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FRACTIONAL INTEGRAL ESTIMATES OF HERMITE-HADAMARD TYPE IN GLOBAL NONPOSITIVE CURVATURE SPACES

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Abstract. We extend the notion of convexity of functions defined on global nonpositive curvature spaces by introducing (geodesically) h-convex functions. Using Katugampola's integral operators, we establish Hermite-Hadamard-type estimates. From these results, we derive an important corollary that provides a sharp estimate involving squared distance mappings between points in a global NPC space. This work contributes to analysis on spaces with curved geometry.

1. Introduction

Convexity plays a central role in analysis, geometry, and optimisation. A fundamental inequality satisfied by convex function is the *Hermite-Hadamard inequality* which provides a bound for the integral average of a convex function. Suppose that $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$, the inequality states

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Note that the inequalities above are reversed if f is concave. The Hermite-Hadamard inequality also characterises convex functions defined on an interval of $\mathbb{R}$ [5]. Inequality (1) was first proved in the article [19] by Hermite, and since then it has garnered a lot of attention in the literature, with notable improvements, extensions, generalisations, and refinements; see, for instance, the monographs [17, 27] and the references therein.

The problem of extending the Hermite-Hadamard inequality to more general geometric settings, such as metric spaces and Riemannian manifolds, remains a question of significant interest. Some authors have investigated extensions of inequality (1) to functions of multiple variables. In particular, Dragomir [14] obtained some estimates involving triple integrals for convex functions defined on a ball, deriving some interesting properties for a certain convex mapping. He also established inequalities on a disk in $\mathbb{R}^2$ and derived results for mappings naturally connected to these inequalities [15]. By introducing a precise definition of *convex functions on the coordinates*, Dragomir also proved some sharp inequalities for functions defined on rectangles, with some significant applications in convex analysis [16].

A converse of (1) for functions defined on simplices was proved in [26]; the authors established that the Hermite-Hadamard inequality on simplices characterises convex functions under some

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conditions on the measure. In [6], a generalisation of (1) for convex functions defined on simplices is proven by using a volume formula and its higher-dimensional generalisation; this approach completely evades well-known tools from Choquet's theory (see [31], [29] for more details). In [28], Niculescu extended Choquet's theorem to compact metric spaces with a global nonpositive curvature, and by using a Jensen-type inequality, he obtained a generalisation of (1) to spaces with curved geometry. In such spaces, geodesics play the role of segments. By establishing a lemma which gives a unique minimal geodesic between two points on a hemisphere, the author in [3] proved an Hermite-Hadamard type inequality for integrable convex functions defined on hemispheres. There are a few studies of the Hermite-Hadamard inequality on nonpositively curved (NPC) spaces [11,12,28]. NPC spaces are fundamental in several areas of mathematics, especially geometry and topology.

Hadamard [18] pioneered the study of what is now known as NPC spaces, and Cartan investigated generalisations of such spaces in higher dimensions. Subsequently, the contributions of Alexandrov and Busemann became foundational to the theory of metric spaces with upper curvature bounds. [1,2,9,10]. The study of convexity in metric spaces, particularly those with nonpositive curvature, is a rich area of research. Such spaces generalise classical Euclidean convexity and provide a natural setting for extending inequalities like the Hermite-Hadamard inequality. The notion of convexity in these spaces is tied to *geodesic convexity*, where the role of line segments in Euclidean spaces is played by geodesics in NPC spaces.

A metric space (M, d) has *nonpositive Alexandrov curvature* if for any $p \in M$ and any geodesic segment $\gamma_{[x,y]} \in M$ between points x and y, the following inequality holds:

$$d^2(p, \gamma_{[x,y]}(1/2)) \leq \frac{1}{2}(d^2(p, x) + d^2(p, y)) - \frac{1}{4}d^2(x, y)$$

provided the points x and y are sufficiently close to p, and $\gamma_{[x,y]}(1/2)$ is the middle between x and y, that is, $d(x, \gamma_{[x,y]}(1/2)) = d(y, \gamma_{[x,y]}(1/2)) = \frac{1}{2}d(x, y)$. The inequality above is well known as the CN inequality of Bruhat and Tits [8]. Additionally, (M, d) is called a *Hadamard space* if it is complete. Examples include simply connected complete Riemannian manifolds with negative constant curvature, Bruhat-Tits buildings, Hilbert spaces, the upper half-plane endowed with the Poincaré metric, see Section 1.6 in [37] and the book [7] for more examples.

In this paper, we extend the notion of convex functions defined on metric spaces with global nonpositive curvature. Thereafter, we establish Hermite-Hadamard type inequalities for a general class of convex functions via Katugampola's fractional integral operators (which we shall define in due course). The motivation for extending notions of convexity to NPC spaces arises from various applications in geometry, optimisation, probability, statistics, evolutionary biology, robotics, machine learning, data science and analysis on metric spaces [24].

2. Global NPC Spaces and Notions of Convexity

In this section, we recall some facts about global NPC spaces and convex functions on such spaces, details can be found in [9,11,12,20,28]. We also introduce the concept of h-convexity on global NPC spaces.

Definition 2.1. Let $t_1, t_2 \in [0, 1]$, a curve γ is called *geodesic* if there exists $\varepsilon > 0$ such that the length of γ, when restricted to $[t_1, t_2]$, is the metric distance between $\gamma(t_1)$ and $\gamma(t_2)$ provided that $|t_1 - t_2| < \varepsilon$.

A metric space (M, d) is called a *geodesic space*, if for any two points $x, y \in M$, there exists a shortest geodesic arc joining them. In other words, there is a continuous curve $\gamma : [0,1] \to M$ with endpoints $x = \gamma(0)$ and $y = \gamma(1)$ and the length of γ is precisely the distance between the points x and y.

A geodesic space has a *global nonpositive curvature in the sense of Busemann* if for any two shortest geodesics $\gamma, \tilde{\gamma} : [0,1] \to M$ with $\gamma(0) = x = \tilde{\gamma}(0)$, the distance map $t \mapsto d(\gamma(t), \tilde{\gamma}(t))$ is convex. In other words, for every point $x, y, z \in M$, we have the inequality

$$2d(\gamma_{[x,y]}(1/2), \gamma_{[x,z]}(1/2)) \leq d(y, z) \tag{2}$$

provided that x, y are sufficiently close to z. If equality holds in (2), then we say that (M, d) is flat. The space (M, d) has *negative curvature in the sense of Busemann* if the inequality in (2) is strict (this happens when the endpoint of neither geodesic is contained in the other one).

The space (M, d) is called a *global NPC space* if the following conditions are satisfied

1. each pair of points can be connected by a geodesic
2. for $x_0, x_1 \in M$ there exists a point $y \in M$ such that for all $p \in M$

$$d^2(p, y) \leq \frac{1}{2}(d^2(p, x_0) + d^2(p, x_1)) - \frac{1}{4}d^2(x_0, x_1). \tag{3}$$

Generally, the following comparison principle holds: let $\gamma_{[p,x_0]}$, $\gamma_{[x_0,x_1]}$ and $\gamma_{[p,x_1]}$ be three geodesic segments connecting the points $p, x_0, x_1 \in M$, and let x_t be an arbitrary point on $\gamma_{[x_0,x_1]}$ which is a fraction of $d(x_0, x_1)$, then

$$d(x_0, x_t) = td(x_0, x_1) \qquad \text{and} \qquad d(x_t, x_1) = (1-t)d(x_0, x_1)$$

with the following inequality

$$d^2(p, x_t) \leq (1-t)d^2(p, x_0) + td^2(p, x_1) - t(1-t)d^2(x_0, x_1)\,, \qquad t \in [0,1].$$

Denote $x_t := (1-t)x_0 + tx_1$. Clearly, $x_{1/2} := \gamma_{[x_0,x_1]}(1/2)$ is the midpoint of the segment that connects x_0 and x_1, and the mean value of a function on $[0,1]$ exists, thus we can introduce the notion of convexity on global NPC spaces. We see at once that a function $f : K \subseteq M \to \mathbb{R}$ is convex if for all $t \in [0,1]$, we have $f(x_t) \leq (1-t)f(x_0) + tf(x_1)$. Global NPC spaces have global nonpositive curvature in the sense of Busemann, they are also known as CAT(0) spaces.

Theorem 2.2. *[23, 25, 33] Let (M, d) be a global NPC space. Let x_0, x_1, y_0 and y_1 be four points in M. Let x_t be the point which is a fraction of $d(x_0, x_1)$. For any $t \in [0,1]$, the following holds*

$$\begin{aligned} d^2(x_t, y_0) + d^2(x_{1-t}, y_1) \leq{}& d^2(x_0, y_0) + d^2(x_1, y_1) + 2t^2 d^2(x_0, x_1) \\ &+ t(d^2(y_0, y_1) - d^2(x_0, x_1)) - t(d(y_0, y_1) - d(x_0, x_1))^2. \end{aligned} \tag{4}$$

Definition 2.3. A subset $K \subseteq M$ is called convex if for each geodesic $\gamma : [0,1] \to M$ joining two arbitrary points in K, it holds that $\gamma([0,1]) \subseteq K$.

Definition 2.4. A function $f : K \to \mathbb{R}$ is convex if the function $f \circ \gamma : [0,1] \to \mathbb{R}$ is convex whenever $\gamma : [0,1] \to K$ is geodesic, that is, for all $t \in [0,1]$

$$f(\gamma(t)) \leq (1-t)f(x) + tf(y). \tag{5}$$

Note that inequality (5) follows from the convexity of $f \circ \gamma$, that is,

$$f(\gamma(t)) = f(\gamma[(1-t)\cdot 0 + t\cdot 1]) \leq (1-t)f\circ\gamma(0) + tf\circ\gamma(1) = (1-t)f(x) + tf(y).$$

In particular, the distance map $d : M \times M \to \mathbb{R}$ is convex. In other words, given any two geodesics $\gamma, \eta : [0,1] \to M$, we have the inequality

$$d(\gamma(t), \eta(t)) \leq (1-t)d(\gamma(0), \eta(0)) + td(\gamma(1), \eta(1)).$$

This implies that every ball in a global NPC space is a convex set. Let $k > 1$, for every $y \in M$, the map $G_y(x) := d^k(x, y)$ is strictly convex. That is, for every nonconstant geodesic $\gamma : [0,1] \to M$ and $t \in (0,1)$ we have the inequality $G_y(\gamma(t)) < (1-t)G_y(\gamma(0)) + tG_y(\gamma(1))$.

If $-f$ is convex then f is *concave*. The function f is said to be *affine* if f is both concave and convex. There are several notions of convexity on metric spaces [22, 30, 35]. Let $p > 0$, a function is said to be p-convex if f^p is convex. If $f(\gamma(t)) \leq \max\{f(\gamma(0)), f(\gamma(1))\}$, then we say that f is *quasi convex*. A function $f : K \to \mathbb{R}$ is *geodesically φ-convex* if there is a function $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $f(\gamma(t)) \leq f(x) + t\varphi(f(y), f(x))$ for all $x, y \in K$ and $t \in [0,1]$. For example, let $M = \mathbb{R} \times \mathbb{S}^1$ and $\varphi(x, y) := x^3 - y^3$, the function $f : K \subset M \to \mathbb{R}$ defined by $f(x, \cdot) = x^3$ is geodesically φ-convex but not convex. The notion of geodesically invex sets and geodesically pre-invex functions can be similarly defined on Riemannian manifolds (see [4]).

The notion of h-convexity for functions defined on an interval of $\mathbb{R}$ was introduced by Varošanec in [39]. It is known that the class of h-convex functions unifies existing classes of convex functions such as s-convex functions, Godunova-Levin functions, and P-functions. Motivated by the results in [12, 39], we extend the notion of h-convex functions to global NPC spaces.

Definition 2.5. Let $K \subseteq M$ be a convex subset of a global NPC space, and let $h : \mathbb{R} \to (0, \infty)$. A function $f : K \to \mathbb{R}$ is geodesically h-convex if the function $f \circ \gamma : [0,1] \to \mathbb{R}$ is h-convex whenever $\gamma : [0,1] \to K$ is geodesic, that is, for all $t \in [0,1]$ we have

$$f(\gamma(t)) \leq h(1-t)f(\gamma(0)) + h(t)f(\gamma(1)).$$

Remark 2.6. *Observe that if $h(t) \geq t$ for all $t \in [0,1]$, then non-negative (geodesically) convex functions are h-convex.*

Let $h_k(x) = x^k$, $x > 0$. It is known that the function $g(x) = x^r$ where $x > 0$, is h_k-convex if $r \in (-\infty, 0] \cup [1, \infty)$ and $k \leq 1$. Also, g is h_k-convex if $r \in (0,1)$ and $k \leq r$. If we define $f \circ \gamma := g$ and restrict the domain of g to $[0,1]$, then f is geodesically h_k-convex on $K \subseteq M$ if g is h_k-convex on $[0,1]$.

Theorem 2.7. *[36] Let $f : [a,b] \to \mathbb{R}$ be an h-convex function, then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq [f(a) + f(b)] \int_0^1 h(t)dt. \tag{6}$$

Theorem (2.7) is known as the Hermite-Hadamard inequality for h-convex functions. With $h(x) = x$, the inequality (6) reduces to (1).

The following lemma shows some properties of geodesics and convex functions in a global NPC space.

Lemma 2.8. *[12] Let (M,d) be a global NPC space, $K \subseteq M$ a convex set and $\gamma : [0,1] \to K$ a geodesic connecting $\gamma(0)$ and $\gamma(1)$. Then*

(1) For $t_1, t_2 \in [0,1]$ the curve $\gamma\big|_{[t_1,t_2]}(\lambda) = \gamma((1-\lambda)t_1 + \lambda t_2)$ is the unique geodesic connecting $\gamma(t_1)$ with $\gamma(t_2)$.

(2) For any $t_0 \in [0,1]$ the midpoint between $\gamma(t_0)$ and $\gamma(1-t_0)$ is given by $\gamma(1/2)$.

(3) If $f : K \to \mathbb{R}$ is convex, then $\int_0^1 f(\gamma(u))du = \int_0^1 f(\gamma(1-t))dt$.

With the help of the lemma above, Conde [12] proved the following Hermite-Hadamard inequality for convex functions on global NPC space.

Theorem 2.9. *Let (M,d) be a global NPC space, $K \subseteq M$ a convex subset and $f : K \to \mathbb{R}$ a convex function. Then*

$$f(\gamma(1/2)) \leq \int_0^1 f(\gamma(t))dt \leq \frac{f(\gamma(0)) + f(\gamma(1))}{2}$$

for all geodesic $\gamma : [0,1] \to K$.

3. Fractional Integral Inequalities for h-Geodesically Convex Functions

The results contained in this section are extensions and generalisations of the Hermite-Hadamard inequality, and related results. Indeed, we obtain Hermite-Hadamard type inequalities for the class of h-geodesically convex functions, which naturally generalises the class of convex functions. Also, we employ Katugampola's fractional integral operators which are generalisations of the well known Riemann-Liouville integral operators and the Hadamard integral operators.

Definition 3.1. [32] Let $\alpha > 0$ be such that $n-1 < \alpha < n$, $n \in \mathbb{N}$. The left and right sided Riemann-Liouvile fractional integrals of order α are given by

$$J_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt$$

and

$$J_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt$$

respectively, where $a < x < b$ and Γ is the well known Euler's gamma function defined by $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt$.

Definition 3.2. [34] The left and right sided Hadamard fractional integrals of order $\alpha > 0$ are defined by

$$H_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t}dt$$

and

$$H_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} \frac{f(t)}{t}dt$$

Definition 3.3. [21] Let $c \in \mathbb{R}$ and $1 \leq p \leq \infty$. The space $X_c^p(a,b)$ is the set of all complex-valued Lebesgue measurable functions f equipped with norm

$$\|f\|_{X_c^p} = \begin{cases} \left(\int_a^b \frac{|t^c f(t)|^p}{t} dt\right)^{\frac{1}{p}} & , \quad 1 \leq p < \infty \\ \operatorname{esssup}_{a \leq t \leq b} |t^c f(t)| & , \quad p = \infty. \end{cases}$$

The space $X_c^p(a,b)$ is the classical $L^p(a,b)$ space when $c = \frac{1}{p}$.

Definition 3.4. [13,21] Let $[a,b] \subset \mathbb{R}$ be a finite interval. The left and right side Katugampola fractional integrals of order $\alpha > 0$ of $f \in X_c^p(a,b)$ are defined by

$$^{\rho}I_{a+}^{\alpha} f(x) := \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}} f(t)dt$$

and

$$^{\rho}I_{b-}^{\alpha} f(x) := \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1}}{(t^\rho - x^\rho)^{1-\alpha}} f(t)dt$$

with $a < x < b$ and $\rho > 0$, provided the integrals exists.

The fractional integral operators $^{\rho}I_{a+}^{\alpha}$ and $^{\rho}I_{b-}^{\alpha}$ are well defined on $X_c^p(a,b)$ for $\rho \geq c$, as shown in [21]. There is a relationship among the integral operators defined above. Let $\alpha > 0$ and $\rho > 0$, then for $x > a$, it can be shown [21] that

$$\lim_{\rho\to 1} {}^{\rho}I_{a+}^{\alpha} f(x) = J_{a+}^{\alpha} f(x) \qquad \text{and} \qquad \lim_{\rho\to 0} {}^{\rho}I_{a+}^{\alpha} f(x) = H_{a+}^{\alpha} f(x).$$

Similar identities hold for the right sided integrals.

Theorem 3.5. *Let $\alpha > 0$ and $\rho > 0$. Let (M,d) be a global NPC space, $K \subseteq M$ a convex set and $f : K \to [0,\infty)$ a geodesic h-convex function with $h \in L^q[0,1]$, $q > 1$. Then the following inequalities hold*

$$\begin{aligned} &f\left(\gamma\Big|_{[a^\rho,b^\rho]}\left(\frac{1}{2}\right)\right) \\ &\leq \frac{\rho^\alpha \Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha} h\left(\frac{1}{2}\right)\left({}^{\rho}I_{a+}^{\alpha} f\left(\gamma\left(b^\rho\right)\right) + {}^{\rho}I_{b-}^{\alpha} f\left(\gamma\left(a^\rho\right)\right)\right) \\ &\leq h\left(\frac{1}{2}\right)\left[f(\gamma(a^\rho)) + f(\gamma(b^\rho))\right]\left[\alpha\left(\frac{q-1}{\alpha q-1}\right)^{\frac{q-1}{q}} \|h\|_{L^q[0,1]} + \rho^\alpha\Gamma(\alpha+1)\ {}^{\rho}I_{0+}^{\alpha} h(1)\right] \end{aligned} \tag{7}$$

where $0 \leq a, b \leq 1$.

Proof. First, note that since f is geodesically h-convex, we have

$$f\left(\gamma\left(\frac{x^\rho + y^\rho}{2}\right)\right) \leq h\left(\frac{1}{2}\right)\left[f(\gamma(x^\rho)) + f(\gamma(y^\rho))\right].$$

Using this fact, and the change of variables $x^\rho = t^\rho a^\rho + (1-t^\rho)b^\rho$ and $y^\rho = t^\rho b^\rho + (1-t^\rho)a^\rho$, we find for all $t \in [0,1]$ and $0 \leq a \leq x, y \leq b \leq 1$ the estimate

$$\begin{aligned} f\left(\gamma\Big|_{[a^\rho,b^\rho]}\left(\frac{1}{2}\right)\right) &= f\left(\gamma\left(\frac{x^\rho + y^\rho}{2}\right)\right) \\ &\leq h\left(\frac{1}{2}\right)\left[f(\gamma(t^\rho a^\rho + (1-t^\rho)b^\rho)) + f(\gamma(t^\rho b^\rho + (1-t^\rho)a^\rho))\right]. \end{aligned}$$

Multiplying the latter by $t^{\alpha\rho-1}$ and integrating over $t \in [0,1]$ yields

$$\begin{aligned} \frac{1}{\alpha\rho} f\left(\gamma\Big|_{[a^\rho,b^\rho]}\left(\frac{1}{2}\right)\right) &\leq h\left(\frac{1}{2}\right)\int_a^b \left(\frac{b^\rho - x^\rho}{b^\rho - a^\rho}\right)^{\alpha-1} \frac{x^{\rho-1}}{b^\rho - a^\rho} f(\gamma(x^\rho))dx \\ &\quad + h\left(\frac{1}{2}\right)\int_a^b \left(\frac{y^\rho - a^\rho}{b^\rho - a^\rho}\right)^{\alpha-1} \frac{y^{\rho-1}}{b^\rho - a^\rho} f(\gamma(y^\rho))dy \\ &= \frac{\rho^{\alpha-1}\Gamma(\alpha)}{(b^\rho - a^\rho)^\alpha} h\left(\frac{1}{2}\right)\left({}^{\rho}I_{a+}^{\alpha} f\left(\gamma\left(b^\rho\right)\right) + {}^{\rho}I_{b-}^{\alpha} f\left(\gamma\left(a^\rho\right)\right)\right) \end{aligned}$$

which proves the first inequality in (7). To prove the second inequality, we use the geodesic h-convexity of f to obtain

$$\begin{aligned}&f(\gamma(t^\rho a^\rho+(1-t^\rho)b^\rho))+f(\gamma(t^\rho b^\rho+(1-t^\rho)a^\rho))\\&\leq h(t^\rho)f(\gamma(a^\rho))+h(1-t^\rho)f(\gamma(b^\rho))+h(t^\rho)f(\gamma(b^\rho))+h(1-t^\rho)f(\gamma(a^\rho)).\end{aligned}$$

Multiplying both sides by $h\left(\frac{1}{2}\right)\alpha\rho t^{\alpha\rho-1}$, and integrating over $[0,1]$ with respect to t yields

$$\begin{aligned}&\frac{\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}h\left(\frac{1}{2}\right)\left({}^\rho I^\alpha_{a+}f\left(\gamma\left(b^\rho\right)\right)+{}^\rho I^\alpha_{b-}f\left(\gamma\left(a^\rho\right)\right)\right)\\&\leq h\left(\frac{1}{2}\right)\alpha\rho[f(\gamma(a^\rho))+f(\gamma(b^\rho))]\int_0^1 t^{\alpha\rho-1}(h(t^\rho)+h(1-t^\rho))dt.\end{aligned}\tag{8}$$

For all $t\in[0,1]$ and $q>1$, we use Hölder inequality to find

$$\int_0^1 t^{\alpha\rho-1}h(t^\rho)dt\leq\frac{1}{\rho}\left(\frac{q-1}{\alpha q-1}\right)^{\frac{q-1}{q}}\|h\|_{L^q[0,1]}.\tag{9}$$

On the other hand, we use the change of variable $u^\rho=1-t^\rho$ to find

$$\int_0^1 t^{\alpha\rho-1}h(1-t^\rho)dt=\int_0^1(1-u^\rho)^{\alpha-1}u^{\rho-1}h(u^\rho)du=\frac{\Gamma(\alpha)}{\rho^{1-\alpha}}{}^\rho I^\alpha_{0+}h(1).\tag{10}$$

Noting that h is nonnegative by definition, we combine (10), (9) and (8) to prove the second inequality in (7). This completes the proof.

□

Remark 3.6. Note that the change of variable $u=t^\rho$ yields

$$\Gamma(\alpha)J^\alpha_{1-}h(0)=\frac{1}{\rho}\int_0^1 u^{\alpha-1}h(u)du=\int_0^1 t^{\alpha\rho-1}h(t^\rho)dt\overset{(9)}{\leq}\frac{1}{\rho}\left(\frac{q-1}{\alpha q-1}\right)^{\frac{q-1}{q}}\|h\|_{L^q[0,1]}.$$

Thus we can remove the condition that $h\in L^q[0,1]$ and refine the estimate (7). We have the following theorem.

Theorem 3.7. *Let $\alpha>0$ and $\rho>0$. Let (M,d) be a global NPC space, $K\subseteq M$ a convex set and $f:K\to\mathbb{R}$ a geodesic h-convex function. Then the following inequalities hold*

$$\begin{aligned}f\left(\gamma\Big|_{[a^\rho,b^\rho]}\left(\frac{1}{2}\right)\right)&\leq\frac{\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}h\left(\frac{1}{2}\right)\left({}^\rho I^\alpha_{a+}f\left(\gamma\left(b^\rho\right)\right)+{}^\rho I^\alpha_{b-}f\left(\gamma\left(a^\rho\right)\right)\right)\\&\leq h\left(\frac{1}{2}\right)[f(\gamma(a^\rho))+f(\gamma(b^\rho))]\Gamma(\alpha+1)\left[\rho J^\alpha_{1-}h(0)+\rho^\alpha\,{}^\rho I^\alpha_{0+}h(1)\right]\end{aligned}\tag{11}$$

where $0\leq a,b\leq 1$.

Theorem 3.8. *Let $\alpha>0$ and $\rho>0$. Let (M,d) be a global NPC space, $K\subseteq M$ a convex set and $f:K\to\mathbb{R}$ a geodesic h-convex function. Then the following inequalities hold*

$$\begin{aligned}f\left(\gamma\left(\frac{1}{2}\right)\right)&\leq\frac{\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}h\left(\frac{1}{2}\right)\left({}^\rho I^\alpha_{a+}f(\gamma(b^\rho))+{}^\rho I^\alpha_{c-}f(\gamma(s^\rho))\right)\\&\leq\frac{f(\gamma(0))+f(\gamma(1))}{(b^\rho-a^\rho)^\alpha}\rho^\alpha\Gamma(\alpha+1)h\left(\frac{1}{2}\right)\left({}^\rho I^\alpha_{a+}h(b^\rho)+{}^\rho I^\alpha_{c-}h(s^\rho)\right)\end{aligned}\tag{12}$$

where $c:=(1-a^\rho)^{\frac{1}{\rho}}$, $s:=(1-b^\rho)^{\frac{1}{\rho}}$ and $0\leq a,b\leq 1$.

Proof. Since f is geodesically h-convex, we have

$$f(\gamma(1/2)) \leq h(1/2)\left(f(\gamma(x^\rho)) + f(\gamma(1-x^\rho))\right). \tag{13}$$

Setting $x^\rho = t^\rho a^\rho + (1-t^\rho)b^\rho$ where $t \in [0,1]$, multiplying both sides of (13) by $t^{\alpha\rho-1}$ and integrating over $[0,1]$ gives

$$\begin{aligned}
\frac{f(\gamma(1/2))}{\alpha\rho} &\leq h(1/2)\int_0^1 t^{\alpha\rho-1}\Big(f(\gamma(t^\rho a^\rho + (1-t^\rho)b^\rho)) + f(\gamma((1-b^\rho)+t^\rho(b^\rho-a^\rho)))\Big)dt \\
&= h(1/2)\int_a^b \left(\frac{b^\rho - x^\rho}{b^\rho - a^\rho}\right)^{\alpha-1}\frac{x^{\rho-1}}{b^\rho-a^\rho}f(\gamma(x^\rho))dx \\
&\quad + h(1/2)\int_{(1-b^\rho)^{\frac{1}{\rho}}}^{(1-a^\rho)^{\frac{1}{\rho}}}\left(\frac{u^\rho-(1-b^\rho)}{b^\rho-a^\rho}\right)^{\alpha-1}\frac{u^{\rho-1}}{b^\rho-a^\rho}f(\gamma(u^\rho))du \\
&= \frac{h(1/2)}{(b^\rho-a^\rho)^\alpha}\Gamma(\alpha)\rho^{\alpha-1}\Big({}^\rho I^\alpha_{a+}f(\gamma(b^\rho)) + {}^\rho I^\alpha_{c-}f(\gamma(s^\rho))\Big). \qquad (14)
\end{aligned}$$

The first inequality in (12) follows from (14). Next is to prove the second inequality in (12). Since f is geodesically h convex, we have

$$\begin{aligned}
&f(\gamma(t^\rho a^\rho + (1-t^\rho)b^\rho)) + f(\gamma((1-b^\rho)+t^\rho(b^\rho-a^\rho))) \\
&= f(\gamma(x^\rho)) + f(\gamma(1-x^\rho)) \\
&\leq [h(x^\rho)+h(1-x^\rho)][f(\gamma(0))+f(\gamma(1))] \\
&= [h(t^\rho a^\rho + (1-t^\rho)b^\rho) + h((1-b^\rho)+t^\rho(b^\rho-a^\rho))][f(\gamma(0))+f(\gamma(1))]. \qquad (15)
\end{aligned}$$

Multiplying both sides of (15) by $t^{\alpha\rho-1}$ and integrating over $[0,1]$, we have

(16)

$$\frac{\Gamma(\alpha)\rho^{\alpha-1}}{(b^\rho-a^\rho)^\alpha}\Big({}^\rho I^\alpha_{a+}f(\gamma(b^\rho)) + {}^\rho I^\alpha_{c-}f(\gamma(s^\rho))\Big) \leq \frac{f(\gamma(0))+f(\gamma(1))}{(b^\rho-a^\rho)^\alpha}\Gamma(\alpha)\rho^{\alpha-1}\Big({}^\rho I^\alpha_{a+}h(b^\rho) + {}^\rho I^\alpha_{c-}h(s^\rho)\Big)$$

The second inequality in (12) follows from (16). This completes the proof. □

Let $k \geq 1$, recall from Definition 2.4 that the function $G_y(x) := d^k(x,y)$ is convex. Let $h : [0,1] \to (0,\infty)$ be a map satisfying $h(t) \geq t$ for all $t \in [0,1]$, then

$$G_y(\gamma(t)) \leq (1-t)G_y(\gamma(0)) + tG_y(\gamma(1)) \leq h(1-t)G_y(\gamma(0)) + h(t)G_y(\gamma(1))$$

so that G_y is geodesically h-convex. In particular, the function $g_y(t) = d^k(y, \gamma_{[x_1,x_2]}(t))$ is h-convex. Consequently, by Theorem 3.8, we have

$$\begin{aligned}
d^k(y,\gamma_{[x_1,x_2]}(1/2)) &\leq \frac{\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}h\left(\frac{1}{2}\right)\Big({}^\rho I^\alpha_{a+}d^k(y,\gamma_{[x_1,x_2]}(b^\rho)) + {}^\rho I^\alpha_{c-}d^k(y,\gamma_{[x_1,x_2]}(s^\rho))\Big) \\
&\leq \frac{d^k(y,x_1)+d^k(y,x_2)}{(b^\rho-a^\rho)^\alpha}\rho^\alpha\Gamma(\alpha+1)h\left(\frac{1}{2}\right)\Big({}^\rho I^\alpha_{a+}h(b^\rho) + {}^\rho I^\alpha_{c-}h(s^\rho)\Big)
\end{aligned}$$

where $c := (1-a^\rho)^{\frac{1}{\rho}}$, $s := (1-b^\rho)^{\frac{1}{\rho}}$ and $0 \leq a,b \leq 1$. By assuming that $h(t) \geq t$ for all $t \in [0,1]$, we use the h-convexity of $g : [0,1] \to \mathbb{R}$, $g(t) = d^2(\gamma(t),\tilde{\gamma}(t))$ (where γ and $\tilde{\gamma}$ are geodesics) to obtain the following corollary, for $k = 2$.

Corollary 3.9. *Let $\alpha > 0$ and $\rho > 0$. Let (M,d) be a global NPC space, and let $\gamma := \gamma_{[x_1,x_2]}$ and $\tilde{\gamma} := \tilde{\gamma}_{[y_1,y_2]}$ be two geodesics connecting the points $x_1, x_2 \in M$ and $y_1, y_2 \in M$ respectively.*

Suppose that $h : [0,1] \to (0,\infty)$ is a function satisfying $h(t) \ge t$, for all $t \in [0,1]$. Then the following inequalities hold

$$\begin{aligned} d^2(\tilde{\gamma}(1/2),\gamma(1/2)) &\le \frac{\rho^\alpha \Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha} h\left(\frac{1}{2}\right)\left({}^\rho I^\alpha_{a+} d^2(\tilde{\gamma}(b^\rho),\gamma(b^\rho)) + {}^\rho I^\alpha_{c-} d^2(\tilde{\gamma}(s^\rho),\gamma(s^\rho))\right) \\ &\le \frac{d^2(y_1,x_1)+d^2(y_2,x_2)}{(b^\rho-a^\rho)^\alpha}\mathcal{E}(h) - C(\alpha,\rho)[d(y_1,y_2)d(x_1,x_2)]^2 \\ &\le \frac{d^2(y_1,x_1)+d^2(y_2,x_2)}{(b^\rho-a^\rho)^\alpha}\mathcal{E}(h) \end{aligned}$$

where $c := (1-a^\rho)^{\frac{1}{\rho}}$, $s := (1-b^\rho)^{\frac{1}{\rho}}$, $0 \le a, b \le 1$, $C(\alpha,\rho) \ge 0$,

$$C(\alpha,\rho) := \frac{(a^\rho\alpha + b^\rho)(2(\alpha+2) - 4b^\rho) - 2a^{2\rho}\alpha(\alpha+1)}{\alpha\rho(\alpha+1)(\alpha+2)}$$

and

$$\mathcal{E}(h) := \rho^\alpha \Gamma(\alpha+1) h\left(\frac{1}{2}\right)\left({}^\rho I^\alpha_{a+} h(b^\rho) + {}^\rho I^\alpha_{c-} h(s^\rho)\right).$$

Proof. We use Corollary 2.5 in [38] (a geodesic comparison result) to write the estimate

$$\begin{aligned} d^2(\tilde{\gamma}_{[y_1,y_2]}(u^\rho),\gamma_{[x_1,x_2]}(u^\rho)) &\le h(1-u^\rho)d^2(y_1,x_1) + h(u^\rho)d^2(y_2,x_2) \\ &\quad - u^\rho(1-u^\rho)[d(y_1,y_2)-d(x_1,x_2)]^2 \\ &\le h(1-u^\rho)d^2(y_1,x_1) + h(u^\rho)d^2(y_2,x_2) \end{aligned}$$

where $u \in [0,1]$ and $\rho > 0$. Hence, we deduce that

$$\begin{aligned} &d^2(\tilde{\gamma}_{[y_1,y_2]}(u^\rho),\gamma_{[x_1,x_2]}(u^\rho)) + d^2(\tilde{\gamma}_{[y_1,y_2]}(1-u^\rho),\gamma_{[x_1,x_2]}(1-u^\rho)) \\ &\le [h(u^\rho)+h(1-u^\rho)]d^2(y_1,x_1) + [h(u^\rho)+h(1-u^\rho)]d^2(y_2,x_2) \\ &\quad - 2u^\rho(1-u^\rho)[d(y_1,y_2)-d(x_1,x_2)]^2 \\ &\le [h(u^\rho)+h(1-u^\rho)]\left[d^2(y_1,x_1)+d^2(y_2,x_2)\right]. \end{aligned} \tag{17}$$

Set $u^\rho = t^\rho a^\rho + (1-t^\rho)b^\rho$ in the estimates in (17), so that u^ρ is the line segment connecting a^ρ and b^ρ. Next, multiply the resulting estimates by $h(1/2)t^{\alpha\rho-1}$ and integrate on $[0,1]$ with respect to t, via Katugampola's fractional integral operators. Finally, apply Theorem 3.8. Note that the constant $C(\alpha,\rho)$ is computed as follows.

Since $0 \le u^\rho \le 1$, we have $2u^\rho(1-u^\rho) \ge 0$. Hence

$$\begin{aligned} 0 \le C(\alpha,\rho) &= \int_0^1 [2(t^\rho a^\rho + (1-t^\rho)b^\rho) - 2(t^\rho a^\rho + (1-t^\rho)b^\rho)^2]t^{\alpha\rho-1}dt \\ &= -\frac{2(b^\rho-a^\rho)}{\rho(\alpha+1)} + \frac{2b^\rho}{\alpha\rho} \\ &\quad - \frac{2a^{2\rho}\alpha(\alpha+1) + 2b^{2\rho}[(\alpha+1)(\alpha+2) - 2\alpha(\alpha+2) + \alpha(\alpha+1)]}{\alpha\rho(\alpha+1)(\alpha+2)} \\ &\quad + \frac{4a^\rho b^\rho[\alpha(\alpha+2)-\alpha(\alpha+1)]}{\alpha\rho(\alpha+1)(\alpha+2)} \\ &= -\frac{2(b^\rho-a^\rho)\alpha(\alpha+2) - 2b^\rho(\alpha+1)(\alpha+2)}{\alpha\rho(\alpha+1)(\alpha+2)} - \frac{2a^{2\rho}\alpha(\alpha+1) + 4b^{2\rho} + 4a^\rho b^\rho \alpha}{\alpha\rho(\alpha+1)(\alpha+2)} \\ &= \frac{(a^\rho\alpha+b^\rho)(2(\alpha+2)-4b^\rho) - 2a^{2\rho}\alpha(\alpha+1)}{\alpha\rho(\alpha+1)(\alpha+2)}. \end{aligned}$$

This concludes the proof. □

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