Su = u\}$.

On what follows we state and prove the main theorem.

Theorem 3.6. *Let $\mathcal{X}$ be a 2-Banach space and let S be an interpolative Matkowski type contraction on $\mathcal{X}$, then S has a fixed point in $\mathcal{X}$.*

Proof. Following the related lines in the proof of Theorem 3.2, without loss of generality, let $x_0 \in \mathcal{X}$, we construct a sequence $\{x_{n+1} = Sx_n\}$ such that $x_n \neq x_{n+1}$. From (31), we get

$$(39) \quad \|x_n - x_{n+1}, z\| \leq \phi(\|x_{n-1} - x_n, z\|) \leq \phi^n(\|x_0 - x_1, z\|).$$

Since $\phi \in \Phi$, we get $\lim_{n \rightarrow \infty} \phi^n(\|x_0 - x_1, z\|) = 0$. So

$$(40) \quad \lim_{n \rightarrow \infty} \|x_n - x_{n+1}, z\| = 0.$$

Now, we demonstrate that $(x_n)_{n \in \mathbb{N}}$ is Cauchy. Let $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$(41) \quad \phi^n(\|x_0 - x_1, z\|) < \varepsilon - \phi(\varepsilon).$$

Using (39), this implies that

$$(42) \quad \|x_n - x_{n+1}, z\| < \varepsilon - \phi(\varepsilon).$$

We claim that

$$(43) \quad \|x_m - x_n, z\| < \varepsilon - \phi(\varepsilon) \quad \text{for all } m \geq n \geq n_0.$$

We show (43) by induction. Since $\varepsilon - \phi(\varepsilon) < \varepsilon$, by (41), we conclude that (43) holds when $m = n + 1$. Now suppose that (43) holds for $m = k$. For $m = k + 1$, we have

$$\begin{aligned} \|x_n - x_{k+1}, z\| &\leq \|x_n - x_{n+1}, z\| + \|x_{n+1} - x_{k+1}, z\| \\ &\leq \varepsilon - \phi(\varepsilon) + \|Sx_n - Sx_k, z\| \\ &\leq \varepsilon - \phi(\varepsilon) + \phi(\|x_n - x_k, z\|^\beta \cdot \|x_n - Sx_n, z\|^\alpha \cdot \|x_k - Sx_k, z\|^\gamma) \\ (44) \quad &\cdot \left[\frac{1}{2}(\|x_n - Sx_k, z\| + \|x_k - Sx_n, z\|)\right]^{1-\alpha-\beta-\gamma} \\ &= \varepsilon - \phi(\varepsilon) + \phi(\|x_n - x_k, z\|^\beta \cdot \|x_n - x_{n+1}, z\|^\alpha \cdot \|x_k - x_{k+1}, z\|^\gamma) \\ &\cdot \left[\frac{1}{2}(\|x_n - x_{k+1}, z\| + \|x_k - x_{n+1}, z\|)\right]^{1-\alpha-\beta-\gamma}. \end{aligned}$$

Since $k \geq n \geq n_0$, we have

$$(45) \quad [\|x_n - x_k, z\|]^\beta \cdot [\|x_n - x_{n+1}, z\|]^\alpha \cdot [\|x_k - x_{k+1}, z\|]^\gamma \cdot \left[\frac{1}{2}(\|x_n - x_{k+1}, z\| + \|x_k - x_{n+1}, z\|)\right]^{1-\alpha-\beta-\gamma} < \varepsilon.$$

Therefore

$$(46) \quad \|x_n - x_{k+1}, z\| \leq \varepsilon - \phi(\varepsilon) + \phi(\varepsilon) = \varepsilon.$$

So (43) holds for $m = k + 1$. Using (43), the sequence (x_n) is Cauchy, so there exists $u \in \mathcal{X}$ such that $\lim_{n \rightarrow \infty} \|x_n - u, z\| = 0$. Suppose that $u \neq Su$. Since $x_n \neq Sx_n$ for each $n \geq 0$, by (38), we have

$$\begin{aligned} \|x_{n+1} - Su, z\| &= \|Sx_n - Su, z\| \\ (47) \quad &\leq \phi(\|x_n - u, z\|^\beta \cdot \|x_n - Sx_n, z\|^\alpha \cdot \|u - Su, z\|^\gamma) \\ &\cdot \left[\frac{1}{2}(\|x_n - Su, z\| + \|u - x_{n+1}, z\|)\right]^{1-\alpha-\beta-\gamma}. \end{aligned}$$

It is obvious that there exists $N \in \mathbb{N}$ such that for each $n \geq N$,

$$(48) \quad \begin{aligned} & [\|x_n - u, z\|]^\beta \cdot [\|x_n - Sx_n, z\|]^\alpha \cdot [\|u - Su, z\|]^\gamma \\ & \cdot \left[\frac{1}{2} (\|x_n - Su, z\| + \|u - x_{n+1}, z\|) \right]^{1-\alpha-\beta-\gamma} < \|u - Su, z\|. \end{aligned}$$

Since ϕ is nondecreasing, by insertion of this last inequality in (48), we obtain

$$(49) \quad \|x_{n+1} - Su, z\| \leq \phi(\|u - Su, z\|) \quad \text{for all } n \geq N.$$

Letting $n \rightarrow \infty$, we obtain

$$(50) \quad \|u - Su, z\| \leq \phi(\|u - Su, z\|).$$

Then $Su = u$. □

Proceeding as the proof of Theorem 3.6, we state the following result.

Corollary 3.7. *Let $\mathcal{X}$ be a 2-Banach space. Let $S : \mathcal{X} \rightarrow \mathcal{X}$ be a self-mapping. Suppose that there exist $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$ and $\phi \in \Phi$ such that*

$$(51) \quad \|Sx - Sy, z\| \leq \phi([\|x - y, z\|]^\beta \cdot [\|x - Sx, z\|]^\alpha \cdot [\|y - Sy, z\|]^{1-\alpha-\beta})$$

for each $x, y \in \mathcal{X} \setminus \text{Fix}(S)$ and $z \in \mathcal{X}$ where $\text{Fix}(S) = \{u \in \mathcal{X} : Su = u\}$. Then S has a fixed point in $\mathcal{X}$.

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