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MULTIPLE POSITIVE SOLUTIONS OF DISCRETE THIRD-ORDER THREE-POINT BVP WITH SIGN-CHANGING GREEN'S FUNCTION

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ABSTRACT. In this article, by using the Leggett-Williams fixed point theorem we research the multiple Positive Solutions for the following third-order three-point boundary value problem (BVP):

$$\begin{cases} \Delta^3 u(t-1) = a(t)f(t,u(t)), \quad t \in [1, T-2]_{\mathbb{Z}}, \\ u(T) = \Delta^2 u(0) = \Delta u(T-1) - \Delta^2 u(\eta) = 0 \end{cases}$$

where $T > 6$ is an integer and $f : [1, T-2]_{\mathbb{Z}} \times [0, +\infty) \to [0, +\infty)$ is continuous $.a : [0, T-2]_{\mathbb{Z}} \to (0, \infty)$, and η satisfies the condition:
F_0 $\eta \in [\frac{T-1}{2}, T-2]$. if T is an odd number or $\eta \in [\frac{T-2}{2}, T-2]$. if T is an even number
The emphasis is mainly that although the cor- responding Green's function is sign-changing, we still obtain the existence of at least $2n-1$ positive solutions for arbitrary positive integer m under suitable conditions on f.

1. INTRODUCTION

The problems of multi-point border values of differential equations have a broad application In computational physics, economics, and modern biological fields [1].Gupta [2] studied the ability to solve the problem of three-point marginal value in a differential equation in 1992. Soon afterwards, there arose many results on multi-point nonlinear boundary value problems At 1999, Ma [7] studied a positive solution to a second-tier differential three-point problem of border value.Subsequently, several conclusions were examined regarding the existence of positive solutions to multi-point border value problems. With the development of the computing science and the computer simulation, multi-point boundary value problems should be

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discretized, so we need to study corresponding difference equation.

In 1998, by using Krasnoselskii's fixed point theorem, Agarwal and Henderson [24] studied the discrete problem

$$\begin{cases} \Delta^3 u(t-1) = \lambda a(t) f(t, u(t)), & t \in [2, T]_{\mathbb{Z}}, \\ u(0) = u(1) = u(T+1) = 0 \end{cases}$$

They obtained the existence of positive solutions in two cases for $\lambda = 1$ and $\lambda \neq 1$. Later, there were many interesting results on the positive solutions to the discrete boundary value problems, see, for instance, $[23-26]$ and the references therein. It is noted that Green's functions are positive in most of these results.However,when the Green's function is sign-changing, could we also obtain the existence of positive solutions to these kinds of problems?
In 2015, by using the Guo-Krasnoselskii fixed point theorem, Wang and Gao [25] studied the existence of positive solutions to the discrete third-order three-point boundary value problem

$$\begin{cases} \Delta^3 u(t-1) = \lambda a(t) f(t, u(t)), & t \in [0, T-1]_{\mathbb{Z}}, \\ u(0) = \Delta u(T) = \Delta^2 u(\eta) = 0 \end{cases}$$

In this paper we study the following third-order three-point BVP :

$$\begin{cases} \Delta^3 u(t-1) = a(t) f(t, u(t)), & t \in [1, T-2]_{\mathbb{Z}}, \\ u(T) = \Delta^2 u(0) = \Delta u(T-1) - \Delta^2 u(\eta) = 0 \end{cases} \tag{1}$$

where $T > 6$ is an integer, $a : [1, T-2]_{\mathbb{Z}} \to (0, +\infty)$ and $f : [1, T-2]_{\mathbb{Z}} \times [0, +\infty) \to [0, +\infty)$ is continuous. Difference equations appear in many mathematical models in diverse fields, such as economy, biology, physics, and finance; see [1-3]. In recent years, the existence and multiplicity of positive solutions of discrete boundary value problems have received much attention from many authors and a great deal of work has been done by using classical methods such as fixed point theory [4-8], lower and upper solutionsmethods [9], critical point theory [10-12], etc.
Theorem 1.1 (Leggett-Williams fixed point theorem) Let $A : \bar{K}_c \to \bar{K}_c$ be completely continuous and β be a nonnegative continuous concave functional on K such that $\beta(u) \leq \|u\|$ for all $u \in \bar{K}_c$ assume that there exist $0 < d < a < b \leq c$ such that
$(i)\{u \in K(\beta, a, b) : \beta(u) > a\} \neq \theta$ and $\beta(Au) > 0$ for $u \in K(\beta, a, b)$
$(ii)\|Au\| < d$ for $\|u\| \leq d$
$(iii)\beta(Au) > a$ for $u \in K(\beta, a, c)$, with $\|Au\| > b$
Then A has at least three fixed points u_1, u_2, u_3 in $\bar{K}_c$ satisfying.

$$\|u_1\| < d, \beta(u_2) > a, \|u_3\| > d, \beta(u_3) < a$$

2. Preliminaries

First, let us consider the following linear problem:

$$\begin{cases} \Delta^3 u(t-1) = y(t), & t \in [1, T-2]_{\mathbb{Z}}, \\ u(T) = \Delta^2 u(0) = \Delta u(T-1) - \Delta^2 u(\eta) = 0. \end{cases} \tag{2}$$

We will convert (2.1) to the equivalent summation equation. To get it, let us define the Green's function $G(t, s)$ as follows.

If $s > \eta$, then

$$G(t,s) = \begin{cases} \frac{(T+s-2t)(T-s-1)}{2}, & s > t-2, \\ \frac{(T+s-2t)(T-s-1)}{2} + \frac{(t-s-1)(t-s)}{2}, & s \le t-2. \end{cases} \tag{3}$$

If $s \le \eta$, then

$$G(t,s) = \begin{cases} \frac{(T-s)(T+s+2t)+(4t-3T)}{2}, & s > t-2, \\ \frac{(T-s)(T+s+2t)+(4t-3T)}{2} + \frac{(t-s-1)(t-s)}{2}, & s \le t-2. \end{cases} \tag{4}$$

Now, we get the following lemma

Lemma 2.1 The problem (2.1) has a unique solution

$$u(t) = \sum_{s=1}^{T-2} G(t,s)y(s), \tag{5}$$

where $G(t,s)$ is defined as (2.2) and (2.3).

Proof. By summing from $s = 1$ to $s = t-1$ at both sides of (2.1), we get

$$\Delta^2 u(t-1) = \sum_{s=1}^{t-1} y(s).$$

Repeating the above process, we obtain

$$\Delta u(t-1) = \Delta u(0) + \sum_{s=1}^{t-2} (t-s-1)y(s).$$

Summing from $s = 1$ to$s = t$ at both sides of the above equation, we have

$$u(t) = u(0) + t\Delta^2 u(0) + \sum_{s=1}^{t-2} \frac{(t-s)(t-s-1)}{2} y(s).$$

By using the boundary condition $u(T) = \Delta^2 u(0) = \Delta u(T-1) - \Delta^2 u(\eta) = 0$, we get

$$\Big\{ \ \Delta u(0) + \textstyle\sum_{s=1}^{T-2} (T-s-1)y(s) - \sum_{s=1}^{\eta} y(s) = 0.$$

Furthermore, we get

$$\Big\{ \ \Delta u(0) = -\textstyle\sum_{s=1}^{T-2} (T-s-1)y(s) + \sum_{s=1}^{\eta} y(s).$$

Then we have

$$u(t) = \sum_{s=1}^{T-2} \frac{(T-s-1)(T+s-2t)}{2} y(s) - (T-t)\sum_{s=1}^{\eta} y(s) + \sum_{s=1}^{t-2} \frac{(t-s)(t-s-1)}{2} y(s),$$

which implies (2.2) holds. □

Obviously, if$u$ is a fixed point of A in K, then u is a nonnegative and decreasing solution of the BVP (1.1).

Lemma 2.2 It is not difficult to verify that $G(t,s)$ has the following characteristics:

(i) If $s \in [1,\eta]$, then $G(t,s)$ is non increasing with respect to $t \in [0,T]$. If $s \in [\eta+1, T-2]$ is nondecreasing with respect to $t \in [0,T]$.

(ii) $G(t,s)$ changes its sign on $[0,T] \times [1,T-2]$. In details, if $(t,s) \in [0,T] \times [0,\eta]$, then

$G(t,s)\geq 0$. If $(t,s)\in[0,T]\times[\eta+1,T-2]$, then $G(t,s)\leq 0$.
(iii) If $s\geq\eta$, then $\max_{t\in[0,T]}G(t,s)=G(T,s)=0$ such that

$$G(t,s)\geq 0 \text{ for } 1\leq s\leq\eta \text{ and } G(t,s)\leq 0 \text{ for } \eta\leq s\leq T-2.$$

Moreover, if $s\geq\eta$, then
$\max G(t,s):t\in[0,T]=G(T,s)=0,$

$$\min G(t,s):t\in[0,T]=G(0,s)=-\frac{(T-s)(T-s-1)}{2}\geq-\frac{(T-\eta)(T-\eta-1)}{2}$$

if $s<\eta$, then

$$\max G(t,s):t\in[0,T]=G(0,s)=\frac{(T-s-1)(T+s)}{2}\leq\frac{(T-\eta-1)(T+\eta)}{2},$$

$$\min G(t,s):t\in[0,T]=G(T,s)=0$$

Now, let

$$E=\{u:[0,T]_z\to R|u(T)=\Delta^2u(0)=\Delta u(T)-\Delta^2u(\eta)=0\}$$

Then E is a Banach space under the norm $\|u\|=\max_{t\in[0,T]_z}|u(t)|$.
let

$$K_0=\{y\in E: y(t)\geq 0,\Delta y(t)\geq 0, t\in[0,T]_z and \quad \Delta^2y(t-1)>0, t\in[\eta+1,T]_z\}.$$

$u(t)$ is nonnegative and decreasing Then K_0 is a cone in E.
Lemma 2.3 Assume $y\in E$,$y(t)\geq 0$ for $t\in[0,T+1]_z$ and $\Delta y(t)\geq 0$ for $t\in[0,T]_z$. Then u is the unique solution of the BVP (2.1) belongs to K_0, where $u(t)$ is defined as (2.4).
Moreover, $u(t)$ is concave on $[\eta+1,T+1]_z$.

Proof. The following proof will be divided into two cases.
Case I. For $0\leq t-2<\eta$, we have

$$\begin{aligned}u(t)=&\sum_{s=1}^{t-2}\frac{(t-T)(T+t)}{2}y(s)\\&+\sum_{s=t-1}^{\eta}\frac{(T-s-1)(T-2t+s)}{2}y(s)\\&-\sum_{s=\eta+1}^{T-2}\frac{(T-s)(T-s-1)}{2}y(s)\end{aligned}$$

$$\begin{aligned}\Delta u(t)&=u(t+1)-u(t)\\&=\sum_{s=1}^{t-2}\frac{2t+1}{2}y(s)+y(t-1)-\sum_{s=1}^{\eta}(T-s-1)y(s)\\&\geq y(\eta)\left[\sum_{s=1}^{t-2}\frac{2t+1}{2}-\sum_{s=1}^{\eta}(T-s-1)\right]\\&\geq y(\eta)\left[(2t+1)(t-2)-2\eta T+\eta(\eta+1)+2\eta\right]\geq 0\end{aligned}$$

and

$$\Delta^2u(t-1)\;=\;\sum_{s=1}^{t-2}y(s)\geq 0$$

Second, if $\eta < t-2 \le T-2$, then

$$u(t) = \sum_{s=1}^{\eta} \frac{(t-T)(T+t)}{2} y(s) + \sum_{s=\eta+1}^{t-2} \frac{(t-T)(T+t-2s-1)}{2} y(s) - \sum_{s=t-1}^{T-2} \frac{(T-s)(T-s-1)}{2} y(s)$$

$$\begin{aligned}
\Delta u(t) &= u(t+1) - u(t) \\
&= \sum_{s=1}^{t-2} \frac{2t+1}{2} y(s) + \sum_{s=\eta+1}^{t-2} (t-s) y(s) + y(t-1) \\
&\ge y(\eta) \left[\sum_{s=1}^{t-2} \frac{2t+1}{2} y(s) + \sum_{s=\eta+1}^{t-2} (t-s) y(s) \right] \\
&\ge y(\eta) \left[(3t+2)(t-2) \right] \ge 0
\end{aligned}$$

$$\Delta^2 u(t-1) = \sum_{s=1}^{t-2} y(s) - \sum_{s=\eta+1}^{t-2} s y(s) \le 0$$

consequently for $t \in [0,T]_z$

$$\Delta u(t) \ge 0$$

which mean that $u(t)$ is increasing ago $\Delta u(T) = 0$,for $t \in [0, T+1]_z$ we have $u(t) \ge 0$ and $u \in K_0$. for $t \in [\eta+1, T]_z$, $\Delta^2 u(t-1) \ge 0$ we get that $u(t)$ is concave on $[\eta+1, T+1]_z$.

□

Lemma 2.4 Suppose that $y \in E$, $y(t) \ge 0$ for $t \in [0, T+1]_z$, $\Delta y(t) \ge 0$ for $t \in [0,T]_z$ and u is the solution of (2.1). Then u satisfies

$$\begin{aligned}
\min_{t \in [\theta, T+1-\theta]} u(t) &\ge u(\theta) \ge \frac{\theta-\eta-1}{T-\eta} \|u\| = \theta^* \|u\| \\
\textit{where} \quad \theta^* &= \frac{\theta-\eta-1}{T-\eta} \quad \textit{and} \quad \theta \in [T+1, \eta+2]
\end{aligned}$$

Proof. From Lemma (2.2), we teach that u is the concave on $t \in [\eta+2, T+1]_z$. thus this

$$u(t) \ge \frac{u(T+1) - u(\eta+1)}{T-\eta} \le \frac{u(t) - u(\eta+1)}{t-\eta-1}, \quad t \in [\eta+1, T+1]_z$$

Finally,by direct account, we get

$$u(t) \ge \frac{(t-\eta-1)}{T-\eta} u(T+1) = \frac{(t-\eta-1)}{T-\eta} \|u\|$$

$$\min_{t \in [\theta, T+1-\theta]} u(t) = u(\theta) \ge \frac{\theta-\eta-1}{T-\eta} \|u\| = \theta^* \|u\|.$$

□

3. Main results

In this section, we conclude the existence of a positive solution of (1.1). To get it, we assume that:
$(F1)$ $f : [1, T-2]_z \times [0, +\infty) \to [0, \infty)$ is continuous and mapping $u \mapsto f(t, u)$ is nondecreasing for each $t \in [1, T-2]_z$;
$(F2)$ $a : [1, T-2]_z \to (0, +\infty)$ is increasing function.
$K = \{u \in K_0 : u(0), \min_{t\in[\mu,T-\mu]_z} u(t) \geq \mu^* \|u\|\}$,
consequently, K is a cone in E define an operator $A : K \to E$ such as

$$Au(t) = \sum_{s=1}^{T-2} G(t, s)a(s)f(s, u(s)) \tag{6}$$

Lemma 3.1 $A : K \to K$ is perfectly continuous.

Proof. It is obvious that $A : K \to E$ is completely continuous since the Banach space E is finite dimensional. Now, let us prove that $A : K \to K$, that is to say, for any $u \in K$, $Au \in K$.
Let $u \in K$. Then $u \in K_0$, which implies that

$$\Delta u(t) \geq 0$$

and u is increasing on t.
Therefore, by $(F1)$, $f(t, u(t))$ is a increasing function of t.
Let $y(t) := a(t)f(t, u(t))$. Then, from $(F1)$ and $(F2)$, we obtain that $y(t) \geq 0$ and y is also a increasing function of t. Thus, $y \in K_0$. moreover, by (3.1), we know that

$$\begin{aligned} &\Delta^3(Au)(t-1) = y(t), \quad t \in [1, T-2]_z, \\ &u(0) = \Delta^2 u(0) = \Delta u(T) - \Delta^2 u(\eta) = 0. \end{aligned} \tag{7}$$

$$(Au)(0) = \Delta^2(Au)(0) = 0 \quad \Delta(Au)(T) - \Delta^2(Au)(\eta) = 0. \tag{8}$$

Therefore, Au satisfies problem (2.1). Now, similar to the proof of Lemma 2.3, and using the fact $y \in K_0$, we obtain that $Au \in K_0$ and Au is concave on $[\eta + 1, T + 1]_z$. Furthermore, by Lemma 2.4 and the fact $Au \in K_0$, we know that

$$\min_{t\in[\mu,T-\mu]} (Au)(t) \quad \geq \quad \mu^* \|Au\|$$

Therefore, $Au \in K$ and $A : K \to K$ is completely continuous set.
From(3.1)and Lemma3.1,we know that if u is afixed point of A inK,then u is a positive solution of (1.1). Let

$$B = \sum_{s=1}^{T-2} \frac{(T-\eta-1)(T+\eta)}{2} a(s), \qquad D = \sum_{s=\mu}^{T-\mu} G(T-\mu, s)a(s)$$

□

Theorem 3.2 Assume that there exist numbers d,a and c with
$0 < d < a < \frac{a}{\mu^*} \leq c$ such that $(H1) f(t, u) < \frac{d}{B}, for (t, u) \in [1, T-2]_z \times [0, d]$,
$(H2) f(t, u) > \frac{a}{D}, for (t, u) \in [\mu, T-\mu]_z \times [a, \frac{a}{\mu^*}]$

$(H3) f(t,u) < \frac{c}{B}, for(t,u) \in [1, T-2]_z \times [0,c]$,
then boundary value problem (1.1) has at least three positive solutions u,v and w satisfying

$$\|u\| < d, \quad \min_{t\in[\mu,T-\mu]} v(t) > a, \quad \|w\| > d, \quad \min_{[\mu,T-\mu]} w(t) < a$$

Proof. for $u \in K$.we define

$$\beta(u) = \min_{[\mu,T-\mu]} u(t).$$

It is simple to check that β is a nonnegative continuous concave functional on K with $\beta(u) \leq \|u\|$ for $u \in k$ and that $A : k \to k$ is completely continuous.

We confirm first that if there exists a positive number r such that $f(t,u) < \frac{r}{B}$ for $t \in [1, T-2]$ $u \in [0,r]$ then $A : \bar{k}_r \to \bar{k}_r$ in effect,if $u \in \bar{k}_r$ then

$$\begin{aligned}
\|Au\| &= \max_{t\in[0,T]_z} \left| \sum_{s=1}^{T-2} G(t,s)a(s)f(s,u(s)) \right| \\
&\leq \max_{t\in[0,T]_z} \sum_{s=1}^{T-2} |G(t,s)|\, a(s)f(s,u(s)) \\
&\leq \frac{(T-\eta-1)(T+\eta)}{2} \sum_{s=1}^{T-2} a(s)f(s,u(s)) \\
&< \frac{(T-\eta-1)(T+\eta)}{2} \sum_{s=1}^{T-2} \frac{r}{B} a(s) \\
&= r = \|u\|.
\end{aligned}$$

that is,$Au \in K_r$.
Subsequently,we have shown that if $(H1)$ and $(H3)$ hold, then $A : \bar{K}_d \to K_d$ and $A : \bar{K}_c \to K_c$.
Next, we assure that $\left\{u \in K(\beta, a, \frac{a}{\mu^*}) : \beta(u) > a\right\} \neq 0$ and $\beta(Au) > a$ for all $u \in K(\beta, a, \frac{a}{\mu^*})$.
In fact, the constant function $\frac{\frac{a}{\mu^*}+a}{2}$belongs to
$\left\{u \in K(\beta, a, \frac{a}{\mu^*}) : \beta(u) > a\right\}$. Then,for $u \in K(\beta, a, \frac{a}{\mu^*})$, we have

$$a < \beta(u) = \min_{t\in[0,\mu]} u(t) \leq u(t) \leq \|u\| \leq \frac{a}{\mu^*} \tag{9}$$

for all $t \in [0,\mu]$. Also,
we know that $G(t,s) \geq 0$for $t-2 < s \leq \eta$. for any $u \in K$ and $t \in [0,\mu]$, we have

$$\begin{aligned}
&\sum_{s=1}^{T-\mu-1} G(t,s)a(s)f(s,u(s)) + \sum_{s=\mu+1}^{\eta} G(t,s)a(s)f(s,u(s)) \\
&\quad + \sum_{s=\eta+1}^{T-2} G(t,s)a(s)f(s,u(s)) \\
&\geq \sum_{s=1}^{T-\mu-1} \frac{(t-T)(T+t)}{2} a(s)f(s,u(s)) - \sum_{s=\mu+1}^{T-2} \frac{(T-s)(T-s-1)}{2} a(s)f(s,u(s)) \\
&\geq a(\eta)f(\eta,u(\eta)) \left[\sum_{s=1}^{T-\mu-1} \frac{(t-T)(T+t)}{2} - \sum_{s=\mu+1}^{T-2} \frac{(T-s)(T-s-1)}{2} \right] \\
&\geq a(\eta)f(\eta,u(\eta)) \left[(2-(T-\mu))(T-\mu-1) \right] \geq 0
\end{aligned}$$

This indicates that $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$. Therefore,$A$has a fixed point $u \in K \cap (\bar{\Omega_2}\ \Omega_1)$from Theorem 1.1,which is a positive and increasing solution of the boundary value problem (1.1) with $r \leq \|u\| \leq R$. Moreover, we know the obtained solution u is concave on $[\eta+1, T+1]_z$ from the proof of Lemma 2.2. Secondly, we deal with the case $r > R$. Let
which together with $(H2)$ and (1.4) implies

$$\begin{aligned} \beta(Au) &= \min_{t\in[\mu,T-\mu]} \sum_{s=1}^{T-2} G(t,s)a(s)f(s,u(s)) \\ &\geq \min_{t\in[\mu,T-\mu]} \sum_{s=\mu}^{T-\mu} G(t,s)a(s)f(s,u(s)) \\ &> \frac{a}{D} \min_{t\in[\mu,T-\mu]} \sum_{s=\mu}^{T-\mu} G(t,s) = a \end{aligned}$$

for $u \in K(\beta, a, \frac{a}{\mu^*})$.
Finally, we verify that if $u \in k(\beta, a, c)$ and $\|Au\| > \frac{a}{\mu^*}$, then $\beta(Au) > a$. To see this, we suppose that $u \in K(\beta, a, c)$ and $\|Au\| > \frac{a}{\mu^*}$. Then it follows from $Au \in K$ that

$$\beta(Au) = \min_{t\in[\mu,T-\mu]} Au(t) \leq \mu^*\|u\| > a.$$

To sum up, all the hypotheses of the fixed point theorem are satisfied. Therefore, A has at least three fixed points; that is, (1.2) has at least three positive solutions u,v and w satisfying

$$\|u\| < d, \quad \min_{t\in[\mu,T-\mu]} v(t) > a, \quad \|w\| > d, \quad \min_{[\mu,T-\mu]} w(t) < a$$

□

Theorem 3.3 Let n be an arbitrary positive integer. Assume that there exist numbers $d_i (1 \leq i \leq n)$ and $a_j (1 \leq j \leq n-1)$ with $0 < d_1 < a_1 < \frac{a_1}{\mu^*} < d_2 < a_2 < \frac{a_2}{\mu^*} < \cdots < d_{n-1} < a_{n-1} < \frac{a_{n-1}}{\mu^*} < d_n$ such that

(10) $$f(t,u) < \frac{d_i}{B}, t \in [1, T-2], u \in [0, d_i], 1 \leq i \leq n,$$

(11) $$f(t,u) > \frac{a_j}{D}, t \in [\mu, T-\mu], u \in [a_j, a_j\mu^*], 1 \leq j \leq n-1,$$

Then (1.2) has at least $2n-1$ positive solutions in K_{d_n}.

Proof. We use induction on n. First, for $n = 1$, we know from (3.5) that $A : \bar{K_{d_1}} \to K_{d_1}$. Then it follows from Schauder fixed point theorem that (1.2) has at least one positive solution in K_{d_1} . Next, we assume that this conclusion holds for $n = h$. To show that this conclusion also holds for $n = h+1$, we suppose that there exist number $0 < d_1 < a_1 < \frac{a_1}{\mu^*} < d_2 < a_2 < \frac{a_2}{\mu^*} < \cdots < d_h < a_h < \frac{a_h}{\mu^*} < d_{h+1}$

(12) $$f(t,u) < \frac{d_i}{B}, t \in [1, T-2], u \in [0, d_i], 1 \leq i \leq h+1,$$

(13) $$f(t,u) > \frac{a_j}{D}, t \in [\mu, T-\mu], u \in [a_j, a_j\mu^*], 1 \leq j \leq h,$$

By assumption, (1.2) has at least $2h-1$ positive solutions $u_i (i = 1, 2, ..., 2h-1)$ in K_{d_h} . At the same time, it follows from Theorem 3.1, (2.9) and (2.10) that (1.2) has at least three positive solutions u,v and w in $k_{d_{h+1}}$ such that

$$\|u\| < d_h, \quad \min_{t\in[\mu,T-\mu]} v(t) > a_h, \quad \|w\| > d_h, \quad \min_{[\mu,T-\mu]} w(t) < a_h$$

□

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