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VARIOUS NOTIONS OF SUBSPACE HYPERCYCLIC POWER OPERATORS AND THEIR DIRECT SUMS IN OPERATOR SPACES

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Abstract. Subspace hypercyclic operators forms a very important class of operators in operator spaces.A lot of their properties have been studied over along period of time, however, complete characterization of this property has not been done. In fact, a lot of open questions remain unanswered with regard to subspace hypercyclicity. Most of these studies have been done in special cases of finite dimensional operator spaces. It is therefore interesting to address these questions in general operator spaces. In this research therefore we extend an investigation on subspace hypercyclicity by investigating different notions of the subspace hypercyclicity. In particular, we consider subspace hypercyclic operators, their powers and direct sums and show that operators under direct sum satisfies various subspace-hypercyclicity criteria and has a lot of interesting properties.

1. Introduction

In the realm of functional analysis, there exists a profound interplay between invariant subspaces and orbits of hypercyclic operators. A key concept in this domain is hypercyclicity, which characterizes operators whose orbits densely cover the entire space [46]. The notion of hypercyclicity traces back to Beauzamy who was inspired by the well established concept of cyclicity in Functional analysis [10]. This study laid the groundwork by demonstrating that translation operators have dense orbits in spaces of complete functions that uniformly converge on the compact sets. Building upon Birkhoff's work in [10], the findingsof Maclane [25] explored entire functions further and established that $\{f, f', f'', f''', ...\}$ fills the entire $H(\mathbb{C})$. The author also established that differential operators D in a complex space $\mathbb{C}$ are dense in $H(\mathbb{C})$. Kim and Song [22] researched on numerically hypercyclic operators and established the link between several operators satisfying general hypercyclicity criterion. Shkarin [45] extended this work by creating operators whose square is not numerically hypercyclic but are numerically hypercyclic on their own. They also confirmed the existence of numerically hypercyclic operators on $\mathbb{C}^2$. Moreover, they further characterized certain diagonal operator $S \in B(\mathbb{C}^4)$ and proved that diagonal operators have two orbits that are not numerically hypercyclic, but the union of these two orbits is dense in $\mathbb{C}$. In the process of the investigation they further restricted an operator on a finite dimensional invariant subspaces, hence providing the necessary and

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sufficient conditions of weak and strong numerically hypercyclic operators. This was accomplished by establishing the relationship between the point spectrum and strong numerically hypercyclic operators as well as the relationship between the spectrum and weak numerically hypercyclic operators. Ansari [2] proved that if S is a hypercyclic operator and for any $k \geq 1$, then it follows that S^k is also hypercyclic. consequently $HC(S^k) = HC(S)$. Saavendra and Muller [41] established that if an operator is hypercyclic then so is its rotation. They proved that $\overline{Orb(S,x)} = \overline{\{S^n x : n \in \mathbb{N}\}} = X$ iff $\overline{Orb(\lambda_n S, x)} = \overline{\{\lambda_n S^n x : n \in \mathbb{N}, \lambda \in \mathbb{S}\}} = X$, more particulaly they proved that for any $\phi \in \mathbb{R}$ then we have $HC(S) = HC(e^{i\phi})$. Bayart and Costakis [5] studied operators which are rotated by complex numbers whose modular is unit with polynimial phase and further demonstrated that when the phase grows at a geometric rate to infinity, hypercyclicity fails. The aforementioned findings make it abundantly evident that weakly and hypercyclic operators share many characteristics. These properties include, for example, the spectrum being empty. De la Rosa [12] further examined existing examples and proved that difference also exist. Chan [11] examined separable infinite dimensions $\mathbb{H}$. The investigation's conclusion was that the strong operator topology has an orbit that fills the entire complex Hilbert space. Furthermore, the operator norm topology has a dense linear span in $\mathbb{H}$. In addition the research established that a set of bounded linear operators $B(\mathbb{H})$ did posses non-hypercyclic operators [40]. Matache [27] proved that contractions can sometimes be hypercyclic if multiplied by a scalar that is strictly greater than 1. The author further researched on contractions and proved that if those operators have finite defect indices, then they have hypercyclic scalar multiple. The interest to consider contraction was because the well known fact that contractions have a bounded orbit hence non-hypercyclic. Further operators that are similar to contraction are not hypercyclic. Kitai [21] contributed significantly on the study of hypercyclicity by formulating conditions for continuous linear operators to exhibit hypercyclicity. This research was extended to invertible operators and the above results were also admitted. Additionally the above property was investigated in $\oplus^{\infty} S_{k=1}$ on $\oplus_{k=1}^{\infty} H$ and it was proved that $S^i X \cap Z \neq \emptyset$ and $S^i Z \cap Y \neq \emptyset$ holds for hereditarily hypercyclic operators. Bes and Peris [9] in their investigations proved that $S \oplus S$ is hereditarily hypercyclic when S satisfy hypercyclicity criterion. Their work provided insights into the behavior of hypercyclic operators of the composite operator on the separable Frechet Space $\mathbb{F}$. Herrero [18] addressed the issue of the connection between a hypercyclic operator and its inverse and established that $S \oplus S$ is hypercyclic if S satisfies the hypercyclicity criterion. Further in the research the study established that if the power operator S^n is hypercyclic then $S^n \oplus S^n = (S \oplus S)^n$ is also hypercyclic. Grosse-Erdmann and Bernal-Gonzale [15] investigated hypercyclicity properties of almost-commuting sequence S_m operators on an $\mathbb{F}$-space. They further strengthened their work by proving that $S_m \oplus S_m$ is hypercyclic on $X \times X$. This result motivated them and they investigated the hypercyclicity properties of almost commuting sequences of operators, more particularly, they investigated hypercyclicity of operators in X with hereditary subsequences and found out that $S_m \oplus .... \oplus S_m$ (M-fold) is densely hereditarily hypercyclic on X^M. The work of Rolewicz [40] investigated hypercyclicity properties of $(\lambda_m S^m)$. Saavendra-Leon and Muller in [41] established that if $S \in B(X)$ and (λ_m) is a sequence of complex numbers and if $(\lambda_m S^m)$ satisfies conditions (C) and the $\sup_m |\lambda_m| d^m < \infty$ where $d = dist(0, \delta_\epsilon(S))$, then $(\lambda_m S^m)$ has a subspace hypercyclic subspace on a closed infinite dimensional space. It is generally recognized that condition (C) is the weakest approach of testing for hypercyclicity. Manuel and Charles [26] showed we have operator S that has dense orbits but $S \oplus S$ does not have a dense orbit. They demonstrated that if an operator S is M-hypercyclic then $\delta(S)$ intersects with the unit circle, but elements of $\delta(S)$ do not. It should be noted that the notion of M-hypercyclicity is strictly infinite dimensional. Madore and Martinez-Avendano [24] work motivated Bamerni and Kilicman [4] to research on diskcyclic vector subspaces. They proved that if T is diskcyclic then the product of the $B[0,1]$ and the union of the numerical range of all iterations of T is dense in Hilbert space. Furthermore, they proved that in some instances there exists diskcyclic operators that have non-trivial closed invariant subspaces. Furthermore, they proved these operators have dense linear subspace that are in particular infinite dimensional with nonzero diskcyclic vectors. Bamerni and Kilicman [3] established the relationship between orthogonal projections onto a closed subspace M of $\mathbb{H}$. In their research they proved the hypercyclicity of the product an orthogonal projection with the orbit of subspace hypercyclic operator and established the link between the invariant subspace $M^{\perp}$ under and an operator S and the projection P. In the research on $M^{\perp}$-hypercyclicity and orthogonal projection, they proved that $M^{\perp} \subseteq \overline{Orb(PS, x) \cap M^{\perp})}$. Saavendra-Leon and Muller in [42] established conditions that guarantee hypercyclicity for a sequence of operators and further proved that i-sequence of operators $S_k \oplus ... \oplus S_k$ is also hypercyclic. Grosse-Erdmann and Bernal-Gonzale [15] who developed and proved the hypercyclicity criteria for a sequence of operators. They further proved that $S_k \oplus S_k \oplus ....$ is hypercyclic. Building on the above results, Moosapoor [35] developed the concept of M-hypercyclicity for a sequence of operators and established the equivalence relationship between $\{S_i\}$, and the invariance of $S_i \subseteq B(X)$ and M-hypercyclicity and also proved that M-hypercyclic operator is M-hypercyclic. Bes and Peris in [9] researched on the notion of herditarily hypercyclic operators and proved that of S is hereditarily hypercyclic then so is $S \oplus S$. They further proved that if S is hereditarily hypercyclic then so is S^i and extended this results and proved that $S^i \oplus S^i = (S \oplus S)^i$ is also hereditarily hypercyclic. They expanded their investigations and established that if S is hypercyclic then S and S^{k_i} share the same hypercyclic vector. Moosapoor [34] building on the above results introduced and investigated the notion of hereditarily subspace-hypercyclic operators. In their research, they discovered sufficient requirements for an operator to be hereditary subspace-hypercyclic [50]. Furthermore Moosapoor [32] went on to give the subspace-supercyclicity criterion. They researched on hereditary hypercyclic operators and came up with the notion of hereditarily subspace-hypercyclic operators with all the necessary and sufficient conditions an operator must satisfy for it be hereditarily subspace-hypercyclic and the research established that hereditarily subspace-hypercyclic operators are subspace hypercyclic. Herrero [18] proved that $S \oplus S$ hypercyclic when T satisfy the hypercyclicity criterion. This was only possible if the operator is hereditarily hypercyclic and thus $S \oplus S$. This notion was extended to two power operators and proved that if $S \oplus S$ and S^n satisfies the hypercyclicity criterion than $S^n \oplus S^n = (S \oplus S)^n$ is hypercyclic. Bayart and Matheron in [6] researched on the weakly supercyclic operators. In their research they established that weakly supercyclic hyponormal operators are generally a multiple of unitary operators and hyponormal operators are not N-supercyclic. They further established the equivalence in bilateral weighted shifts where they proved that N-supercyclicity is equivalent to supercyclicity. Nathan [37] introduced the concept of n-supercyclicity. During the investigation, they demonstrated that on a Hilbert space $\mathbb{H}$ there exist bounded linear operators that have dense orbits on Hilbert space that have n-dimensional subspaces. This research resulted in a new set of operators known as n-supercyclic operators and create a set of conditions that operators must meet in order to be n-supercyclic. This criteria was extended to direct sum and conclude that $(\oplus_{k=1}^{n} S_k)$ is n-supercyclic on $\oplus_{k=1}^{n} H_k$. This notion was extended to the direct sum of infinite number of separable Hilbert spaces and proved that $(\oplus_{k=1}^{\infty} S_k)$ is ∞-supercyclic on $(\oplus_{k=1}^{\infty} H_k)$. The author furthermore utilized the knowledge of spectral theory to establish that an operator S is 2-supercyclic and proved that if S has a decomposition property then S and S^* are 2-supercyclic. The study of [14] create a subspace-supercyclicicity criteria and provide some equivalent criterion. They provided examples of direct sum in backward shifts resulting in subspace hyperclic operators. This notion motivated Bamerni and Kilicman [4] who proved that the direct sum of two different unilateral backward weighted shifts $\mathbb{B}_1 \oplus \mathbb{B}_2$ in the Hilbert space $l^2(\mathbb{N})$ is $M_1 \oplus M_2$-subspace hypercyclic. In [6], they established that if $S \oplus S$ subspace hypercyclic then the two individual operators are subspce hypercyclic. Further from there research they proved that the converse does not hold. In their investigation they come up with following questions; Suppose $S \oplus S$ is M-hypercyclic, are both operators M-hypercyclic?, additionally suppose that the two operators are M-hypercyclic does that imply that $S \oplus S$ M-hypercyclic?, again suppose that S satisfies M-hypercyclic criterion, does it mean that $S \oplus S$ is also M-hypercyclic?, finally suppose $S \oplus S$ is M-hypercyclic, does S satisfy hypercyclic criterion? The first two questions positively were answered while the last two were given partial answers. Motivated by the work of Nathan [38] that is if $S = S_1, S_2, ..., S_m$ is m-tuple of operators that commute then S is hypercyclic and the work of [40] on scaled hypercyclic operators, Yousefi and Sharifi [49] investigated subspace supercyclicity for a tuple of operators. In their investigation, they established that $S = \lambda S_1, S_2, ..., S_m$ is subspace supercyclic. They further developed a subspace-supercyclicity criterion for tuples of operators. They also established that $S_d^{(2)} = (S = S_1, S_2, ..., S_m) \oplus (S = S_1, S_2, ..., S_m)$ is $M \oplus M$ subspace-hypercyclic. The work of [5] introduced the concept of frequently hypercyclic operators and stated off by giving examples of frequently hypercyclic operators. Such operators were the translation operators. Jeneker [20] research on various methods that can be used to determine whether an operator is hypercyclic or frequent hypercyclic. The author also analysed a wide range of operators on F-space and established that if S is hypercyclic, then the set of hypercyclic vectors is dense in G_δ and of interest was that if S is hypercyclic the $S \oplus S \oplus ... \oplus S$ is also hypercyclic and finally proved that hypercyclicity does not necessarily imply frequent hypercyclicity. Menet [29] extended the scope of research on frequently hypercyclic operators by researching on the properties U-frequently hypercyclic operators and proved that the exists U-frequently hypercyclic operators whose inverse is not dense. The author utilized the C-type operator which operator with four parameters. Menet [28] investigated hypercyclicity properties of invertible operators and particularly, invertible frequently hypercyclic operators. The research answered a long standing question of Bayart and Grivaux in [7] which to establish hypercyclicity properties if frequent invertible hypercyclic operator. Bayart and Ruzsa [8] established the link between an invertible hypercyclic operator S and U-frequently hypercyclicity. Grosse-Erdmann in [16] researched on bilateral weghted shifts and proved that T_w hypercyclic under invertibility. Building on this concept of frequently hypercyclic operators Moosapoor in [33] introduced and investigated the notion of M-frequently hypercyclic operators and proved that subspace-frequently hypercyclic operators are M-hypercyclic. Further S^p where $p \in \mathbb{N}$ and $S \oplus S$ are subspace hypercyclic provided that S is subspace-frequently hypercyclic. Heo, Kim and Kim in [17] researched on q-frequently hypercyclic operators. In their research they developed and proved the q-hypercyclicity criterion and also investigated properties of q-frequently hypercyclic subspaces of bounded linear operator in F-spaces and also established that q-frequently hypercyclicity subspaces also admits the infinite dimensional property. Moosapoor [32] using hypercyclic bounded operators and scaled identity operators from Hilbert spaces constructed an operator; $S = A \oplus \lambda_1 I \oplus \lambda_2 I \oplus ... \oplus \lambda_n I$ which was subspace-hypercyclic. Tajmouti et al in [47] come up with the notion of subspace-hypercycliciity of C_0-semigroup and investigated semigroups and in the process of the research they developed and proved the necessary and sufficient conditions this semigroup satisfies for it to be subspace-hypercyclic. They partially characterized the notion of supercyclic C_0-semigroup, developed and proved supercyclicity criterion and provided equivalent results for this criterion. El Berrag and Tojmouati [13] further extended this knowledged proved that if $(S_t \oplus S_s)$ is $M_0 \oplus M_1$-supercyclic C_0-semigroup, then S_t and S_s are M_1-supercyclic and M_2 -supercyclic C_0-semigroups respectively. Moosapoor [30] established equivalence relationship between M-supercyclicity criterion and the notion of invariant subspaces while Moosapoor [33] established that hypercyclic operators have invariant subspaces that are dense exception of zero of subspace-hypercyclic vectors. Further, during the research process, it was determined s that all members are made up of M-hypercyclic vectors for any operator from this family. Tajmouati et al in [48] studied the M-hypercyclicity considering substantially strong continuous cosine functions in a separable complex Banach space, gave conditions the cosine function must satisfy for it to be M-hypercyclic, and established the relationship between M-hypercyclicity of cosine operators and M-transitivity. In our research it was interesting to investigate M-hypercyclicity of other trigonometric functions, develop the M-hypercyclicity criteria and establish M-transitivity criteria. Building on the work of [40], El Berrag in [13] characterized the concept of subspace-hypercyclicity of Cesaro operators and developed subspace Cesaro-hypercyclic criterion and proved that it is subspace mixing. It is worth noting that there is a link between and subspsce-hypercyclicity, subspace-transitivity and subspace-mixing in direct sum of two operators. The above research findings were not expanded to $S_1 \oplus ... \oplus S_n$. Furthermore, the study did not take into account the subspace-hypercyclicity of the $S_1 \oplus ... \oplus S_n$ of different classes of operators that individually meet the hypercyclicity condition. The study hypercyclic and supercyclic operators has been of great interest in the recent time because they are common in familiar classes of operators. Using a disk, Nathan in [38] proved that T^* is 2-supercyclic but not supercyclic when $T = T_1 \oplus T_2$ and if $T = \oplus_{i=1}^{n} T_i$ is T^* is n-supercyclic and provided sufficient conditions that guarantee N-supercyclicity. Ahmadi in [1] research on conditions for supercyclicity. This played an important role in characterizing supercyclic operators. The research further explored properties of adjoints of composition of operators in Hardy spaces. In so doing Ahmadi [1] established that composition operators are not supercyclic. This results was extended to adjoints of composition operators that are contractions in Hilbert spaces because $\|C_\varphi^*\| \leq \|C_\varphi\| \leq 1$. The finding was extended to holomorphic functions with fixed points on the Bergman space and Dirichlet space. Moosapoor [36] concluded that S can be subspace-hypercyclic when $\delta_p(S^*)$ is empty or when $\delta_p(S^*)$ is non empty. The findings established the fact that the emptyness of the spectrum does not necessarily imply that the operator S is not M-hypercyclic. Salas [44] proved that there are bilateral weighted shift operators whose adjoints are hypercyclic but $S \oplus S^*$ is not even cyclic. This implies that S or S^* or both operators do not satisfy hypercyclic operators criterion. Of interest was to determine whether there are conditions if satisfied then the direct sum of an operator and its adjoint is subspace-hypercyclic if the operator and its adjoint are individually subspace-hypercyclic. It was proved that hypercyclic operators are not hypornormal, which indicates that hypercyclic operators cannot be quasinormal and, more specifically, they cannot be normal. Nathan in [39] did the characterization of hypornormal operators with hypercyclic adjoints. These conditions were extended to hypornormal operators and it was proved that S^* is hypercyclic. Nathan [39] did not generalize the results by considering other operators that are not in this class of operators. In our study, we will characterize different types of operators and evaluate whether they or their adjoints are subspace-hypercyclic. If they are subspace-hypercyclic, we will investigate these operators if their adjoints are subspace-hypercyclic. Herrero in [18] investigated on the link between hyperclicity and hyperinvariance and proved that if S is hyperinvariant, then the restriction of S on M is also hypercyclic. The results above motivates us to characterize other known operators restricted to certain subspaces and determine whether there adjoints are subspace-hypercyclic. If any characterized adjoint of any operator is subspace hypercyclic, then we will consider if the direct sum of such an operator and its adjoint is subspace hypercyclic. Herrero in [18] proved restriction of S on M is also hypercyclic. Salas in [39] extended hypercyclicity property to bilateral weight shifts of adjoints, showcasing the breadth of hypercyclic phenomena. Nathan in [38] did the characterization of hypornormal operators whose adjoints are hypercyclic. Hypornormal operators are not hypercyclic but it can be attained by applying the concept of separated sequence to obtain dense orbits. The use of separated sequence was necessary because orbits of hypornormal operators either strictly increase or decrease in norm or strictly decrease up-to a certain point and increase there after hence no dense orbits. The above research did not address the topic of when an operator and its adjoint is subspace-hypercyclic or subspace-supercyclic under direct sum. The question on the criterion to be applied on direct sum of an operator and its adjoint for them to be subspace-hypercyclic remains open. Bamerni and Kilicman in [3] provided sufficient conditions for bilateral shift to be subsapce-hypercyclic. As a result, they proved that an operator S exists such that S and S^* are both subspace-hypercyclic. This was achieved by constructing a positive weight sequence x_n that satisfies the following conditions; $\lim_{k\to\infty} \Pi_{j=m_i}^{m_i-n_k+1} x_i = 0$ and $\lim_{k\to\infty} \Pi_{j=m_i+1}^{m_i+n_k} \frac{1}{x_j} = 0$ Moosapoor in [31] expanded their study to analytic Toeplitz operators and established such operators cannot be multi-subspace-hypercyclic because the identity operator I obtained from the that that $T^k = I$ cannot be multi subspace-hypercyclic.

2. Preliminaries

Certain preliminaries are given here since they are important for proofs in the results section.

Definition 2.1. *([23], Definition 2.18). Let* X *be a Banach space and let* $S : X \to X$ *be bounded linear operator. We define the orbit of a vector* x *in* X *with respect to* S *by* $Orb(\mathrm{S,x}) = \{S^n x : n \in \mathbb{N}\}$.

Definition 2.2. *([19], Definition 2.19). A vector* x *in a Banach space* X *is said to be hypercyclic for an operator* S *in* B(X) *if the set* $Orb(\mathrm{S,x}) = \{S^n x : n \in \mathbb{N}\}$ *is norm dense in the whole space. And is supercyclic if* $\overline{Orb(S,x)} = \overline{\{cS^n x : n \in \mathbb{N}, c \in \mathbb{C}\}}$ =X *That is to say* $\overline{Orb(\mathrm{S,x})} = \overline{\{S^n x : n \in \mathbb{N}\}} = \mathrm{X}$

Definition 2.3. *([43], Definition 2.1). A bounded linear operator* $T\colon X \to \mathrm{X}$ *is said to be subspace-hypercyclic for a subspace* M *of* X *if there exists a vector* x∈X *such that* $orb(\mathrm{T,x}) \cap \mathrm{M}$ *is dense in* M. *Such a vector* x *is called a* M*-hypercyclic vector for* T.

3. Main Results

We start off by introducing the notion of numerical subspace-hypercyclicity of operators.

Proposition 3.1. *Let* T_i *be a set of operators in an operator space then operators under direct sum are weakly numerically subspace-hypercyclic if* $f_1 + ... + f_m \in X_1^* \oplus ... \oplus X_m^*$ *such that* $Orb(T_1 \oplus ... \oplus T_m, x_1 + ... + x_m, f_1 + ... + f_m) \cap M_1 \oplus ... \oplus M_n$ *is dense and compact in* $M_1 \oplus ... \oplus M_m$.

Proof. Let $x_1 + ... + x_m$ be a weakly numerically subspace-hypercyclic vector of $T_1 \oplus ... \oplus T_m$ with respect to $M_1 \oplus ... \oplus M_m$. Let $U_1 \oplus ... \oplus U_m$ be a nonzero open set of $M_1 \oplus ... \oplus M_m$ and $f_1 \oplus ... \oplus f_m \in M_1^* \oplus ... \oplus M_m^*$, then by definition we have; $\{(f_1 + ... + f_m)(T_1 \oplus ... \oplus T_m)^n (x_1 + ... x_m) : n \in \mathbb{N}\} \cap M_1 \oplus ... \oplus M_m$ is dense in $M_1 \oplus ... \oplus M_m$. Thus $x_1 \oplus ... \oplus x_m$ is a weakly numerically subspace hypercyclic vector of $T_1 \oplus ... \oplus T_m$ with respect to $M_1 \oplus ... \oplus M_m$. Thus, $T_1 \oplus ... \oplus T_m$ satisfies weakly numerically subspace-hypercyclicity. □

We now show that power bounded operators preserve numerical subspace-hypercyclicity property in operator spaces. Thus, the direct sum of power bounded operators is also numerically subspace-hypercyclic.

Lemma 3.2. *Let* T_i^p *be numerically subspace-hypercyclic, then* $T_1^p \oplus ... \oplus T_m^p$ *is numerically subspace-hypercyclic.*

Proof. Suppose $p \in \mathbb{N}$ is such that T_i^p is numerically subspace-hypercyclic. Let $x_1 + ... + x_m$ be a numerically subspace-hypercyclic vector of $T_1^p \oplus ... \oplus T_m^p$ with respect to $M_1 \oplus ... \oplus M_m$. Let $f_1 + ... + f_m \in M_1^* \oplus ... \oplus M_m^*$ and let $U_1 \oplus ... \oplus U_m$ be nonzero open subset of $M_1 \oplus ... \oplus M_m$, then we note that $\{n \in \mathbb{N} : (f_1 + ... + f_m)(T_1^p \oplus ... \oplus T_m^p)(x_1 + ... + x_m) \in U_1 \oplus ... \oplus U_m\} \subseteq \{n \in \mathbb{N} : (f_1 + ... + f_m)(T_1 \oplus ... \oplus T_m)(x_1 + ... + x_m) \in U_1 \oplus ... \oplus U_m\}$. Since $\{n \in \mathbb{N} : (f_1 + ... + f_m)(T_1 \oplus ... \oplus T_m)(x_1 + ... + x_m) \in U_1 \oplus ... \oplus U_m\}$ is numerically subspace-hypercyclic then it follows that $\{n \in \mathbb{N} : (f_1 + ... + f_m)(T_1^p \oplus ... \oplus T_m^p)(x_1 + ... + x_m) \in U_1 \oplus ... \oplus U_m\}$ is numerically subspace-hypercyclic. Thus $x_1 + ... + x_m$ is numerically subspace-hypercyclic vector of $T_1 \oplus ... \oplus T_m$ with respect to $M_1 \oplus ... \oplus M_m$. □

Theorem 3.3. *For strongly numerically subspace-hypercyclic operators there exists* $\lambda_1, ..., \lambda_n \in \delta_p(T_1 \oplus ... \oplus T_n)$ *and* $c_1, ..., c_n \in (R_{+1} \oplus ... \oplus \mathbb{R}_{+n})$, *then* $T_1 \oplus ... \oplus T_n$ *is numerically subspace-hypercyclic if* $\sum\{(c_1, ..., c_n)(\lambda_i^k) : k \in \mathbb{N}\} \cap (M_1 \oplus .... \oplus M_n$ *is dense in* $M_1 \oplus ... \oplus M_n$.

Proof. Let M_i be nonzero closed subspaces of X and M_i^* be nonzero closed subspaces of X^*. Suppose that $(\lambda_1 + ... + \lambda_n) \in \delta_p(T_1 \oplus ... \oplus T_n)$ and $(c_1 + ... + c_n) \in \mathbb{R}_{1+} \oplus ... \oplus \mathbb{R}_{n+}$ and if $(c_1 + ... + c_n) = 1$ then there exists $(x_1 + ... + x_n, f_1 + ... + f_n \in (M_1 \oplus ... \oplus M_n) \times (M_1^* \oplus ... \oplus M_n^*$ such that $(f_1 + ... + f_n)(T_1 \oplus ... \oplus T_n)^k, (x_1 + ... + x_n) = (c_1)(\lambda_1^k) + ... + (c_n)(\lambda_n^k)$ for all $k \in \mathbb{Z}$. Thus the $Orb((T_1 + ... + T_n), (x_1 + ... + x_n), (f_1 + ... + f_n)) \cap (M_1 + ... + M_n)$ is dense in $(M_1 + ... + M_n)$ and thus $(T_1 + ... + T_n) \in B(M_1 \oplus ... \oplus M_n)$ is numerically subspace-hypercyclic. We note that the operator similar to $(T_1 \oplus ... \oplus T_n)$ satisfy the same condition hence $(T_1 \oplus ... \oplus T_n)$ is strongly numerically subspace-hypercyclic. □

In the next corollary we provide conditions for diagonal operators to be numerically subspace-hypercyclic.

Corollary 3.4. *For operators that are subspace-hypercyclic numerically, subspace-hypercyclic in a stronger sense is guaranteed.*

Proof. Suppose $\{(c_1)(\lambda_1^k) + ... + (c_n)(\lambda_n^k) : k \in \mathbb{Z}\} \cap M_1 \oplus ... \oplus M_n$ is dense in $M_1 \oplus ... \oplus M_n$, then $T_1 \oplus ... \oplus T_n$ is strongly numerically subspace-hypercyclic. Thus $T_1 \oplus ... \oplus T_n$ is numerically subspace-hypercyclic. Now assume that $T_1 \oplus ... \oplus T_n$ is numerically subspace-hypercyclic, then there exists $x_1 \oplus ... \oplus x_n \in M_1 \oplus ... \oplus M_n$ for which the numerical $\overline{Orb((T_1 \oplus ... \oplus T_n), (x_1 \oplus ... \oplus x_n) \cap M_1 \oplus ... \oplus M_n} = M_1 \oplus ... \oplus M_n$. We note that $Orb((T_1 \oplus ... \oplus T_n), (x_1 + ... + x_n) \cap M_1 \oplus ... \oplus M_n = |x_1|^2 \mid \lambda_1^k \mid + ... + |x_n|^2 \mid \lambda_n^k \mid : k \in \mathbb{Z} \cap M_1 \oplus ... \oplus M_n$. Thus by setting $(c_i) = \mid x_i \mid^2$, we have, $\mid c_1 \mid^2 \mid \lambda_1^k \mid + ... + \mid c_n \mid^2 \mid \lambda_n^k \mid : k \in \mathbb{Z} \cap M_1 \oplus ... \oplus M_n$ is dense in $M_1 \oplus ... \oplus M_n$ □

We now introduce the notion of subspace multidiskcyclicity. The following result characterizes the concept of subspace muiltidiskcyclity in Banach spaces.

Proposition 3.5. *Let $T_i, i = 1, 2, ..., n$ be subspace multidiskcyclic. Let M_i be nonzero closed subspaces of X, then there exists $x_i \in M_i$ such that the subspace disk orbit of $x_1 + ... + x_n$ under $T_1 \oplus ... \oplus T_n$ is dense in $M_1 \oplus ... \oplus M_n$.*

Proof. Let $M_1 \oplus ... \oplus M_n$ be a nonzero closed subset of the disk K under $T_1 \oplus ... \oplus T_n$. Let $\mathbb{D}orb((T_1 \oplus ... \oplus T_n), (x_1 + ... + x_n)) \cap (M_1 \oplus ... \oplus M_n)$ be nowhere dense for every $(x_1 + ... + x_n) \in (M_1 \oplus ... \oplus M_n)$, then we have $(x_1 + ... + x_n)_k \in (M_1 \oplus ... \oplus M_n)$ such that $(M_1 \oplus ... \oplus M_n)$ is nowhere dense. Thus $\cup_{i=1, i \neq k}^m \mathbb{D}orb((T_1 \oplus ... \oplus T_n), (x_1 + ... + x_n)_i) \cap (M_1 \oplus ... \oplus M_n)$ is dense in $(M_1 \oplus ... \oplus M_n)$. Thus $(T_1 \oplus ... \oplus T_n)$ is subspace multidiskcyclic with respect to $(M_1 \oplus ... \oplus M_n)$. □

We now provide the link between subspace multidiskcyclicity and subspace diskcyclicity

Proposition 3.6. *Let T_i be subspace multidiskcyclic with respect to nonzero closed subspace of M_i of the disk K, then $T_1 \oplus ... \oplus T_n$ is subspace diskcyclic.*

Proof. Suppose n is a positive integer. Let $M_1 \oplus ... \oplus M_n = \{(x_1 + ... + x_n)_1, ..., (x_1 + ... + x_n)_n\}$ be subspace diskcyclic with respect to $T_1 \oplus ... \oplus T_n$, then we have

$$\cup_{i=1}^n \overline{\mathbb{D}Orb\{(T_1 \oplus ... \oplus T_n), (x_1 + ... + x_n)_i)\} \cap (M_1 \oplus ... \oplus M_n)}$$

is dense in $M_1 \oplus ... \oplus M_n$.

Suppose that $n > 1$ and $(x_1 + ... + x_n)_1, ..., (x_1 + ... + x_n)_n \in (M_1 \oplus ... \oplus M_n)$ with

$$int\overline{\mathbb{D}Orb\{(T_1 \oplus ... \oplus T_n), (x_1 + ... + x_n)_i\} \cap (M_1 \oplus ... \oplus M_n)} \neq \emptyset.$$

Then we have $(x_1 + ... + x_n)_h \in M_1 \oplus ... \oplus M_n$ such that the intersection between

$$\overline{\{int\mathbb{D}Orb\{(T_1 \oplus ... \oplus T_n), (x_1 + ... + x_n)\}}$$

and

$$(M_1 \oplus ... \oplus M_n) \cap \overline{\{int\mathbb{D}Orb\{(T_1 \oplus ... \oplus T_n), (x_1 + ... + x_n)_h\}} \cap (M_1 \oplus ... \oplus M_n)\}$$

is not equal to the empty set. Thus,

$$\{int\mathbb{D}Orb\{(T_1 \oplus ... \oplus T_n), (x_1 + ... + x_n)\} \cap (M_1 \oplus ... \oplus M_n)\}$$

is equal to

$$\{int\mathbb{D}Orb\{(T_1 \oplus ... \oplus T_n), (x_1 + ... + x_n)_h\} \cap (M_1 \oplus ... \oplus M_n)\}$$

. □

In the next result we show that MS-hypercyclicity where MS denotes multi-subspace.

Lemma 3.7. *Let $T_i \in B(X)$. If T_i are MS-hypercyclic with respect to M_i, then $T_1^m \oplus ... \oplus T_n^m$ is MS-hypercyclic with respect to $M_1 \oplus ... \oplus M_n$ for any $m \in \mathbb{N}$ and $i = 1, 2, ..., n$.*

Proof. When $m = 1$ and T_i are MS-hypercyclic and consider $\{(x_1 + ... + x_n)_1, ..., (x_1 + ... + x_n)\}$ in $X_1 \oplus ... \oplus X_n$ such that $\cup_{j=1}^{i} Orb\{(T_1 \oplus ... \oplus T_n), (x_1 + ... + x_n)_j\} \cap (M_1 \oplus ... \oplus M_n)$ is dense in $M_1 \oplus ... \oplus M_n$. Suppose $(y_1 + ... + y_n)_{j,k} = (T_1^k + ... + T_n^k)(x_1 + ... + x_n)_j$ where $1 \leq j \leq i$ and $1 \leq k \leq m - 1$. Now, since $\cup_{j=1}^{i} Orb\{(T_1 \oplus ... \oplus T_n), (x_1 + ... + x_n)\} \cap (M_1 \oplus ... \oplus M_n) = \cup_{1 \leq j \leq i, 1 \leq k \leq m-1} Orb\{(T_1^m \oplus ... \oplus T_n^m), (y_1 + ... + y_n)\}_{j,k} \cap (M_1 \oplus ... \oplus M_n)$. We note that

$$\cup_{1 \leq j \leq i, 1 \leq k \leq m-1} \overline{(Orb(T_1^m \oplus ... \oplus T_n^m), (y_1 + ... + y_n)_{j,k} \cap (M_1 \oplus ... \oplus M_n)}$$

is equal to $(M_1 \oplus ... \oplus M_n)$. Hence, we have that

$$\cup_{1 \leq j \leq i} \overline{(Orb(T_1 \oplus ... \oplus T_n), (x_1 + ... + x_n)_j \cap (M_1 \oplus ... \oplus M_n)}$$

is equal to $(M_1 \oplus ... \oplus M_n)$. Thus, $(T_1^m \oplus ... \oplus T_n^m)$ is multi subspace-hypercyclic with respect to $M_1 \oplus ... \oplus M_n$. □

In the next result we established then link between subspace-hypercyclicity and MS-hypercyclicity.

Theorem 3.8. *Let $T_1 \oplus ... \oplus T_i \in B(X_1 \oplus ... \oplus X_i)$ be $M_1 \oplus ... \oplus M_i$-hypercyclic then $T_1^m \oplus ... \oplus T_i^m$ is MS-hypercyclic for any $m \in \mathbb{N}$.*

Proof. If $m = 1$ then it follows by hypothesis. Now suppose that $m \geq 2$, let $x_1 + ... + x_i$ be an $M_1 \oplus ... \oplus M_i$-hypercyclic vetor for $T_1 \oplus ... \oplus T_i$, then it follows that: $\overline{Orb((T_1 \oplus ... \oplus T_i), (x_1 + ... + x_i)) \cap (M_1 + ... + M_i)} = M_1 + ... + M_i$. Now put $x_1 + ... + x_1 - i - times = x + ... + x - i - times, x_2 + ... + x_2 - i - times = (T_1 \oplus ... \oplus T_i)(x + ... + x), ..., x_n + ... + x_n - i - times = (T_1^{m-1} \oplus ... \oplus T_i^{m-1})(x + ... + x)$. Then $= \cup_{k=1}^{m} Orb((T_1^m \oplus ... \oplus T_i^m), (T_1^{k-1} \oplus ... \oplus T_i^{m-1})(x + ... + x) = (Orb((T_1^m \oplus ... \oplus T_i^m), (x + ... + x)) \cup (Orb((T_1^m \oplus ... \oplus T_i^m), (T_1 \oplus ... \oplus T_i)(x + ... + x)) \cup ... \cup (Orb((T_1^m \oplus ... \oplus T_i^m), (T_1^{m-1} \oplus ... \oplus T_i^{m-1})(x + ... + x)) = \{(x + ... + x), (T_1 \oplus ... \oplus$

$T_i)(x+...+x), ..., (T_1^{m-1} \oplus ... \oplus T_i^{m-1})(x+...+x), (T_1^{m+1} \oplus ... \oplus T_i^{m+1})(x+...x), ...\} \cap M_1 \oplus ... \oplus M_i$. $= Orb((T_1 \oplus ... \oplus T_i), (x+...+x)) \cap (M_1 \oplus ... \oplus M_i$. Thus, the result follows. □

Lemma 3.9. *If $T_1 \oplus ... \oplus T_n$ satisfies $(M_1 \oplus ... \oplus M_n)$-hypercyclicity criterion, then $T_1, T_2, ..., T_n$ satisfies $M_1, M_2, ...M_n$-hypercyclicity criterion respectively hence $T_1, T_2, ..., T_n$ are individually hypercyclic.*

Proof. Consider

(i). $(T_1 \oplus ... \oplus T_n)^{n_k}(x_1', ..., x_n') \to (0, 0, ..., 0) \forall (x_1', ..., x_n') \in U_1 \oplus ... \oplus U_n$.

(ii). For all $(y_1', y_2', ..., y_n') \in V_1 \oplus ... \oplus V_n$ there exists a sequence $(a_k, b_k, ...g_k) \subset M_1 \oplus ... \oplus M_n$ such that $(a_k, b_k, ...g_k) \to (0, 0, ..., 0)$.

and $(T_1 \oplus ... \oplus T_n)^{n_k}(a_k, b_k, ...g_k) \to (y_1', y_2', ..., y_n')$. Since U_1 and V_1 are dense in M_1, U_2 and V_2 are dense in M_2 and U_n and V_n are dense in M_n, so it follows that $T_1, T_2, ..., T_n$ are M_1-hypercyclic, M_2-hypercyclic up to M_n-hypercyclic respectively. Thus $T_1, T_2, ...T_n$ are individaully hypercyclic. □

In the next theorem we establish that T_i $i = 1, 2, ..., n$ satisfies the subspace hypercyclicity criterion, under direct sum.

Theorem 3.10. *Suppose $T_1, ..., T_n$ satisfies $M_1, ..., M_n$-hypercyclicity criterion respectively, then $T_1 \oplus ... \oplus T_n$ satisfies $M_1 \oplus ... \oplus M_n$-hypercyclicity criterion.*

Proof. Now, let $(y_1, y_2, ..., y_n \in D_2 \oplus ... \oplus D_2$ then $y_1 \in D_2, y_2 \in D_2, ..., y_n \in D_2$. such that for some sequences $x_k \to 0$ and $T_1^{n_k} x_k \to y_1$, $y_k \to 0$ and $T_2^{n_k} y_k \to y_2, ...,$ and $z_k \to 0$ and $T_n^{n_k} z_k \to y_n$ Therefore, since $(x_k, y_k, ..., z_k) \to (0, 0, ..., 0)$ and

$$(T_1 \oplus T_2 \oplus ... \oplus T_n)(x_k, y_k, ..., z_k) \to (y_1, y_2, ..., y_n) \tag{1}$$

Finally, since $T^{n_k} M \subseteq M$, then

$$(T_1 \oplus ... \oplus T_n)(M_1 \oplus ... \oplus M_n) \subseteq (M_1 \oplus ... \oplus M_n) \tag{2}$$

Thus, by Equation 1 and Equation 2, it follows that $(T_1 \bigoplus T_2 \oplus ... \oplus T_n)$ satisfies $(M_1 \oplus M_2 \oplus ... \oplus M_n)$-hypercyclicity criterion and thus hypercyclic. □

In the next result, we establish the equivalence relationship between subspace-transitivity and subspace-hypercyclicity.

Corollary 3.11. *For the equivalent conditions:*

(i). *$C_{t_{n1}} \oplus C_{t_{n2}} \oplus ... \oplus C_{t_{nn}}$ is $M_1 \oplus M_2 \oplus ... \oplus M_n$-transitive.*

(ii). *$\exists t > 0 : (C_{t_{n1}} \oplus C_{t_{n2}} \oplus ... \oplus C_{t_{nn}})^{-1}(V_1 \oplus ... \oplus V_n) \cap (U_1 \oplus ... \oplus U_n)$ is not empty.*

(iii). *$\exists t > 0$ such that $(C_{t_{n1}} \oplus ... \oplus C_{t_{nn}})(M_1 \oplus ... \oplus M_n) \subseteq (M_1 \oplus ... \oplus M_n)$*

We have $(C_{t_{n1}} \oplus C_{t_{n2}} \oplus ... \oplus C_{t_{nn}})$ being M-transitive, M-hypercyclic and hence hypercyclic.

Proof. Consider $M_1 \oplus M_2 \oplus ... \oplus M_n$ such that $(V_1 \oplus ... \oplus V_n) \cap (D_1 \oplus ... \oplus D_n) \neq \emptyset$ and $(D_{n+1} \oplus ... \oplus D_{n+n}) \cap (U_1 \oplus ... \oplus U_n) \neq \emptyset$. Let $a = a_1 + a_2 + ... + a_n \in (D_{n+1} \oplus ... \oplus D_{n+n}) \cap (U_1 \oplus ... \oplus U_n)$ and $b = b_1 + b_2 + ... + b_n \in (V_1 \oplus ... \oplus V_n) \cap (D_1 \oplus ... \oplus D_n)$,then there exists $t > 0 : B(a, \epsilon) \subset (U_1 \oplus ... \oplus U_n)$ and $B(b, \epsilon) \subseteq (V_1 \oplus ... \oplus V_n)$. From $b \in \oplus_{i=1}^n D_i$ and $a \in \oplus_{i=1}^n D_{n+i}$, we have: $C_{t_{n1}} \oplus ... \oplus C_{t_{nn}} \to b_1 + ... + b_n$ and there exists $(x_n)_n = (x_{n1} + x_{n2} + ... + x_{nn})_n \subset M_1 \oplus ... \oplus M_n$ such that $(x_{n1} + x_{n2} + ... + x_{nn}) \to 0$ and $(C_{t_{n1}} \oplus ... \oplus C_{t_{nn}})(x_{n1} + x_{n2} + ... + x_{nn}) \to a_1 + a_2 + ... + a_n$.

Consequently, there exists $N \in \mathbb{N}$ such that $\| x_{n1} + x_{n2} + ... + x_{nn} \| < \epsilon$, Thus, $\| (C_{t_{n1}} \oplus ... \oplus C_{t_{nn}})(x_{n1}+x_{n2}+...+x_{nn}) - (a_1+a_2+...+a_n) \| < \frac{\epsilon}{n}+...+\frac{\epsilon}{n} = \epsilon$. Thus, $\| C_{t_{n1}} - a_1 \| < \frac{\epsilon}{n} + ... + \| C_{t_{nn}} - a_n \| < \frac{\epsilon}{n}$ and $\| (C_{t_{n1}} \oplus ... \oplus C_{t_{nn}})(x_{n1}+...+x_{nn}) - (b_1+...+b_n) \| < \frac{\epsilon}{n}+...+\frac{\epsilon}{n} = \epsilon \forall n \geq N$. Therefore, $\| (b_1 + ... + b_n) + (x_{n1} + ... + x_{nn}) - (a_1 + a_2 + ... + a_n) \| = \| (x_{n1} + ... + x_{nn}) \| < \epsilon$. This implies that $(b_1 + ... + b_n) + (x_{n1} + ... + x_{nn}) \in B(b, \epsilon) \subset (V_1 \oplus ... \oplus V_n)$. Thus, $(b_1+...+b_n)+(x_{n1}+...+x_{nn}) \subset (V_1 \oplus ... \oplus V_n)$ On the other hand, $\| (C_{t_{n1}} \oplus ... \oplus C_{t_{nn}})(b_1+...+b_n+x_{n1}+...+x_{nn}) - (a_1+a_2+...+a_n) \| = \| C_{t_{n1}}(b_1) + C_{t_{n1}}x_{n1} - a_1 \| + ... + \| C_{t_{nn}}(b_n) + C_{t_{nn}}x_{nn} - a_n \| < \frac{\epsilon}{n} + ... + \frac{\epsilon}{n} = \epsilon$ This implies that $(C_{t_{n1}} \oplus ... \oplus C_{t_{nn}})(b_1 + ... + b_n + x_{n1} + ... + x_{nn}) \in B(a, \epsilon) \in (U_1 \oplus ... \oplus U_n)$ Thus, $(C_{t_{n1}} \oplus ... \oplus C_{t_{nn}})(b_1 + ... + b_n + x_{n1} + ... + x_{nn}) \in (U_1 \oplus ... \oplus U_n)$. So, $(b_1 + ... + b_n) + (x_{n1} + ... + x_{nn}) \in (C_{t_{n1}} \oplus ... \oplus C_{t_{nn}})^{-1}(U_1 \oplus ... \oplus U_n)$ and we obtain $(b_1 + ... + b_n) + (x_{n1} + ... + x_{nn}) \in (C_{t_{n1}} \oplus ... \oplus C_{t_{nn}})^{-1}(U_1 \oplus ... \oplus U_n) \cap (V_1 \oplus ... \oplus V_n)$ and $(C_{t_{n1}} \oplus ... \oplus C_{t_{nn}})^{-1}(U_1 \oplus ... \oplus U_n) \cap (V_1 \oplus ... \oplus V_n) \neq \emptyset$. By hypothesis $(C_{t_{n1}} \oplus C_{t_{n2}} \oplus ... \oplus C_{t_{nn}})(M_1 \oplus ... \oplus M_n) \subseteq (M_1 \oplus ... \oplus M_n)$, then $(C_{t_{n1}} \oplus C_{t_{n2}} \oplus ... \oplus C_{t_{nn}})_{t \geq 0}$ is $M_1 \oplus ... \oplus M_n$-transitive, let $(\oplus_{i=1}^{n} C_{ti})_{t \in \mathbb{R}}$ be on $X_1 \times ... \times X_n$ and consider C_{tn}. Fix $X_{01} \times ... \times X_{0n} = \{x_1 + ... + x_n \in X_1 \times ... \times X_n / \lim_{n \to \infty}(C_{t_{n1}} \oplus ... \oplus C_{t_{nn}})(x_1 + ... + x_n) = 0\}$. □

Lemma 3.12. *Every subspace-hypercyclic operator is hypercyclic with regard to direct sum.*

Proof. For any nonempty set $U_1 \oplus ... \oplus U_n \subseteq (M_1 \oplus ... \oplus M_n)$ and $V_1 \oplus ... \oplus V_n \subseteq (M_1 \oplus ... \oplus M_n)$, both relatively open in $(M_1 \oplus ... \oplus M_n)$, consider $x_{01} + ... + x_{0n} \in (V_1 \oplus ... \oplus V_n)$ and $y_{01} + ... + y_n \in (U_{01} \oplus ... \oplus U_n)$. Since $J_S((T_1 \oplus ... \oplus T_n), M_1 \oplus ... \oplus M_n, x_1 + ... + x_n) = M_1 \oplus ... \oplus M_n$, there exists $n \geq 1$ and $\lambda \in \mathbb{C} \backslash \{0\}$ such that $\lambda(T_1 \oplus ... \oplus T_n)^n(V_1 \oplus ... \oplus V_n) \cap (U_1 \oplus ... \oplus U_n) \neq \emptyset$ and $(T_1 \oplus ... \oplus T_n)(M_1 \oplus ... \oplus M_n) \subseteq M_1 \oplus ... \oplus M_n$. □

Lemma 3.13. *Let $T_i \in B(X)$, suppose T_i are all invertible and M_i are nonzero closed subspaces of X. If for all $x_1 + ... + x_n \in M_1 \oplus ... \oplus M_n$, $J_S(T_1^{-1} \oplus ... \oplus T_n^{-1}, M_1 \oplus ... \oplus M_n, x_1 + ... + x_n) = M_1 \oplus ... \oplus M_n$, then $(T_1 \oplus ... \oplus T_n)^{-1}$ is also subspace supercyclic with respect to $M_1 \oplus ... \oplus M_n$.*

Proof. By 10 $T_1 \oplus ... \oplus T_n$ is subspace-supercyclic for $M_1 \oplus ... \oplus M_n$. By assumption $J_S(T_1^{-1} \oplus ... \oplus T_n^{-1}, M_1 \oplus ... \oplus M_n, x_1 + ... + x_n) = M_1 \oplus ... \oplus M_n$. For any nonempty sets $U_1 \oplus ... \oplus U_n \subseteq M_1 \oplus ... \oplus M_n$ and $V_1 \oplus ... \oplus V_n \subseteq M_1 \oplus ... \oplus M_n$ both relatively open and which contains $x_{01} + ... + x_{0n}, y_{01} + ... + y_{0n}$ respectively, then we have $n > 1$ and $\lambda \in \mathbb{C} \backslash \{0\}$ such that $\lambda(T_1^{-1} \oplus ... \oplus T_n^{-1})^n(V_1 \oplus ... \oplus V_n) \cap (U_1 \oplus ... \oplus U_n) \neq \emptyset$ and $(T_1^{-1} \oplus ... \oplus T_n^{-1})(M_1 \oplus ... \oplus M_n) \subseteq (M_1 \oplus ... \oplus M_n)$. Thus for every $y_{01} + y_{02} + ... + y_{0n} \in M_1 \oplus ... \oplus M_n$; $J_S(T_1^{-1} \oplus ... \oplus T_n^{-1}, M_1 \oplus ... \oplus M_n, x_1 + ... + x_n) = M_1 \oplus ... \oplus M_n$ Hence, $(T_1^{-1} \oplus ... \oplus T_n^{-1}$ is also subspace-supercycic for $(M_1 \oplus ... \oplus M_n)$. □

The following proposition provides conditions for a vector to be 1-weakly subspace-hypercyclic.

Proposition 3.14. *For operators, being 1-weakly subspace supercyclic implies they are cyclic.*

Proof. Let $x = x_1 + ... + x_n \in M_1 \oplus ... \oplus M_n$. If $x = x_1 + ... + x_n$ is 1-weakly subspace-hypercyclic, then $\mathbb{F}.Orb((T_1 \oplus ... \oplus T_n), (x_1 + ... + x_n)) \cap (M_1 \oplus ... \oplus M_n)$ is dense in $M_1 \oplus ... \oplus M_n$. Additionally, by hypothesis, if $x_1 + ... + x_n$ is cyclic then the linear span. Thus, $\mathbb{F}.Orb((T_1 \oplus ... \oplus T_n), (x_1 + ... + x_n)) \cap (M_1 \oplus ... \oplus M_n)$ has a dense linear span. Thus,

$$\overline{\mathbb{F}.Orb((T_1 \oplus ... \oplus T_n), (x_1 + ... + x_n)) \cap (M_1 \oplus ... \oplus M_n)} = M_1 \oplus ... \oplus M_n.$$

□

4. Conclusion

Subspace hypercyclic operators forms a very important class of operators in operator spaces.A lot of their properties have been studied over along period of time, however, complete characterization of this property has not been done. In fact, a lot of open questions remain unanswered with regard to subspace hypercyclicity. Most of these studies have been done in special cases of finite dimensional operator spaces. It is therefore interesting to address these questions in general operator spaces. In this research therefore we extended an investigation on subspace hypercyclicity by investigating different notions of the subspace hypercyclicity. In particular, we considered subspace hypercyclic operators, their powers and direct sums and show that operators under direct sum satisfies various subspace-hypercyclicity criteria and has a lot of interesting properties. An open question that needs to be addressed states: Can one characterize these notions of subspace hypercyclic operators in norm-attainable class?

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