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ON A LINEAR COMBINATION OF q-STARLIKE AND q-CONVEX EXPRESSIONS

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Abstract. This investigation is about a set of analytic-univalent function-kind

$$\mathfrak{f}(z) = z + \sum_{j \geqq 2} a_j z^j, \quad |z| < 1$$

analytically defined in the two novel sets: $\Omega_{q,\delta}(s, t; \eta)$ and $\mho_{q,\delta}(s, t; \gamma)$. The definition of functions $\mathfrak{f}$ in set $\Omega_{q,\delta}(s, t; \eta)$ involves the generalizations of the sets of q-convex and q-starlike functions; while $\mathfrak{f}$ in set $\mho_{q,\delta}(s, t; \gamma)$ involves the generalizations of the sets of strongly-q-convex and strongly-q-starlike funtions. In addition to these, the two sets have association with the Carathéodory functions and some real parameters. It is well-known that the subsets: q-convex, q-starlike, strongly-q-convex, and strongly-q-starlike have significant, empirical and classical features in the well-known set of analytic-univalent functions. Some mathematical principles employed in the methodology are the concepts of q-calculus, infinite series generation, and the linear combination of some geometric expressions. Thus, the reported results for the two novel sets span through some estimates for the Fekete-Szegö coefficient functionals (with real and complex parameters) and some initial coefficient bounds. Generally speaking, the two novel sets (and their results) reduce to a number of recognized sets (and their results) when values for certain parameters are altered within their declaration interval.

1. Background Details

Let the set of all analytic functions with normalization $\mathfrak{f}(0) = 0$ and $\mathfrak{f}'(0) = 1$ in the open unit disc $|z| < 1$ be denoted by $\mathfrak{A}$. Such kind of functions can be expressed as

$$\mathfrak{f}(z) = z + \sum_{j \geqq 2} a_j z^j, \quad |z| < 1, \tag{1}$$

where each coefficient a_j $(j = 2, 3, 4, \ldots)$ is a complex constant. The coefficient estimates (for instance see [2, 15]), for analytic-univalent functions are fundamental topic in geometric function theory, particularly in the study of the geometric and analytic properties of the analytic-univalent functions in the complex number field. A univalent function, also known as

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a schlicht function, is a function that is one-to-one (injective) in a given domain, typically the unit disc $|z| < 1$.

The study of these coefficients is central to understanding the geometric and analytic properties of univalent functions. The Bieberbach conjecture, posed in 1916 and proven by Branges [4], is a landmark result in this area. It states that for any analytic-univalent function $\mathfrak{f}$ in (1), the coefficients satisfy the inequality $|a_j| \leqq j$ for all $j \geqq 2$. This conjecture was motivated by the observation that the Koebe function $\mathfrak{f}(z) = z/(1-z)^2$, which maps the unit disc onto the entire complex plane minus a slit $x \leqq -\frac{1}{4}$, achieves equality in this bound. The proof of the Bieberbach conjecture has profound implications for the geometry of conformal mappings and has inspired extensive research into coefficient estimates for related sets and subsets of analytic functions, such as the sets of convex functions of order $\eta \in (0, 1]$ (if $\Re e\,((z\mathfrak{f}''(z)/\mathfrak{f}'(z))+1) > \eta$), starlike functions of order $\eta \in [0, 1)$ (if $\Re e\,(z\mathfrak{f}'(z)/\mathfrak{f}(z)) > \eta$), strongly-convex functions of order $\gamma \in (0, 1]$ (if $|\arg((z\mathfrak{f}''(z)/\mathfrak{f}'(z)) + 1)| < \gamma\pi/2$), strongly-starlike functions of order $\gamma \in [0, 1)$ (if $|\arg(z\mathfrak{f}'(z)/\mathfrak{f}(z))| < \gamma\pi/2$), and several more.

Coefficient estimates are not only of theoretical interest but also of practical applications in fields such as fluid dynamics [19, p. 280–286], where analytic-univalent functions model fluid flow, and in engineering [19, p. 247–258], where they describe shapes with optimal properties. The study of these estimates continues to be an active area of research, with several works exploring generalizations, refinements, and connections to other areas of mathematics.

The Carathéodory set of functions, herein denoted by $\mathfrak{P}$, consists of analytic functions of the infinite series kind

$$\rho(z) = 1 + \sum_{j \geqq 1} \varrho_j z^j, \quad |z| < 1 \tag{2}$$

such that $\rho(0) = 1$ and $\Re e\,\rho(z) > 0$ for all $|z| < 1$. This function set is crucial in complex analysis and in particular, geometric function theory due to its connection with the positive real parts of the complex number field, which often arise in the study of univalent functions, bi-univalent functions, conformal mappings, meromorphic functions, and harmonic functions' analysis. Their properties are foundational in understanding the behavior of analytic functions in various applications, including potential theory and operator theory, for instance see [1]. Further, Libera and Livingston [13] introduced the set $\mathfrak{P}(\eta)$ of Carathéodory functions of order η which are subsets of $\mathfrak{P}$ and consists of functions of the infinite series kind

$$\rho_\eta(z) = 1 + (1-\eta)\sum_{j \geqq 1} \varrho_j z^j, \quad \eta \in [0, 1), \text{ and } |z| < 1. \tag{3}$$

Note that if $\eta = 0$, then (3) becomes (2) and $\mathfrak{P}(\eta) \subseteq \mathfrak{P}(0) \equiv \mathfrak{P}$. In fact, it is observable that $\rho \in \mathfrak{P}$ and $\rho_\eta \in \mathfrak{P}(\eta)$ are interrelated such that

$$\frac{\rho_\eta(z) - \eta}{1-\eta} = \rho(z) \Longrightarrow \rho_\eta(z) = \eta + (1-\eta)\rho(z). \tag{4}$$

The Jackson's derivative (also known as quantum derivative, q-derivative, or q-difference) of a real or complex function $\mathfrak{f}$ extends the concept of the classical derivative by incorporating a quantum parameter $q \in (0, 1)$ into the classical derivative of any function $\mathfrak{f}$. For a complex function $\mathfrak{f}$ of the kind (1), its q-derivative is defined by

$$\text{(5)}\qquad \begin{cases} \mathfrak{D}_q\mathfrak{f}(z) = \begin{cases} \mathfrak{f}'(0) & = 1 \qquad\qquad\qquad (z=0), \\ \frac{\mathfrak{f}(z)-\mathfrak{f}(qz)}{(1-q)z} & = 1 + \sum\limits_{j\geqq 2}[j]_q a_j z^{j-1} \quad (z\neq 0), \end{cases} \\ \mathfrak{D}_q^2\mathfrak{f}(z) = \mathfrak{D}_q(\mathfrak{D}_q\mathfrak{f}(z)) = \sum\limits_{j\geqq 2}[j-1]_q[j]_q a_j z^{j-2} \end{cases}$$

such that the conditions $q\in(0,1)$ and $\lim\limits_{q\to 1}[j]_q = j$ hold. This operator is particularly useful in q-calculus, a framework that generalizes classical calculus and finds applications in areas like quantum mechanics, operators, algebra, combinatorics, and special functions. The q-derivative clearly reduces to the classical derivative as q approaches 1, hence, bridging the gap between discrete and continuous analysis. Interested readers may see some texts such as [1,3,9–11,14,16] for more details on applications, definitions, and historical background.

2. Some Novel Classes of Analytic-univalent Functions

Inspired by the properties of the above mentioned function sets, the concept of q-derivative, and the principle of linear combination (see [8, p. 122]) of geometric expressions of q-convexity and q-starlikeness, mixed with some real parameters; we therefore, introduce two new function sets $\Omega_{q,\delta}(s,t;\eta)$ and $\mho_{q,\delta}(s,t;\gamma)$ as follows.

Definition 2.1. Let $s,t\geqq 1$, $\delta\in[0,1]$, $\frac{f(z)f'(z)}{z}\neq 0$, $q\in(0,1)$,

$$\Upsilon_{q,\delta}(s,t;\mathfrak{f}) = \frac{1-\delta}{s}\left\{s-1+\frac{z\mathfrak{D}_q\mathfrak{f}(z)}{\mathfrak{f}(z)}\right\} + \frac{\delta}{t}\left\{t+\frac{z\mathfrak{D}_q^2\mathfrak{f}(z)}{\mathfrak{D}_q\mathfrak{f}(z)}\right\},$$

and $\mathfrak{D}_q\mathfrak{f}(z)$ is as defined in (5). Then $\mathfrak{f}$ of the infinite series kind (1) is said to be in the set $\Omega_{q,\delta}(s,t;\eta)$ if the conditions

$$\text{(6)}\qquad \mathfrak{Re}\,\Upsilon_{q,\delta}(s,t;\mathfrak{f}) > \eta\in[0,1) \text{ and } |z|<1$$

hold. And it is said to be in the set $\mho_{q,\delta}(s,t;\gamma)$ if the conditions

$$\text{(7)}\qquad \left|\arg(\Upsilon_{q,\delta}(s,t;\mathfrak{f}))\right| < \frac{\pi}{2}\gamma, \quad \gamma\in(0,1], \text{ and } |z|<1$$

hold.

Some careful variations of the parameters in (6) shows that

(1) $\mathfrak{C}_q(\eta) = \Omega_{q,1}(s,1;\eta)$, the set of q-convex functions of order η, for instance see [17].
(2) $\mathfrak{C}_q = \Omega_{q,1}(s,1;0)$, the set of q-convex functions, for instance see [17].
(3) $\mathfrak{C}(\eta) = \lim\limits_{q\to 1}\Omega_{q,1}(s,1;\eta)$, the set of convex functions of order η, for instance see [7].
(4) $\mathfrak{C} = \lim\limits_{q\to 1}\Omega_{q,1}(s,1;0)$, the set of convex functions, for instance see [7].
(5) $\mathfrak{S}_q(\eta) = \Omega_{q,0}(1,t;\eta)$, the set of q-starlike functions of order η, for instance see [17].
(6) $\mathfrak{S}_q = \Omega_{q,0}(1,t;0)$, the set of q-starlike functions, for instance see [17].
(7) $\mathfrak{S}(\eta) = \lim\limits_{q\to 1}\Omega_{q,0}(1,t;\eta)$, the set of starlike functions of order η, for instance see [7].
(8) $\mathfrak{S} = \lim\limits_{q\to 1}\Omega_{q,0}(1,t;0)$, the set of starlike functions, for instance see [7].
(9) $\mathfrak{B}_\delta(s,t) = \lim\limits_{q\to 1}\Omega_{q,\delta}(s,t;0)$, the set investigated by Ravikumar and Latha [18].

Note, there are also many subsets of set $\mho_{q,\delta}(s,t;\gamma)$ not mentioned here. Interested researchers can easily harvest some subsets of $\mho_{q,\delta}(s,t;\gamma)$ when the involving parameters are varied within their interval of declaration.

The key properties observed in this work are some upper estimates for the coefficient bounds and Fekete-Szegö functionals, see [6].

3. Lemmas

The following lemmas hold for ρ in (2). We shall need them to proof the results.

Lemma 3.1. ([7, p. 80]). *If $\rho \in \mathfrak{P}$, then $|\varrho_j| \leqq 2\ \forall j = 1,2,3,\dots$.*

Lemma 3.2. ([5, Corollary 2.5]). *If $\rho \in \mathfrak{P}$, then*

$$\left|\varrho_2 - \varkappa\frac{\varrho_1^2}{2}\right| \leqq \begin{cases} 2(1-\varkappa), & (\varkappa \leqq 0). \\ 2, & (\varkappa \in [0,2]). \\ 2(\varkappa-1), & (\varkappa \geqq 2). \\ 2\max\{1, |\varkappa-1|\}, & (\varkappa \in \mathbb{C}). \end{cases}$$

4. Main Results

4.1. Estimates for $\mathfrak{f} \in \Omega_{q,\delta}(s,t;\eta)$.

Theorem 4.1. *If $\mathfrak{f} \in \Omega_{q,\delta}(s,t;\eta)$, then*

$$|a_2| \leqq \frac{2(1-\eta)}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]}, \tag{8}$$

$$|a_3| \leqq \frac{2(1-\eta)}{\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]} + \frac{4(1-\eta)^2\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q^2\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}, \tag{9}$$

and

$$|a_3-\alpha a_2^2| \leqq \begin{cases} \frac{(1-\eta)}{\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]} \times (-4\omega), & (\alpha \leqq u), \\ \frac{(1-\eta)}{\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]} \times 2, & (\alpha \in [u,U]), \\ \frac{(1-\eta)}{\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]} \times 4\omega, & (\alpha \geqq U), \\ \frac{(1-\eta)}{\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]} \times 2\max\{1, 2|\omega|\}, & (\alpha \in \mathbb{C}), \end{cases} \tag{10}$$

where

$$\omega = \left[\frac{\alpha(1-\eta)\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2} - \frac{(1-\eta)\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q^2\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2} - \frac{1}{2}\right],$$

$$u = \frac{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q^2\right]}{\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]},$$

and

$$U = u + \frac{\left[\left(\frac{1-\delta}{s}\right)([2]_q - 1) + \frac{\delta}{t}[2]_q\right]^2}{(1-\eta)\left[\left(\frac{1-\delta}{s}\right)([3]_q - 1) + \frac{\delta}{t}[2]_q[3]_q\right]}.$$

Proof. For $\mathfrak{f} \in \Omega_{q,\delta}(s,t;\eta)$, we write (6) in view of (4) as

$$\Upsilon_{q,\delta}(s,t;\mathfrak{f}) = \eta + (1-\eta)\rho(z) \tag{11}$$

where ρ is as defined in (2). Note that

$$\frac{z\mathfrak{D}_q\mathfrak{f}(z)}{\mathfrak{f}(z)} = 1 + \{[2]_q - 1\}a_2 z + \{([3]_q - 1)a_3 - ([2]_q - 1)a_2^2\}z^2 + \cdots,$$
$$\frac{z\mathfrak{D}_q^2\mathfrak{f}(z)}{\mathfrak{D}_q\mathfrak{f}(z)} = [2]_q a_2 z + \{[2]_q[3]_q a_3 - [2]_q^2 a_2^2\}z^2 + \cdots,$$

so that LHS of (11) becomes

$$\begin{aligned}\Upsilon_{q,\mathfrak{f}}(s,t,\delta) = 1 &+ \left\{\left(\frac{1-\delta}{s}\right)([2]_q - 1) + \frac{\delta}{t}[2]_q\right\} a_2 z \\ &+ \left\{\left[\left(\frac{1-\delta}{s}\right)([3]_q - 1) + \frac{\delta}{t}[2]_q[3]_q\right] a_3 - \left[\left(\frac{1-\delta}{s}\right)([2]_q - 1) + \frac{\delta}{t}[2]_q^2\right] a_2^2\right\} z^2 + \cdots\end{aligned} \tag{12}$$

and RHS of (11) becomes

$$\eta + (1-\eta)\rho(z) = 1 + (1-\eta)\varrho_1 z + (1-\eta)\varrho_2 z^2 + \cdots. \tag{13}$$

Clearly, the corresponding terms of (12) and (13) gives

$$a_2 = \frac{(1-\eta)\varrho_1}{\left(\frac{1-\delta}{s}\right)([2]_q - 1) + \frac{\delta}{t}[2]_q} \tag{14}$$

and

$$|a_2| \leqq \frac{(1-\eta)|\varrho_1|}{\left(\frac{1-\delta}{s}\right)([2]_q - 1) + \frac{\delta}{t}[2]_q}$$

so that the use of Lemma 3.1 gives inequality (8). Secondly,

$$\left[\left(\frac{1-\delta}{s}\right)([3]_q - 1) + \frac{\delta}{t}[2]_q[3]_q\right] a_3 - \left[\left(\frac{1-\delta}{s}\right)([2]_q - 1) + \frac{\delta}{t}[2]_q^2\right] a_2^2 = (1-\eta)\varrho_2 \tag{15}$$

so that the use of (14) in (15) with some demonstrations yields

$$a_3 = \frac{(1-\eta)\varrho_2}{\left[\left(\frac{1-\delta}{s}\right)([3]_q - 1) + \frac{\delta}{t}[2]_q[3]_q\right]} + \frac{(1-\eta)^2\left[\left(\frac{1-\delta}{s}\right)([2]_q - 1) + \frac{\delta}{t}[2]_q^2\right]\varrho_1^2}{\left[\left(\frac{1-\delta}{s}\right)([2]_q - 1) + \frac{\delta}{t}[2]_q\right]^2\left[\left(\frac{1-\delta}{s}\right)([3]_q - 1) + \frac{\delta}{t}[2]_q[3]_q\right]} \tag{16}$$

and

$$|a_3| \leqq \frac{(1-\eta)|\varrho_2|}{\left[\left(\frac{1-\delta}{s}\right)([3]_q - 1) + \frac{\delta}{t}[2]_q[3]_q\right]} + \frac{(1-\eta)^2\left[\left(\frac{1-\delta}{s}\right)([2]_q - 1) + \frac{\delta}{t}[2]_q^2\right]|\varrho_1|^2}{\left[\left(\frac{1-\delta}{s}\right)([2]_q - 1) + \frac{\delta}{t}[2]_q\right]^2\left[\left(\frac{1-\delta}{s}\right)([3]_q - 1) + \frac{\delta}{t}[2]_q[3]_q\right]}$$

where the consideration of Lemma 3.1 implies that inequality (9) holds. Thirdly, using (14) and (16) implies

$$a_3-\alpha a_2^2=\frac{(1-\eta)\varrho_2}{\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}+\frac{(1-\eta)^2\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q^2\right]\varrho_1^2}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}-\alpha\left(\frac{(1-\eta)\varrho_1}{\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q}\right)^2$$

where some simplifications yield

$$a_3-\alpha a_2^2=\frac{(1-\eta)}{\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}\Bigg\{\varrho_2-\left(\frac{\alpha(1-\eta)\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}-\frac{(1-\eta)\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q^2\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}\right)\varrho_1^2\Bigg\}$$

and

$$|a_3-\alpha a_2^2|\leqq\frac{(1-\eta)}{\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}\left|\varrho_2-2\left(\frac{\alpha(1-\eta)\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}-\frac{(1-\eta)\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q^2\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}\right)\frac{\varrho_1^2}{2}\right|$$

or

$$|a_3-\alpha a_2^2|\leqq\frac{(1-\eta)}{\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}\left|\varrho_2-\varkappa\frac{\varrho_1^2}{2}\right| \tag{17}$$

where

$$\varkappa=\frac{2\alpha(1-\eta)\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}-\frac{2(1-\eta)\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q^2\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}. \tag{18}$$

Now, using Lemma 3.2 in (17) and (18) shows that for the expression $\left|\varrho_2-\varkappa\frac{\varrho_1^2}{2}\right|$,

$$2(1-\varkappa)=-4\left[\frac{\alpha(1-\eta)\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}-\frac{(1-\eta)\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q^2\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}-\frac{1}{2}\right]$$

and

$$\varkappa\leqq 0\implies\alpha\leqq\frac{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q^2\right]}{\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}.$$

More so,

$$2(\varkappa-1)=4\left[\frac{\alpha(1-\eta)\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}-\frac{(1-\eta)\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q^2\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}-\frac{1}{2}\right]$$

and

$$\varkappa\geqq 2\implies\alpha\geqq\frac{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}{(1-\eta)\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}+\frac{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q^2\right]}{\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}.$$

And lastly,

$$|\varkappa-1| = \left| \frac{2\alpha(1-\eta)\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2} - \frac{2(1-\eta)\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q^2\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2} - 1 \right|$$
$$= 2\left| \frac{\alpha(1-\eta)\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2} - \frac{(1-\eta)\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q^2\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2} - \frac{1}{2} \right|.$$

Therefore, with these results, (17) metamorphosis into (10). □

Remark 4.2. Setting $\delta = 0$, $s = 1 = t$ and $\eta = 0$ means that $\lim_{q\to 1}\Omega_{q,\delta}(s,t;\eta) = \mathfrak{S}$ and

$$|a_2| \leqq 2, \quad |a_3| \leqq 3, \quad \text{and} \quad |a_3-\alpha a_2^2| \leqq \begin{cases} 3-4\alpha & (\alpha \leqq \frac{1}{2}) \\ 1 & (\alpha \in [\frac{1}{2},1]) \\ 4\alpha-3 & (\alpha \leqq \frac{1}{2}) \\ \max\{1,|4\alpha-3|\} & (\alpha\in\mathbb{C}). \end{cases}$$

The set $\mathfrak{S}$ is the well-known set of starlike functions. See [7,8,12] for the results.

Remark 4.3. Setting $\delta = s = t = 1$ and $\eta = 0$ means that $\lim_{q\to 1}\Omega_{q,\delta}(s,t;\eta) = \mathfrak{C}$ and

$$|a_2| \leqq 1, \quad |a_3| \leqq 1, \quad \text{and} \quad |a_3-\alpha a_2^2| \leqq \begin{cases} 1-\alpha & (\alpha \leqq \frac{2}{3}) \\ \frac{1}{3} & (\alpha \in [\frac{2}{3},\frac{4}{3}]) \\ \alpha-1 & (\alpha \leqq \frac{4}{3}) \\ \max\{\frac{1}{3},|\alpha-1|\} & (\alpha\in\mathbb{C}). \end{cases}$$

The set $\mathfrak{C}$ is the well-known set of convex functions. See [7,8,12] for the results.

4.2. Estimates for $\mathfrak{f}\in\mho_{q,\delta}(s,t;\gamma)$.

Theorem 4.4. *If $\mathfrak{f}\in\mho_{q,\delta}(s,t;\gamma)$, then*

$$|a_2| \leqq \frac{2\gamma}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]}, \tag{19}$$

$$|a_3| \leqq \frac{2\gamma}{\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]} + \frac{4\gamma^2\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q^2\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}, \tag{20}$$

and

$$|a_3-\beta a_2^2| \leqq \begin{cases} \frac{\gamma}{\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}\times(-4\varpi), & (\beta\leqq v), \\ \frac{\gamma}{\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}\times 2, & (\beta\in[v,V]), \\ \frac{\gamma}{\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}\times 4\varpi, & (\beta\geqq V), \\ \frac{(1-\eta)}{\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}\times 2\max\{1,2|\varpi|\}, & (\beta\in\mathbb{C}), \end{cases} \tag{21}$$

where

$$\begin{aligned}\varpi =&\gamma\left[\frac{\beta\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}\right.\\&\left.-\frac{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q^2\right]\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}-\frac{1}{2}\right],\\ \upsilon =&\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q^2\right]-\frac{(1-\gamma)\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}{2\gamma\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]},\end{aligned}$$

and

$$V =\upsilon+\frac{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}{\gamma\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}.$$

Proof. For $\mathfrak{f}\in\mho_{q,\delta}(s,t;\gamma)$, we write (7) as

$$\Upsilon_{q,\delta}(s,t;\mathfrak{f})=(\rho(z))^{\gamma} \tag{22}$$

where ρ is as defined in (2) and by simple computation, we know that

$$(\rho(z))^{\gamma}=1+\gamma\varrho_1 z+\left(\gamma\varrho_2-\frac{\gamma(1-\gamma)\varrho_1^2}{2}\right)z^2+\cdots. \tag{23}$$

Now, the corresponding terms in (12) and (23) gives

$$a_2=\frac{\gamma\varrho_1}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]} \tag{24}$$

and

$$|a_2|\leqq\frac{\gamma|\varrho_1|}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]}$$

so that the use of Lemma 3.1 gives inequality (19). Secondly,

$$\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]a_3-\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q^2\right]a_2^2=\gamma\varrho_2-\frac{\gamma(1-\gamma)\varrho_1^2}{2} \tag{25}$$

so that the use of (24) in (25) with some demonstrations yields

$$a_3=\frac{\gamma\left(\varrho_2-(1-\gamma)\frac{\varrho_1^2}{2}\right)}{\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}+\frac{\gamma^2\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q^2\right]\varrho_1^2}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]} \tag{26}$$

and

$$|a_3|\leqq\frac{\gamma\left|\varrho_2-(1-\gamma)\frac{\varrho_1^2}{2}\right|}{\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}+\frac{\gamma^2\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q^2\right]|\varrho_1|^2}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}$$

where the consideration of Lemmas 3.1 and 3.2 implies that inequality (20) holds. Thirdly, using (24) and (26) implies

$$a_3 - \beta a_2^2 = \frac{\gamma\left(\varrho_2 - (1-\gamma)\frac{\varrho_1^2}{2}\right)}{\left[\left(\frac{1-\delta}{s}\right)([3]_q - 1) + \frac{\delta}{t}[2]_q[3]_q\right]} + \frac{\gamma^2\left[\left(\frac{1-\delta}{s}\right)([2]_q - 1) + \frac{\delta}{t}[2]_q^2\right]\varrho_1^2}{\left[\left(\frac{1-\delta}{s}\right)([2]_q - 1) + \frac{\delta}{t}[2]_q\right]^2\left[\left(\frac{1-\delta}{s}\right)([3]_q - 1) + \frac{\delta}{t}[2]_q[3]_q\right]} - \beta\left(\frac{\gamma\varrho_1}{\left(\frac{1-\delta}{s}\right)([2]_q - 1) + \frac{\delta}{t}[2]_q}\right)^2$$

where some simplifications yield

$$a_3 - \beta a_2^2 = \frac{\gamma\varrho_2}{\left[\left(\frac{1-\delta}{s}\right)([3]_q - 1) + \frac{\delta}{t}[2]_q[3]_q\right]} - \frac{\gamma(1-\gamma)\frac{\varrho_1^2}{2}}{\left[\left(\frac{1-\delta}{s}\right)([3]_q - 1) + \frac{\delta}{t}[2]_q[3]_q\right]} + \frac{\gamma^2\left[\left(\frac{1-\delta}{s}\right)([2]_q - 1) + \frac{\delta}{t}[2]_q^2\right]\varrho_1^2}{\left[\left(\frac{1-\delta}{s}\right)([2]_q - 1) + \frac{\delta}{t}[2]_q\right]^2\left[\left(\frac{1-\delta}{s}\right)([3]_q - 1) + \frac{\delta}{t}[2]_q[3]_q\right]} - \frac{\beta\gamma^2\varrho_1^2}{[\left(\frac{1-\delta}{s}\right)([2]_q - 1) + \frac{\delta}{t}[2]_q]^2}$$

and further simplifications yield

$$a_3 - \beta a_2^2 = \frac{\gamma}{\left[\left(\frac{1-\delta}{s}\right)([3]_q - 1) + \frac{\delta}{t}[2]_q[3]_q\right]}\left\{\varrho_2 - \left(\frac{(1-\gamma)}{2} - \frac{\gamma\left[\left(\frac{1-\delta}{s}\right)([2]_q - 1) + \frac{\delta}{t}[2]_q^2\right]\left[\left(\frac{1-\delta}{s}\right)([3]_q - 1) + \frac{\delta}{t}[2]_q[3]_q\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q - 1) + \frac{\delta}{t}[2]_q\right]^2} + \frac{\beta\gamma\left[\left(\frac{1-\delta}{s}\right)([3]_q - 1) + \frac{\delta}{t}[2]_q[3]_q\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q - 1) + \frac{\delta}{t}[2]_q\right]^2}\right)\varrho_1^2\right\}.$$

Now,

$$|a_3 - \beta a_2^2| \leqq \frac{\gamma}{\left[\left(\frac{1-\delta}{s}\right)([3]_q - 1) + \frac{\delta}{t}[2]_q[3]_q\right]}\left|\varrho_2 - 2\left(\frac{(1-\gamma)}{2} - \frac{\gamma\left[\left(\frac{1-\delta}{s}\right)([2]_q - 1) + \frac{\delta}{t}[2]_q^2\right]\left[\left(\frac{1-\delta}{s}\right)([3]_q - 1) + \frac{\delta}{t}[2]_q[3]_q\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q - 1) + \frac{\delta}{t}[2]_q\right]^2} + \frac{\beta\gamma\left[\left(\frac{1-\delta}{s}\right)([3]_q - 1) + \frac{\delta}{t}[2]_q[3]_q\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q - 1) + \frac{\delta}{t}[2]_q\right]^2}\right)\frac{\varrho_1^2}{2}\right|$$

or for brevity we let

$$|a_3 - \beta a_2^2| \leqq \frac{\gamma}{\left[\left(\frac{1-\delta}{s}\right)([3]_q - 1) + \frac{\delta}{t}[2]_q[3]_q\right]}\left|\varrho_2 - \varkappa\frac{\varrho_1^2}{2}\right| \tag{27}$$

where

$$\varkappa=(1-\gamma)-\frac{2\gamma\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q^2\right]\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}$$

(28)
$$+\frac{2\beta\gamma\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}.$$

So, using Lemma 3.2 in (27) and (28) shows that for the expression $\left|\varrho_2-\varkappa\frac{\varrho_1^2}{2}\right|$,

$$2(1-\varkappa)=-4\gamma\left[\frac{\beta\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}-\frac{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q^2\right]\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}-\frac{1}{2}\right]$$

and

$$\varkappa\leqq 0\implies\beta\leqq\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q^2\right]-\frac{(1-\gamma)\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}{2\gamma\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}.$$

More so,

$$2(\varkappa-1)=4\gamma\left[\frac{\beta\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}-\frac{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q^2\right]\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}-\frac{1}{2}\right]$$

and

$$\varkappa\geqq 2\implies\beta\geqq\frac{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}{\gamma\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}+\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q^2\right]-\frac{(1-\gamma)\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}{2\gamma\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}.$$

And lastly,

$$|\varkappa-1|=\left|(1-\gamma)-\frac{2\gamma\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q^2\right]\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}+\frac{2\beta\gamma\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}-1\right|$$

where simple rearrangement gives

$$|\varkappa-1|=2\gamma\left|\frac{\beta\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}-\frac{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q^2\right]\left[\left(\frac{1-\delta}{s}\right)([3]_q-1)+\frac{\delta}{t}[2]_q[3]_q\right]}{\left[\left(\frac{1-\delta}{s}\right)([2]_q-1)+\frac{\delta}{t}[2]_q\right]^2}-\frac{1}{2}\right|.$$

Therefore, with these results, (27) metamorphosis into (21). □

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