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RESULTS OF SEMIGROUP OF LINEAR EQUATIONS GENERATING LIPSCHITZ PERTURBATIONS OF LINEAR EVOLUTION EQUATIONS

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Abstract. In this paper, results of ω-order preserving partial contraction mapping generating Lipschitz perturbations of linear evolution equation was presented. A certain semilinear value problem was studied where A is the infinitesimal generator of a C_0-semigroup $\{T(t),\ t \geqslant 0\}$ on a Banach space X and $f : [t_0, T] \times X \to X$ is continuous in t and satisfies a Lipschitz condition in u. We assume A to be independent of t and was extended to the case where A depends on t in a way that insure the existence of an evolution system $U(t, s)$, $0 \leqslant s \leqslant t \leqslant T$, for the family $\{A(t)\}_{t\in[0,T]}$ and shows that the initial value problem have a mild solution.

1. Introduction

The solution of the inhomogeneous initial value problem, *i.e.*, the problem with $f \not\equiv 0$ can be represented in terms of the solutions of homogeneous initial value problem via the formula of variation of constants. A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator, and perturbation theory comprises methods for finding an approximate solution to a problem. In perturbation theory, the solution is expressed as a power series in a small parameter ε. The first term is the known solution to the solvable problem. Successive terms in the series at higher powers of ε usually become smaller. Suppose X is a Banach space , $X_n \subseteq X$ is a finite set, $\omega - OCP_n$ the ω-order preserving partial contraction mapping, M_m be a matrix, $L(X)$ be a bounded linear operator on X, P_n a partial transformation semigroup, $\rho(A)$ a resolvent set, $\sigma(A)$ a spectrum of A and $A \in \omega - OCP_n$ is a generator of C_0-semigroup. This paper consist of results of ω-order preserving partial contraction mapping generating a Lipschitz perturbations of linear evolution equations. Agmon *et al.* [1], estimated some boundary problems for solutions of elliptic partial differential equation. Akinyele *et al.* [2], established some perturbation results of the infinitesimal generator in the semigroup of the linear operator. Balakrishnan [3], introduced an operator calculus for infinitesimal generators of semigroup. Banach [4], established and introduced the concept of Banach spaces. Batty *et al.* [5], showed some asymptotic behavior of semigroup of operators. Brezis and Gallouet [6], investigated nonlinear Schrodinger evolution equation. Chill and Tomilov [7], deduced some resolvent approach to stability operator semigroup. Davies [8],

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introduced linear operators and their spectra. Engel and Nagel [9], presented one-parameter semigroup for linear evolution equations. Omosowon *et al.* [10], proved some analytic results of semigroup of linear operator with dynamic boundary conditions, and also in [11], Omosowon *et al.*, established dual Properties of ω-order Reversing Partial Contraction Mapping in Semigroup of Linear Operator. Pazy [12], introduced asymptotic behavior of the solution of an abstract evolution and some applications and also in [13], established a class of semi-linear equations of evolution. Prüss [14], proves some semilinear evolution equations in Banach spaces. Rauf and Akinyele [15], obtained ω-order preserving partial contraction mapping and established its properties, also in [16], Rauf *et al.*, introduced some results of stability and spectra properties on semigroup of linear operator. Vrabie [17], proved some results of C_0-semigroup and its applications. Yosida [18], established some results on differentiability and representation of one-parameter semigroup of linear operators.

2. Preliminaries

Definition 2.1 (C_0-Semigroup) [17]
A C_0-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.
Definition 2.2 (ω-OCP_n) [15]
A transformation $\alpha \in P_n$ is called ω-order preserving partial contraction mapping if $\forall x, y \in \text{Dom}\alpha : x \leq y \implies \alpha x \leq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.
Definition 2.3 (Perturbation) [2]
Let $A : D(A) \subseteq X \to X$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ and consider a second operator $B : D(B) \subseteq X \to X$ such that the sum $A + B$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$. We say that A is perturbed by operator B or that B is a perturbation of A.
Definition 2.4 (Mild Solution) [12]
A continuous solution u of the integral equation

$$u(t) = T(t - t_0)u_0 + \int_{t_0}^{t} T(t - s)f(s, u(s))ds \tag{2.1}$$

will be called a mild solution of the initial value problem

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = f(t, u(t)), \ t > t_0 \\ u(t_0) = u_0 \end{cases} \tag{2.2}$$

if the solution is a Lipschitz continuous function.
Example 1
2×2 matrix $[M_m(\mathbb{N} \cup \{0\})]$
Suppose

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^{2t} & e^{I} \\ e^{t} & e^{2t} \end{pmatrix}.$$

Example 2
3×3 matrix $[M_m(\mathbb{N} \cup \{0\})]$
Suppose

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^{2t} & e^{2t} & e^{3t} \\ e^{2t} & e^{2t} & e^{2t} \\ e^{t} & e^{2t} & e^{2t} \end{pmatrix}.$$

Example 3
3×3 matrix $[M_m(\mathbb{C})]$, we have
for each $\lambda > 0$ such that $\lambda \in \rho(A)$ where $\rho(A)$ is a resolvent set on X.
Suppose we have

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA_\lambda}$, then

$$e^{tA_\lambda} = \begin{pmatrix} e^{2t\lambda} & e^{2t\lambda} & e^{3t\lambda} \\ e^{2t\lambda} & e^{2t\lambda} & e^{2t\lambda} \\ e^{t\lambda} & e^{2t\lambda} & e^{2t\lambda} \end{pmatrix}.$$

Theorem 2.1 Hille-Yoshida [15]
A linear operator $A : D(A) \subseteq X \to X$ is the infinitesimal generator for a C_0-semigroup of contraction if and only if

i. A is densely defined and closed,
ii. $(0, +\infty) \subseteq \rho(A)$ and for each $\lambda > 0$, we have

$$\|R(\lambda, A)\|_{L(X)} \leq \frac{1}{\lambda}. \tag{2.3}$$

3. Main Results

This section present results of semigroup of linear operator by using ω-OCP_n to generate Lipschitz perturbations of linear evolution equations:

Theorem 3.1
Assume $f : [t_0, T] \times X \to X$ is continuous in t on $[t_0, T]$ and uniformly Lipschitz (with constant L) on X. If $A \in \omega - OCP_n$ is the infinitesimal generator of a C_0-semigroup $\{T(t);\ t \geqslant 0\}$ on X, then for every $u_0 \in X$ the initial value problem (2.2) has a unique mild solution $u \in C([t_0, T] : X)$. Moreover, the mapping $u_0 \to u$ is Lipschitz continuous from X into $c([t_0, T] : X)$.
Proof:
For a given $u_0 \in X$ we define a mapping

$$F : C([t_0, T] : X) \to C([t_0, T] : X)$$

by

$$(Fu)(t) = T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s)f(s,u(s))ds \quad t_0 \leqslant t \leqslant T. \tag{3.1}$$

Denoting by $\|u\|_\infty$ the norm of u as an element of $C([t_0, T] : X)$ it follows readily from the definition of F that

$$\|(Fu)(t) - (Fv)(t)\| \leqslant ML(t-t_0)\|u-v\|_\infty \tag{3.2}$$

where M is a bound of $\|T(t)\|$ on $[t_0, T]$ and $A \in \omega - OCP_n$ in the generator of $T(t)$. Using (3.1), (3.2) and the induction on n, then it follows easily that

$$\|(F_u^n)(t) - (F_v^n)(t)\| \leqslant \frac{(ML(t-t_0)^n)}{n!}\|u-v\|_\infty$$

from which

$$\|F_u^n - F_v^n\| \leqslant \frac{(MLT)^n}{n!}\|u-v\|_\infty. \tag{3.3}$$

For n large enough $(MLT)^n/n! < 1$ and by a well known extension of the condition principle F has a unique fixed point u in $C([t_0, T] : X)$. This fixed point is the desired solution of the integral equation (2.1).

The uniqueness of u and the Lipschitz continuity of the map $u_0 \to u$ are consequences of the following argument. Let v be a mild solution of (2.1) on $[t_0, T]$ with the initial value v_0. Then,

$$\begin{aligned} \|u(t) - v(t)\| &\leqslant \|T(t-t_0)u_0 - T(t-t_0)v_0\| + \int_{t_0}^{t} \|T(t-s)(f(s,u(s))) - f(s,v(s))ds\| \\ &\leqslant M\|u_0 - v_0\| + ML\int_{t_0}^{t} \|u(s) - v(s)\|ds \end{aligned} \tag{3.4}$$

which implies, by Gronwall's inequality, that

$$\|u(t) - v(t)\| \leqslant Me^{ML(T-t_0)}\|u_0 - v_0\|$$

and therefore

$$\|u - v\|_\infty \leqslant Me^{ML(T-t_0)}\|u_0 - v_0\|$$

which yields both the uniqueness of u and the Lipschitz continuity of the map $u_0 \to u$. Hence the proof is completed.

Theorem 3.2

Suppose $f : [0, \infty) \times X \to X$ be continuous in t for $t \geqslant 0$ and locally Lipschitz continuous in u, uniformly in t on bounded intervals. If A is the infinitesimal generator of a C_0-semigroup $\{T(t);\ t \geqslant 0\}$ on X, then for every $A \in \omega - OCP_n$ and $u_0 \in X$ there is a $t_{max} \leqslant \infty$ such that the initial value problem

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = f(t, u(t)), & t \geqslant 0 \\ u(0) = u_0 \end{cases} \tag{3.5}$$

has a unique mild solution u on $[0, t_{max})$. Moreover, if $t_{max} < \infty$, then

$$\lim_{t \to t_{max}} \|u(t)\| = \infty.$$

Proof:

We start by showing that for every $t_0 \geqslant 0$, $u_0 \in X$, the initial value problem (2.2) has, under assumptions of our theorem, a unique mild solution u on an interval $[t_0, t]$ whose length is bounded below by

$$\delta(t_0, \|u_0\|) = \min\left\{1, \frac{\|u_0\|}{K(t_0)L(K(t_0), t_0 + 1) + N(t_0)}\right\} \tag{3.6}$$

where $L(c, t)$ is the local Lipschitz constant if and only if

$$\|f(t, u) - f(t, v)\| \leqslant L(c, t')\|u - v\|, \tag{3.7}$$

$$M(t_0) = \max\{\|T(t)\| : 0 \leqslant t \leqslant t_0 + 1\},$$

$$K(t_0) = 2\|u_0\|M(t_0)$$

and

$$N(t_0) = \max\{\|f(t, 0)\| : 0 \leqslant t \leqslant t_0 + 1\}.$$

Indeed, let

$$t_1 = t_0 + \delta(t_0, \|u_0\|)$$

where $\delta(t_0, \|u_0\|)$ is given by (3.6). The mapping F defined by (3.1) maps the ball of radius $K(t_0)$ centered at O of $C([t_0, t_1] : X)$ into itself. This follows from the estimate

$$\begin{aligned}
\|(Fu)(t)\| &\leqslant M(t_0)\|u_0\| + \int_{t_0}^{t} \|T(t - s)\|(\|f(s, u(s)) - f(s, 0)\| + \|f(s, 0)\|)ds \\
&\leqslant M(t_0)\|u_0\| + M(t_0)K(t_0)L(K(t_0), t_0 + 1)(t - t_0) + M(t_0)N(t_0)(t - t_0) \\
&\leqslant M(t_0)\{\|u_0\| + K(t_0)L(K(t_0), t_0 + 1)(t - t_0) + N(t_0)(t - t_0)\} \\
&\leqslant 2M(t_0)\|u_0\| = K(t_0) \qquad (3.8)
\end{aligned}$$

where the last inequality follows from the definition of t_1. In this ball, F satisfies a uniform Lipschitz condition with constant $L = L(K(t_0), t_0 + 1)$ and thus as in the proof of Theorem 3.1 it possesses a unique fixed point u in the ball. This fixed point is the desired solution of (2.1) on the interval $[t_0, t_1]$.

From what we have just proved, it follows that if u is a mild solution of (3.5) on the interval $[0, \tau]$. It can be extended to the interval $[0, \tau + \delta]$ with $\delta > 0$ by defining on $[\tau, \tau + \delta]$, $u(t = w(t))$ where $w(t)$ is the solution of the integral equation

$$w(t) = T(t - \tau)u(\tau) + \int_{\tau}^{t} T(t - s)f(s, w(s))ds, \quad \tau \leqslant t \leqslant \tau + \delta. \tag{3.9}$$

Moreover, δ depends only on $\|u(\tau)\|$, $K(\tau)$ and $N(\tau)$. Let $[0, t_{max}]$ be the maximum interval of existence of the mild solution u of (3.5). If $t_{max} < \infty$, then

$$\lim_{t \to t_{max}} \|u(t)\| = \infty.$$

Since otherwise there is a sequence $t_n \to t_{max}$ such that $\|u(t_n)\| \leqslant C$ for all n. This would imply by what we have just proved for each t_n, near enough to t_{max}, u defined on $[0, t_n]$ can be extended to $[0, t_n + \delta]$ where $\delta > 0$ is independent of t_n and hence u can be extended beyond t_{max} contradicting the definition of t_{max}. To prove the uniqueness of the local mild solution u of (3.5), we note that if v is a mild solution of (3.5), then on every closed interval $[0, t_0]$ on which both u and v exists they coincide by the uniqueness argument given at the end of the proof of Theorem 3.1. Therefore, both u and v have the same t_{max} and on $[0, t_{max})$, $u = v$, and this achieves the proof.

Theorem 3.3

Assume $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup $\{T(t),\ t \geqslant 0\}$ on X. Suppose $f : [t_0, T] \times X \to X$ is continuously differentiable from $[t_0, T] \times X$ into X then the mild solution of (2.2) with $u_0 \in D(A)$ is a classical solution of the initial value problem for all $A \in \omega - OCP_n$.

Proof:

We note first that the continuous differentiability of f from $[t_0, T] \times X$ into X implies that f is continuous in t and Lipschitz continuous in u, uniformly in t on $[t_0, T]$. Therefore the initial value problem (2.2) possesses a unique mild solution u on $[t_0, T]$ by Theorem 3.1. Next we show that this mild solution is continuously differentiable on $[t_0, T]$. To this end, we set

$$B(s) = \left(\frac{\partial}{\partial u}\right) f(s, u)$$

and

$$g(t) = T(t-t_0)f(t_0, u(t_0)) - AT(t-t_0)u_0 + \int_{t_0}^{t} T(t-s)\frac{\partial}{\partial s} f(s, u(s))ds, \tag{3.10}$$

for all $A \in \omega - OCP_n$.

From our assumptions, it follows that $g \in C([t_0, T] : X)$ and that the function $h(t, u) = B(t)u$ is continuous in t from $[t_0, T]$ into X and uniformly Lipschitz continuous in u since $s \to B(s)$ is continuous from $[t_0, T]$ into $B(X)$. Let w be the solution of the integral equation:

$$w(t) = g(t) + \int_{t_0}^{t} T(t-s)B(s)w(s)d(s). \tag{3.11}$$

The existence and uniqueness of $\omega \in C([t_0, T] : X)$ follows that for every $g \in ([t_0, T] : X)$, the integral equation (3.11) has a unique solution $w \in C([t_0, T] : X)$. Moreover, from our assumptions we have

$$f(s, u(s+h)) - f(s, u(s)) = B(s)(u(s+h) - u(s)) + w_1(s, h) \tag{3.12}$$

and

$$f(s+h, u(s+h)) - f(s, u(s+h)) = \left(\frac{\partial}{\partial s}\right) f(s, u(s+h))h + w_2(s, h) \tag{3.13}$$

where $h^{-1}\|w_1(s, h)\| \to 0$ as $h \to 0$ uniformly on $[t_0, T]$ for $i = 1, 2$. If $w_h(t) = h^{-1}(u(t+h) - u(t) - w(t))$, then from the definition of u, (3.11), (3.12) and (3.13). We obtain

$$\begin{aligned} w_h(t) =& [h^{-1}(T(t+h-t_0)u_0 - T(t-t_0)u_0) + AT(t-t_0)u_0] \\ &+ \frac{1}{h}\int_{t_0}^{t} T(t-s)(w_1(s, h) + w_2(s, h))ds \\ &+ \int_{t_0}^{t} T(t-s)\left(\frac{\partial}{\partial s} f(s, u(s+h)) - \frac{\partial}{\partial s} f(s, u(s))\right) ds \\ &+ \left[\frac{1}{h}\int_{t_0}^{t_0+h} T(t+h-s)f(s, u(s))d - T(t-t_0)f(t_0, u(t_0))\right] \\ &+ \int_{t_0}^{t} T(t-s)B(s)w_h(s)ds. \end{aligned} \tag{3.14}$$

It is not difficult to see that the norm of each of the four first terms on the right-hand-side of (3.14) tends to zero as $h \to 0$. Therefore we have

$$\|w_h(t)\| \leqslant \varepsilon(h) + M\int_{t_0}^{t} \|w_h(s)\|ds \tag{3.15}$$

where $M = \max\{\|T(t-s)\|\|B(s)\| : t_0 \leqslant s \leqslant T\}$ and $\varepsilon(h) \to 0$ as $h \to 0$. From (3.15) it follows by Gronwall's inequality that

$$\|w_h(t)\| \leqslant \varepsilon(h)e^{(T-t_0)M}$$

and therefore

$$\|w_h(t)\| \to 0 \quad \text{as} \quad h \to 0.$$

This implies that $u(t)$ is differentiable on $[t_0, T]$ and its derivative is $w(t)$. Since $w \in C([t_0, T] : X)$, u is continuously differentiable on $[t_0, T]$.

Finally, to show that u is the classical solution of (2.2) we note that from the continuous differentiability of f it follows that $s \to f(s, u(s))$ is continuously differentiable on $[t_0, T]$. Then it follows that

$$v(t) = T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s)f(s, u(s))ds \tag{3.16}$$

is the classical solution of the initial value problem

$$\begin{cases} \frac{dv(t)}{dt} + Av(t) = f(t, u(t)) \\ v(t_0) = u_0 \end{cases} \tag{3.17}$$

for all $u \in X$ and $A \in \omega - OCP_n$. But by definition, u is a mild solution of (3.17) and it follows that $u = v$ on $[t_0, T]$. Thus u is a classical solution of the initial value problem (2.2). Hence the proof is completed.

Theorem 3.4

Let A be the infinitesimal generator of a C_0-semigroup $\{T(t);\ t \leqslant 0\}$ on a reflexive Banach space X. If $f : [t_0, T] \times X \to X$ is Lipschitz continuous in both variables, $u_0 \in D(A)$, $A \in \omega - OCP_n$ and u is the mild solution of the initial value problem (2.2), then u is the strong solution of this initial value problem.

Proof:

Let $\|T(t)\| \leqslant M$ and $\|f(t, u(t))\| \leqslant N$ for $t_0 \leqslant t \leqslant T$ and let f satisfies

$$\|f(t_1, x_1) - f(t_2, x_2)\| \leqslant C(|t_1 - t_2| + \|x_1 - x_2\|), \quad t_1, t_2 \in [t_0, T]. \tag{3.18}$$

For $0 < h < t - t_0$ we have

$$\begin{aligned} u(t+h) - u(t =& T(t+h-t_0)u_0 - T(t-t_0)u_0 \\ &+ \int_{t_0}^{t_0+h} T(t+h-s)f(s, u(s))ds \\ &+ \int_{t_0}^{t} T(t-s)[f(s+h, u(s+h)) - f(s, u(s))]ds \end{aligned}$$

and therefore,

$$\begin{aligned}\|u(t+h)-u(t)\| &\leqslant hM\|Au_0\| + hMN + MC\int_{t_0}^{t}(h+\|u(s+h)-u(s)\|)ds \\ &\leqslant C_1h + MC\int_{t_0}^{t}\|u(s+h)-u(s)\|ds\end{aligned}$$

which by Gronwall's inequality implies

$$\|u(t+h)-u(t)\| \leqslant C_1e^{TMC_h} \tag{3.19}$$

and u is Lipschitz continuous.

The Lipschitz continuity of u combined with the Lipschitz continuity of f implies that $t \to f(t,u(t))$ is Lipschitz continuous on $[t_0,T]$. We have that the initial value problem

$$\begin{cases} \frac{dv}{dt} + Av = f(t,u(t)) \\ v(t_0) = u_0 \end{cases} \tag{3.20}$$

has unique strong solution v on $[t_0,T]$ satisfying

$$v(t) = T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s)f(s,u(s))ds = u(t)$$

and so u is a strong solution of (2.2). Hence the proof is completed.

Conclusion

In this paper, it has been established that ω-order preserving partial contraction mapping generates some results of Lipschitz perturbations of linear evolution equations.

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