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REFINED ENUMERATION OF 2-PLANE TREES

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Abstract. Plane trees and their generalizations have been widely studied. One such generalization is by assigning labels to the vertices of plane trees such that the sum of labels of the endpoints of each edge satisfy a certain condition. Gu, Prodinger and Wagner introduced and enumerated 2-plane trees which are plane trees with vertices coloured either white or black such that there are no black-black edges. In this paper, 2-plane trees are enumerated according to the colour of the first child of the root, degree of the root and level of a vertex of a given colour and degree. The counting formulas are obtained by means of symbolic method to obtain the generating functions and making use of Lagrange inversion formula, bijections and decomposition of trees so as to use Rothe-Hagen identity which is a generalization of Vandermonde identity.

1. Introduction

Plane trees are one of the many structures counted by the famous Catalan numbers. Among the statistics that have been considered for their enumeration are number of vertices, number of leaves, degree of a vertex, degree of the root, degree sequences etc. For the formulas counting these structures and the techniques used for their derivations, we refer the reader to [1, 18] among many other papers and books. Formally, a *plane tree* is a rooted tree drawn in the plane such that the children of each vertex are ordered. In a plane tree, a child to a vertex which appears on the far left is called the *first child*, the one that appears to the right of the first child is the *second child*, and the one that appears to the far right is the *youngest child* of the vertex. The *degree* of a vertex in a plane tree is the number of children of the vertex. An arrangement of degrees of all vertices of the tree is the *degree sequence* of the tree. A *leaf* is a vertex of degree 1 and a non-leaf vertex is an *internal vertex*. The *level* of a vertex is the number of edges that are in a path from the root to that vertex. The number of edges in a path determines the length of a path. A collection of disjoint trees is referred to as a *forest*. Among the results obtained by Dershowitz and Zaks in their paper [1] are:

(i) The number of plane trees on n vertices with k leaves is given by the Narayana number,

$$\frac{1}{n-1}\binom{n-1}{k}\binom{n-1}{k-1}.$$

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(ii) The number of plane trees on n vertices with root degree d is given by

$$\frac{d}{n-1}\binom{2n-d-3}{n-2}.$$

(iii) The number of vertices of degree d in plane trees on n vertices is

$$\binom{2n-d-3}{n-2}.$$

(iv) The number of vertices of degree d at level ℓ in plane trees on n vertices is

$$\frac{2\ell+d}{2n-d-2}\binom{2n-d-2}{n+\ell-1}. \tag{1}$$

Plane trees have been generalized by assigning labels to the vertices. In 2010, Gu, Prodinger and Wagner [6] enumerated *k-plane trees*. These are plane trees in which the vertices of the plane trees are labelled with integers in the set $\{1, 2, \ldots, k\}$ such that the sum of labels of the endpoints of each edge does not exceed $k+1$. In the aforementioned paper, the authors showed that the number of k-plane trees with root labelled by h on $n > 1$ vertices is given by

$$\frac{k+1-h}{kn-h+1}\binom{(k+1)n-h-1}{n-1}. \tag{2}$$

A specialised case called *2-plane trees* was considered in an earlier paper of Gu and Prodinger [5]. These are k-plane trees in which the vertices are labelled with integers 1 and 2 such that there are no edges whose endpoints are both labelled 2. The number of these trees on n vertices with root of label h is obtained by setting $k = 2$ in (2). Instead of the labels 1 and 2, we shall always refer to the vertices as white and black vertices respectively. The set of 2-plane trees has not been enumerated by number of leaves, number of vertices of degree d, number of vertices of degree d that reside at level ℓ, degree sequences, number of forests among many other parameters.

A *noncrossing tree* is a tree drawn in the plane with its vertices on the boundary of a circle and edges are line segments that do not intersect inside the circle [2]. In this work, the vertices are labelled around the circle in counterclockwise direction. These trees are enumeratd by the same formula that counts 2-plane trees with a black root. Let (i, j) be an edge in a noncrossing tree such that i is closest to the root than j. Then the edge is an *ascent* if $i < j$. Otherwise, it is a *descent*. Noncrossing trees in which each edge is oriented from a vertex of lower label towards a vertex of higher label were studied by Okoth and Wagner in [13]. The trees are called *locally oriented noncrossing trees* or simply *lnc-trees* therein. In the same paper, the authors defined an *indegree* (respectively, *outdegree*) of a vertex as the number of edges that are oriented towards (respectively, away from) the given vertex. A vertex of indegree (respectively, outdegree) 0 is a *source* (respectively, *sink*). The (l, r)-representation of noncrossing trees has been one of the major breakthroughs in the enumeration of noncrossing trees and related structures. This representation was invented by Panholzer and Prodinger in [15]. Using the representation, a noncrossing tree is a plane tree developed as follows: Vertex 1 is the root of the plane tree and if (i, j) is an edge of the noncrossing tree then vertex j is assigned label r (respectively, l) given that the edge is an ascent (respectively, a descent). Moreover, the children of each internal vertex of the corresponding plane tree are arranged in a way that a vertex which had the highest label in the nocrossing tree is the youngest child and the vertex that had the lowest label is the first child. In Figure 1, we get a noncrossing tree and its (l, r)-representation.

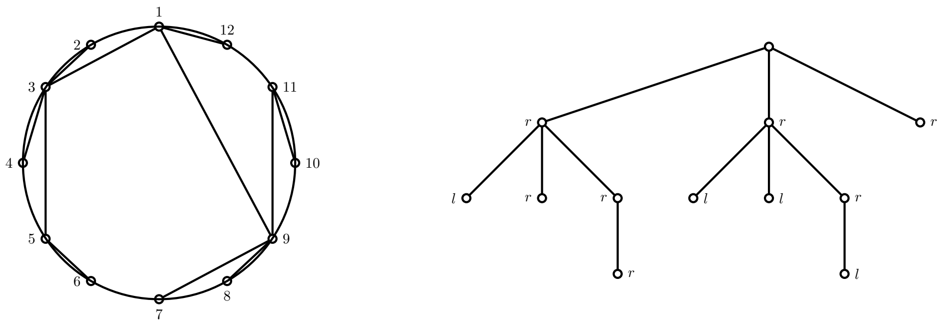

FIGURE 1. A noncrossing tree on 12 vertices with its (l,r)-representation.

If all the vertices of noncrossing trees are coloured black or white such that there are no black-black ascents in the paths from the root then we get a 2-noncrossing tree. These trees were introduced and enumerated by Yan and Liu in [20]. Further refinements of the results of Yan and Liu have been obtained in [9–12, 14]. A generalization of 2-noncrossing trees to k-noncrossing trees was achieved in [16]. Substantial amount of study has since been done on k-noncrossing trees as in [10, 14] among other papers. In the paper [11], Okoth introduced 2-noncrossing increasing trees, which are 2-noncrossing trees in which the labels of vertices increase as one moves away from the root. The author showed that these trees are enumerated by the same formula that counts 2-plane trees and noncrossing trees.

The following theorem is used in Section 4 to extract the coefficient of a variable in a generating function satisfying a certain condition.

Theorem 1.1 (Lagrange Inversion Formula, [18, 19]). *Let $T(x)$ be a generating function that satisfies the functional equation $T(x) = x\phi(T(x))$, where $\phi(0) \neq 0$. Then, $n[x^n]T(x)^k = k[t^{n-k}]\phi(t)^n$.*

This paper is organized as follows. In Section 2, the authors enumerate 2-plane trees by the colour of the first child of the root. Okoth [11] enumerated 2-plane trees with black roots according to degree of the root. The equivalent result for 2-plane trees with white roots is obtained in Section 3. In Section 4, we obtain the number of vertices and leaves that reside at level ℓ in 2-plane trees. The paper is concluded in Section 5 and a few problems are proposed therein.

2. Enumeration by colour of first child

In the sequel, we obtain the number of 2-plane trees in which the colour of the first child of the root is stated. We begin by stating the following result which was obtained by Gu and Prodinger in [5].

Theorem 2.1 ([5, Lemma 3.11]). *The number of 2-plane trees with a white root on $n > 1$ vertices such that the first child of the root is black is given by*

$$\frac{1}{n-1}\binom{3n-3}{n-2}. \tag{3}$$

Corollary 2.2. *The number of 2-plane trees with a white root on $n > 1$ vertices such that the first child of the root is white is given by*

$$\frac{2}{n}\binom{3n-3}{n-2}. \tag{4}$$

Proof. By Lemma 2.3 in [5], the number of 2-plane trees with a white root on n vertices is given by

$$\frac{1}{n}\binom{3n-2}{n-1}. \tag{5}$$

Since the first child of the root is either black or white, then the result follows by Theorem 2.1 and simple algebraic manipulations. □

In [11], Okoth showed that the number of 2-plane trees with a black root on n vertices such that the root has degree d is

$$\frac{d}{n-1}\binom{3n-d-4}{n-d-1}. \tag{6}$$

This formula also counts the number of noncrossing trees on n vertices such that a given vertex has degree d, see for example [7]. Setting $d = 2$ and $n = n + 1$ in (6), we obtain (4). It means that the corollary can as well be proved by constructing a bijection between the set of 2-plane trees on n vertices with a white root such that the first child of the root is white and the set of 2-plane trees on $n + 1$ vertices with a black root such that the root degree is 2. We construct such a bijection to prove Corollary 2.2.

Bijective proof of Corollary 2.2. Let T be a 2-plane tree with a white root on n vertices such that the first child of the root is white. Let e be the edge connecting the root and its first child. Delete this edge. We obtain an ordered pair of 2-plane trees whose roots are white. The first 2-plane tree, T_1, is the subtree of T whose root is the first child of the root of T. The second 2-plane tree, T_2, is the subtree whose root is the root of T. Now, connect the roots of the two subtrees to a new black vertex. The resultant tree is a 2-plane tree on $n + 1$ vertices such that the root is coloured black and it has degree 2. The procedure is easily reversed. The bijection is illustrated in Figure 2.

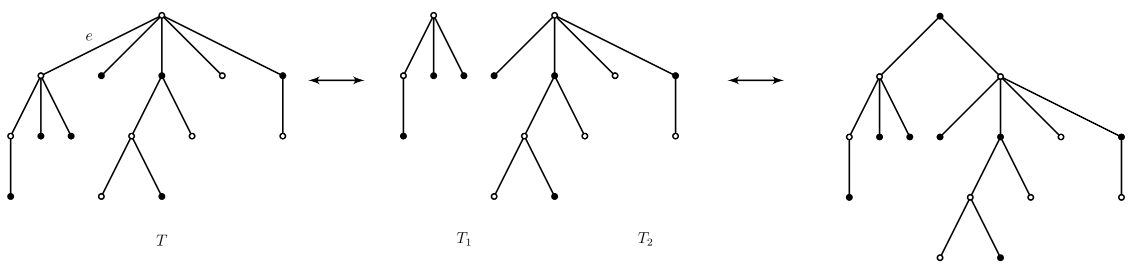

FIGURE 2. Obtaining a 2-plane tree with a black root of degree 2 from a 2-plane tree with a white root and white first child of the root and vice-versa.

□

Proposition 2.3. *There are*

$$\frac{d}{n-1}\binom{3n-d-4}{n-d-1}$$

2-plane trees on $n > 1$ vertices with a white root such that the first child of the root is black and of degree $d - 1$.

Proof. The proof follows as such a tree T, i.e. a 2-plane tree on $n > 1$ vertices with a white root such that the first child of the root is black and of degree $d - 1$, can easily be converted to a 2-plane tree on n vertices with black root of degree d. Let r be the root of T and let v be the first child coloured black. Delete the edge rv and attach the subtree rooted at r as the rightmost subtree of the tree rooted at v. The desired tree is obtained. The reverse process is easily seen. This also provides a bijective proof of Theorem 2.1 by summing over all d. □

Theorem 2.4. *The number of 2-plane trees on $n > 1$ vertices with a white root such that the first child of the root is black and there are a total of d black children of the root is given by*

$$\frac{d}{n-1}\binom{3n-d-4}{n-d-1}. \tag{7}$$

Proof. To prove the formula, we construct a bijection between the set of 2-plane trees with a white root such that the first child of the root is black and the total number of black children of the root is d and the set of noncrossing trees on n vertices such that the root has degree d. Consider a 2-plane tree with a white root on n vertices, such that the first child of the root is black and there are a total of d black children of the root. We obtain the corresponding noncrossing tree on n vertices, with root of degree d by the following steps:

(i) Label all black vertices and non-root white vertices as r and l respectively. Consider a vertex u labelled l. If there is a sequence of vertices labelled l which are children of u such that the vertices are on the immediate right of a child v of u labelled r, then detach the subtrees rooted at the aforesaid children labelled l and attach them in order from left to right as children of v on the right of the youngest child of v. Relabel the new attached children of v as r.

(ii) Repeat the procedure so that for each internal vertex, all children labelled l must be on the left of all children labelled r.

The structure obtained is the (l, r)-representation of a noncrossing tree on n vertices. Each black child of the root in the 2-plane tree has an equivalent child of the root in the corresponding noncrossing tree. The noncrossing tree thus has root degree d. The number of vertices in both 2-plane tree and noncrossing tree is also the same.

We now obtain the reverse procedure. Consider a noncrossing tree on n vertices with root degree d. Obtain its (l, r)-representation. We obtain the corresponding 2-plane tree as follows:

(i) Traversing the tree in preorder, let v be a vertex labelled r with a sequence of children labelled r. Detach subtrees rooted at these children and attach them in order from left to right as siblings of v on the immediate right of v. Relabel the new attached siblings of v as l.

(ii) Repeat the procedure until all the vertices of the noncrossing tree are traversed.

(iii) Colour all vertices labelled l and r as white and black respectively.

The tree obtained is a 2-plane tree. Since each child of the root in the noncrossing tree corresponds to a black child of the root in the 2-plane tree then the root of 2-plane tree has d black children. Moreover, since each vertex of the noncrossing tree has its equivalent in the 2-plane tree, then the number of vertices in the constructed 2-plane tree is also n as for the noncrossing tree. Figure 3 is a depiction of the procedure.

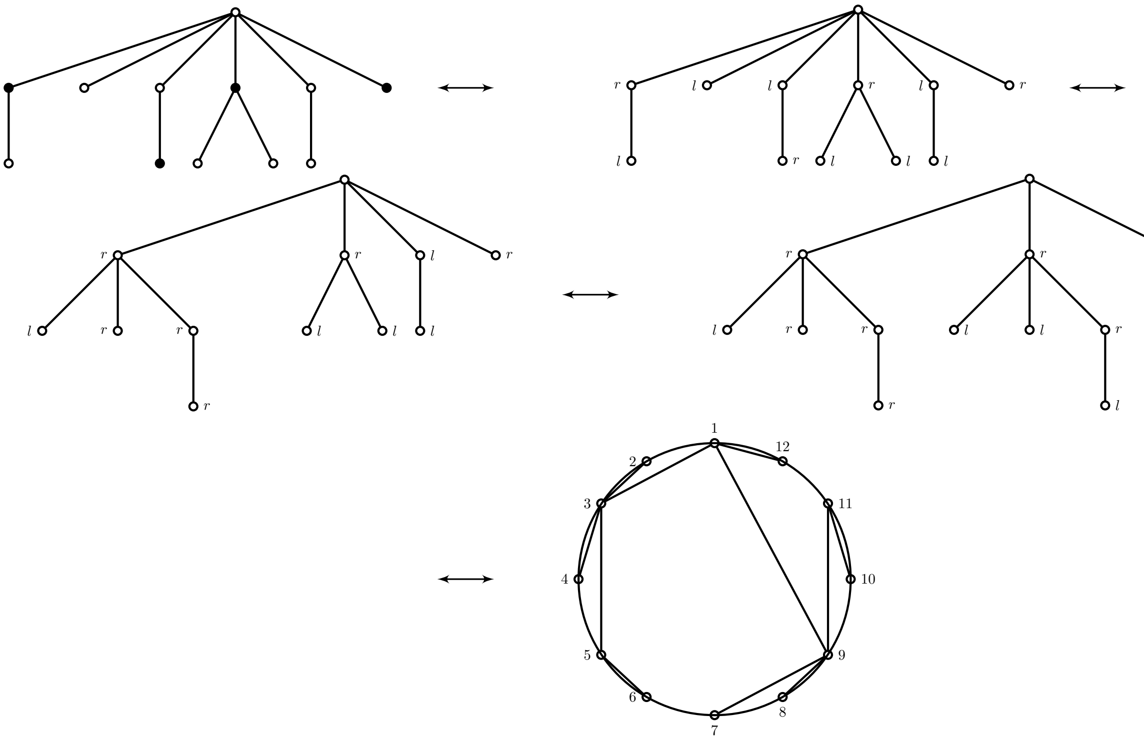

FIGURE 3. Obtaining a noncrossing tree from a 2-plane tree and vice-versa.

□

In [13], Okoth and Wagner proved that the number of locally oriented noncrossing trees with n vertices, and k sinks is given by

$$\frac{1}{n-1}\binom{2n-2}{n-k-1}\binom{n-1}{k}.$$

It is worth noting that by the proof of Theorem 2.4, a sink in a locally oriented noncrossing tree corresponds to a black vertex which is not incident to the root in a 2-plane tree with a white root and the first child of the root is black. We thus obtain the following corollary:

Corollary 2.5. *There are*

$$\frac{1}{n-1}\binom{2n-2}{n-k-1}\binom{n-1}{k}$$

2-plane trees on n vertices with a white root such that the first child of the root is black and the total number of black vertices which are not children of the root is k.

In the said paper [13], the authors constructed bijections between locally oriented noncrossing trees and ternary trees. Thus, the following result.

Corollary 2.6. *There is a bijection between the set of 2-plane trees on n vertices with a white root such that the first child of the root is black and the total number of black vertices which are not children of the root is k, and the set of of ternary trees on $n-1$ vertices with $(k-1)$ right-edges.*

We shall make use of the following identity (Rothe-Hagen Identity) in the proof the subsequent theorem. The identity was obtained independently by Rothe and Hagen [4] as a generalization of Vandermonde identity.

Identity 2.7 (Rothe-Hagen Identity)**.** *Let x, y, z be positive integers and n be a non-negative integer. Then,*

$$\sum_{k=0}^{n} \frac{x}{x+kz}\binom{x+kz}{k}\frac{y}{y+(n-k)z}\binom{y+(n-k)z}{n-k} = \frac{x+y}{x+y+nz}\binom{x+y+nz}{n}. \tag{8}$$

Theorem 2.8. *The number of 2-plane trees on $n > 1$ vertices with a white root such that there are a total of d black children of the root is given by*

$$\frac{2d+1}{2n-1}\binom{3n-d-3}{n-d-1}.$$

Proof. Let $\mathcal{T}(n, d)$ be the set of 2-plane trees on n vertices such that there are a total of d black children of the root. Let $T \in \mathcal{T}(n, d)$ and v be the first black child of the root of T. We decompose T into two 2-plane trees (ordered) such that the first tree T_1 is a 2-plane tree with a white root and all the subtrees with roots the children of the root of T that lie to the left of v. The second tree T_2 is a 2-plane tree with a white root and all the subtrees with roots the children of the root of T being v and the ones that lie to the right of v. The decomposition is illustrated in Figure 4.

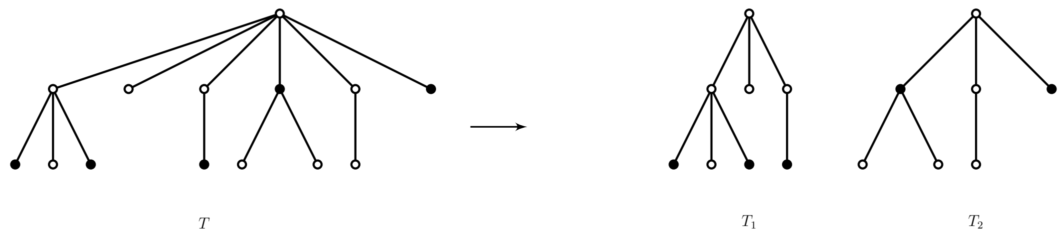

FIGURE 4. Decomposition of the tree T into T_1 and T_2.

Let the number of vertices of T_1 be m. Since the number of vertices in T is n, then the number of vertices in T_2 is $n - m + 1$. Moreover, we can recolour the root of T_1 as black and T_2 is a 2-plane tree with a white root such that the first child of the root is black and the total number of black children of T_2 is d.

By equations (3) and (7), we get

$$|\mathcal{T}(n, d)| = \sum_{m=1}^{n-d} \frac{1}{2m-1}\binom{3m-3}{m-1}\frac{2d}{2n-2m}\binom{3n-3m-d-1}{n-m-d}.$$

Setting $k = m - 1$, we have

$$\begin{aligned} |\mathcal{T}(n, d)| &= \sum_{k=0}^{n-d-1} \frac{1}{2k+1}\binom{3k}{k}\frac{2d}{2n-2k-2}\binom{3n-3k-d-4}{n-k-d-1} \\ &= \sum_{k=0}^{n-d-1} \frac{1}{3k+1}\binom{3k+1}{k}\frac{2d}{3n-3k-d-3}\binom{3n-3k-d-3}{n-k-d-1}. \end{aligned}$$

Now, applying Rothe-Hagen Identity (8) with $x = 1, y = 2d, z = 3$ and $n = n - d - 1$, we get the required result. ∎

Corollary 2.9. *The number of 2-plane trees on $n > 1$ vertices with a white root such that the first child of the root is white and there are a total of d black children of the root is given by*

$$\frac{2d+3}{2n-1}\binom{3n-d-4}{n-d-2}.$$

Proof. The result follows from Theorems 2.4 and 2.8. □

Proposition 2.10. *The number of 2-plane trees on n vertices with a white root such that the eldest black child of the root is its d-th child is given by*

$$\frac{2d-1}{2n-1}\binom{3n-d-2}{n-d}.$$

Proof. Let $\mathcal{S}(n,d)$ be the set of these trees. Moreover, let $T \in \mathcal{S}(n,d)$. We decompose T as in the proof of Theorem 2.8 with the first subtree T_1 on $n-m+1$ vertices with root degree $d-1$ and the second subtree T_2 on m vertices such that the first child of the root is black . According to (6) and Theorem 2.1, the number of such trees is given by

$$\frac{d-1}{n-m}\binom{3n-3m-d}{n-m-d+1} \quad \text{and} \quad \frac{1}{2m-1}\binom{3m-3}{m-1}$$

respectively.

Now,

$$\begin{aligned}|\mathcal{S}(n,d)| &= \sum_{m=1}^{n-d+1} \frac{1}{2m-1}\binom{3m-3}{m-1}\frac{d-1}{n-m}\binom{3n-3m-d}{n-m-d+1}\\ &= \sum_{k=0}^{n-d} \frac{1}{2k+1}\binom{3k}{k}\frac{2d-2}{2n-2k-2}\binom{3n-3k-d-3}{n-k-d}.\end{aligned}$$

Again, applying Rothe-Hagen Identity (8) with $x = 1, y = 2d-2, z = 3$ and $n = n-d$, we get the formula in the statement of the proposition. □

Proposition 2.11. *The number of 2-plane trees on n vertices with a white root such that the first child of the root is black and the second eldest black child is its d-th child, where $d > 1$, is given by*

$$\frac{d}{n-1}\binom{3n-d-4}{n-d-1}.$$

Proof. Let $\mathcal{U}(n,d)$ be the set of 2-plane trees on n vertices with a white root such that the first child of the root is black and the second eldest black child is its d-th child. Moreover, let $T \in \mathcal{S}(n,d)$, We decompose T as in the proof of Theorem 2.8 with the first subtree T_1 on $m \geq 2$ vertices with root degree 1 and the second subtree T_2 on $n-m+1$ vertices such that it's eldest black child is it's d-th child. By (6) and Proposition 2.10, the number of these trees is

$$\frac{1}{2m-3}\binom{3m-6}{m-2} \quad \text{and} \quad \frac{2d-1}{2n-2m+1}\binom{3n-3m-d+1}{n-m-d+1}.$$

respectively.

Now,

$$
\begin{aligned}
|\mathcal{U}(n,d)| &= \sum_{m=2}^{n-d+1} \frac{1}{2m-3}\binom{3m-6}{m-2}\frac{2d-1}{2n-2m+1}\binom{3n-3m-d+1}{n-m-d+1}\\
&= \sum_{k=0}^{n-d-1} \frac{1}{2k+1}\binom{3k}{k}\frac{2d-1}{2n-2k-3}\binom{3n-3k-d-5}{n-k-d-1}\\
&= \sum_{k=0}^{n-d-1} \frac{1}{3k+1}\binom{3k+1}{k}\frac{2d-1}{3n-3k-d-4}\binom{3n-3k-d-4}{n-k-d-1}.
\end{aligned}
$$

Applying Rothe-Hagen Identity (8) with $x = 1, y = 2d-1, z = 3$ and $n = n-d-1$, we get the desired formula. ☐

3. Enumeration by root degree

In this section, we enumerate 2-plane trees with a white root according to root degree. We begin by proving the following result:

Lemma 3.1. *There are*

$$\frac{2d-1}{2n-3}\binom{3n-d-5}{n-d-1}$$

2-plane trees on n vertices with a white root of degree d such that the first child of the root is black and it is the only black child of the root.

Proof. Let $\mathcal{T}(n,d)$ be the set of 2-plane trees in the statement of the lemma. Let $T \in \mathcal{T}(n,d)$. We decompose T as in the proof of Theorem 2.8 with the first subtree T_1 on $m \geq 2$ vertices with root degree 1 and the second subtree T_2 on $n-m+1$ vertices with a white root of degree $d-1$. The decomposition is shown in Figure 5.

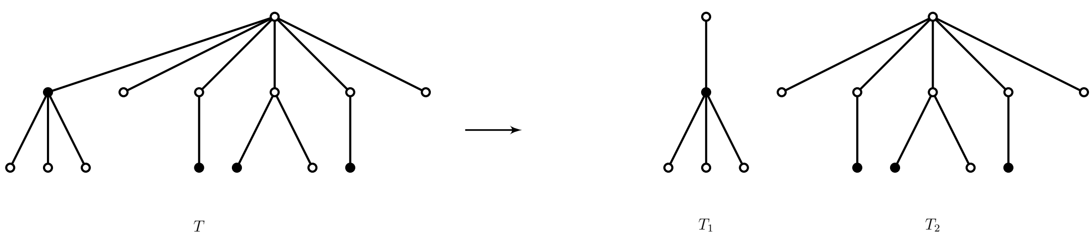

Figure 5. Decomposition of the tree T into T_1 and T_2.

By (6), the number of these trees is

$$\frac{1}{2m-3}\binom{3m-6}{m-2} \quad \text{and} \quad \frac{d}{n-m}\binom{3n-3m-d-1}{n-m-d}.$$

respectively.

We have,

$$|\mathcal{T}(n,d)| = \sum_{m=2}^{n-d+1} \frac{1}{2m-3}\binom{3m-6}{m-2}\frac{d-1}{n-m}\binom{3n-3m-d}{n-m-d+1}$$
$$= \sum_{k=0}^{n-d-1} \frac{1}{2k+1}\binom{3k}{k}\frac{2d-2}{2n-2k-4}\binom{3n-3k-d-6}{n-k-d-1}$$
$$= \sum_{k=0}^{n-d-1} \frac{1}{3k+1}\binom{3k+1}{k}\frac{2d-2}{3n-3k-d-5}\binom{3n-3k-d-5}{n-k-d-1}.$$

Setting $x=1, y=2d-2, z=3$ and $n=n-d-1$ in the Rothe-Hagen Identity (8), we get the desired result. □

We now prove the following result:

Theorem 3.2. *The number of 2-plane trees on n vertices with a white root of degree d such that the first child of the root is black and the total number of black children of the root is j is given by:*

$$\frac{2d-j}{2n-j-2}\binom{d-1}{j-1}\binom{3n-d-j-4}{n-d-1}. \tag{9}$$

Proof. We decompose a tree T with the conditions in the statement of the theorem into j distinct subtrees $T_1, T_2, \ldots, T_j$ such that the black children of the root are the first children of each subtree and all the white siblings of a black child u which are older than the next black child v are in the subtree with the first child u. The decomposition is illustrated in Figure 6.

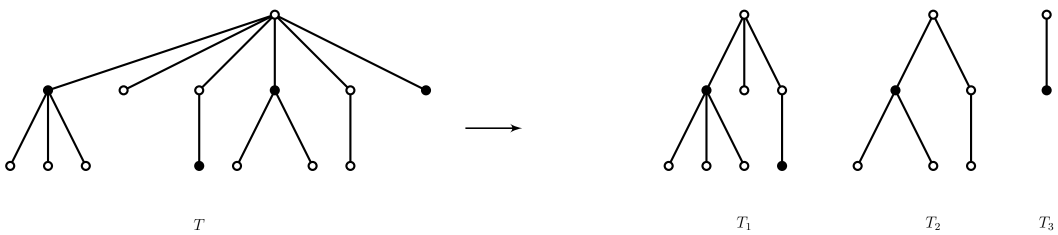

FIGURE 6. Decomposition of the tree T into T_1, T_2 and T_3.

Let n_ℓ and d_ℓ be the number of vertices and degree of the root of the subtree T_ℓ for all $j=1,2,\ldots,j$. It is clear that $n_1+n_2+\cdots+n_j = n+j-1$ and $d_1+d_2+\cdots+d_j = d$. We also have $n_\ell \geq 2$ and $d_\ell \geq 1$ for all $\ell = 1,2,\ldots,j$.

There are $\binom{d-1}{j-1}$ choices for the black children of the root. By Lemma 3.1, if $\mathcal{T}_\ell$ is the set of the subtrees T_ℓ then,

$$|\mathcal{T}_\ell| = \frac{2d_\ell-1}{2n_\ell-3}\binom{3n_\ell-d_\ell-5}{n_\ell-d_\ell-1}.$$

The desired result is thus

$$\binom{d-1}{j-1}\sum_{\substack{n_1+n_2+\cdots+n_j=n+j-1\\ d_1+d_2+\cdots+d_j=d\\ n_1\geq 2, n_2\geq 2,\ldots,n_j\geq 2\\ d_1\geq 1,d_2\geq 1,\ldots,d_j\geq 1}} \prod_{\ell=1}^{j}\frac{2d_\ell-1}{2n_\ell-3}\binom{3n_\ell-d_\ell-5}{n_\ell-d_\ell-1},$$

which reduces to the required formula. □

We get an immediate result upon setting $j = d$ in (12).

Corollary 3.3. *There are*

$$\frac{d}{2n-d-2}\binom{3n-2d-4}{n-d-1} \tag{10}$$

2-plane trees with a white root of degree d such that all its children are black.

David Callan recorded in the Neil Sloane's OEIS [17, A11616], that (10) counts the number of Dyck paths of semilength $n-2$ for which all descents are of even length with no valley vertices at height 1 and with $d-1$ returns to ground level. It would be interesting to construct a bijection between these sets of combinatorial structures.

By setting $i = d - j$ in (9), we obtain the following result:

Corollary 3.4. *The number of 2-plane trees on n vertices with a white root of degree d such that the first child of the root is black and there are i white children of the root is given by*

$$\frac{d+i}{2n-d+i-2}\binom{d-1}{i}\binom{3n-2d+i-4}{n-d-1}. \tag{11}$$

Summing over all values of i in (11) we get that:

Corollary 3.5. *The number of 2-plane trees on n vertices with a white root of degree d such that the first child of the root is black is given by*

$$\sum_{i=0}^{d-1}\frac{d+i}{2n-d+i-2}\binom{d-1}{i}\binom{3n-2d+i-4}{n-d-1}. \tag{12}$$

Combinatorialist Emeric Deutsch recorded in the celebrated encyclopaedia of integer sequences [17, A101409], that (12) counts the number of noncrossing trees on n vertices such that the leftmost leaf is at level d. The result is in the paper [3] where Deutsch and his co-authors gave a generating function. It would be of combinatorial interest to construct a bijection between the set of the 2-plane trees in question and set of the corresponding noncrossing trees.

Proposition 3.6. *Let $\mathcal{W}(n,d,j,r)$ be the set of 2-plane trees on n vertices with a white root of degree d such that there are j black children of the root, the eldest of which is the r-th child of the root where $r \geq 2$. Then*

$$|\mathcal{W}(n,d,j,r)| = \frac{2d-j}{2n-j-2}\binom{d-r}{j-1}\binom{3n-d-j-4}{n-d-1}. \tag{13}$$

Proof. Let $\mathcal{W}(n,d,j,r)$ be the set of 2-plane trees on n vertices with a white root of degree d such that there are j black children of the root, the eldest of which is the r-th child of the root where $r \geq 2$. Let $T \in \mathcal{W}(n,d,j,r)$. We decompose T as in the proof of Theorem 2.8 with the first subtree T_1 on $m \geq r$ vertices such that the root degree is $r-1$ and the second subtree T_2 on $n-m+1$ vertices with root of degree $d-r+1$ such that eldest child of the root is black and in total, T_2 has j black children of the root. The decomposition is given in Figure 7.

By (6) and (9), the number of these trees is

$$\frac{r-1}{m-1}\binom{3m-r-3}{m-r} \quad \text{and} \quad \frac{2d-2r-j+2}{2n-2m-j}\binom{d-r}{j-1}\binom{3n-3m-d+r-j-2}{n-m-d+r-1}$$

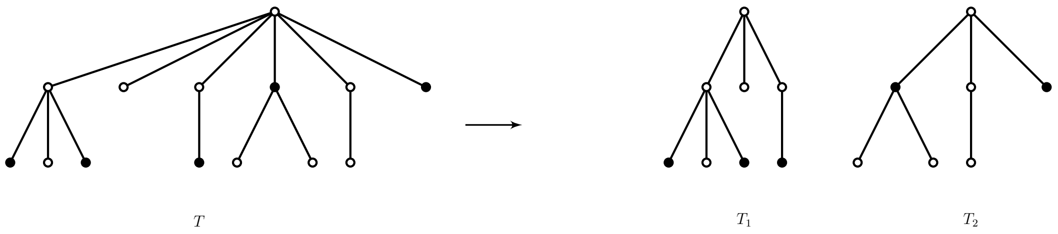

FIGURE 7. Decomposition of the tree T into T_1 and T_2.

respectively.

Thus,

$$|\mathcal{W}(n,d,j,r)|$$

$$= \sum_{m=r}^{n-d+r-1} \frac{r-1}{m-1}\binom{3m-r-3}{m-r}\frac{2d-2r-j+2}{2n-2m-j}\binom{d-r}{j-1}\binom{3n-3m-d+r-j-2}{n-m-d+r-1}$$

$$= \binom{d-r}{j-1}\sum_{k=0}^{n-d-1}\frac{2r-2}{2k+2r-2}\binom{3k+2r-3}{k}\frac{2d-2r-j+2}{2n-2k-2r-j}\binom{3n-3k-2r-d-j-2}{n-k-d-1}.$$

Setting $x = 2r-2, y = 2d-2r-j+2, z = 3$ and $n = n-d-1$ in the Rothe-Hagen Identity (8), we get the required result. □

We rediscover (9) by setting $r = 1$ in (13). By summing over all values of r in (13), we get the following result.

Corollary 3.7. *There are*

$$\frac{2d-j}{2n-j-2}\binom{d}{j}\binom{3n-d-j-4}{n-d-1} \tag{14}$$

2-plane trees on n vertices with a white root of degree d such that there are j black children of the root.

Also, replacing j with $d-i$ in (14), we find that the number of 2-plane trees on n vertices with a white root of degree d such that there are i white children of the root is given by

$$\frac{d+i}{2n-d+i-2}\binom{d}{i}\binom{3n-2d+i-4}{n-d-1}. \tag{15}$$

Corollary 3.8. *The number of 2-plane trees on n vertices with a white root of degree d is given by*

$$\sum_{i=0}^{d}\frac{d+i}{2n-d+i-2}\binom{d}{i}\binom{3n-2d+i-4}{n-d-1}. \tag{16}$$

Proof. The result follows by summing over all values of i in (15). □

4. Enumeration by level of a vertex

Recall that the number of vertices of degree d that reside at level $\ell \geq 0$ in plane trees on n vertices is given by (1). In this section, we obtain the corresponding formulas for 2-plane trees with white roots and black roots.

4.1. **Trees with white roots.** We start by proving the following result which gives the number of possible paths.

Lemma 4.1. *Let* (P_ℓ) *be a sequence of paths of length* $\ell \geq 0$ *that satisfies the conditions that the vertices in the path are coloured either black or white and that there are no black-black edges. If the initial vertex is white then this sequence satisfies the recurrence relation* $P_\ell = P_{\ell-1} + P_{\ell-2}$ *for* $\ell \geq 2$ *and the initial conditions being* $P_0 = 1$ *and* $P_1 = 2$.

Proof. To obtain a path of length ℓ we attach a white vertex at the beginning of a path of length $\ell - 1$. This gives a total of $P_{\ell-1}$ paths of length ℓ such that the second vertex on the path is white. To obtain the paths with the second vertex on the path being black, we consider paths of length $\ell - 2$ and attach white-black edge at the beginning of the path. We thus obtain $P_{\ell-2}$ paths of length ℓ such that the second vertex on the path is black. The initial conditions follow since $P_0 = 1$, i.e. the single vertex coloured white and white-white and white-black edges ensure that $P_1 = 2$. This proves the recurrence relation. □

By solving the recurrence relation, we find that the number of these paths of length ℓ is given as the $(\ell + 1)^{\text{th}}$ Fibonacci number,

$$P_\ell = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{\ell+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{\ell+1}\right].$$

Corollary 4.2. *Let* $(P_{\ell,k})$ *be a sequence of paths of length* $\ell \geq 0$ *with satisfies the conditions that the vertices in the path are coloured either black or white, the total number of black vertices is* k *and that there are no black-black edges. If the initial vertex is white then there are*

$$\binom{\ell - k}{k} \tag{17}$$

such paths.

Proof. The problem is equivalent to finding the number of bit strings of length ℓ which start with a 0, have k 1's and that there are no consecutive 1's which is given by the formula. □

Theorem 4.3. *The number of black vertices of degree* d *that reside at level* $\ell > 0$ *in* 2*-plane trees on* n *vertices with white roots such that there are* $k \geq 1$ *black vertices on a path from the root to the black vertex of degree* d *is given by*

$$\frac{2\ell - k + d + 1}{n + \ell - k}\binom{3n + \ell - 2k - d - 2}{n - \ell - d - 1}\binom{\ell - k - 1}{k - 1}. \tag{18}$$

Proof. Let $B(x)$ and $W(x)$ be the generating functions of 2-plane trees with black and white roots respectively. Here, x marks the number of vertices. Let the root of the trees be white and let a black vertex of degree d reside at level ℓ. Let the number of black vertices on the path from the root to the black vertex at level ℓ be k. By (17) there are

$$\binom{\ell - k - 1}{k - 1}$$

such paths, i.e., paths starting at a white vertex and ending at a black vertex such that the path length is ℓ and the number of black vertices is k.

The generating functions are given by

$$B(x) = \frac{1}{1 - xW} \tag{19}$$

and

$$W(x) = \frac{1}{1 - xW - xB}. \tag{20}$$

Substituting (19) in (20), we obtain

$$W(x) = \frac{1}{(1 - xW)^2}. \tag{21}$$

Equations (19) and (21) give

$$B(x)^2 = \frac{1}{(1 - xW)^2} = W(x). \tag{22}$$

By the decomposition of the trees as shown in Figure 8, we have that the generating function

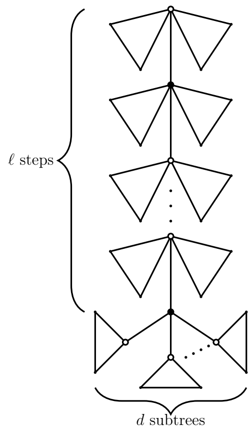

FIGURE 8. Decomposition of 2-plane tree with white root and black vertex of degree d at level ℓ.

for 2-plane trees in the statement of the theorem is given by:

$$(x(W(x)^2))^{\ell-k+1}(xB(x)^2)^{k-1}x(xW(x))^d = x^{\ell+d+1}W(x)^{2\ell-2k+d+2}(B(x)^2)^{k-1}.$$

By (22), the generating function can be rewritten as

$$x^{\ell+d+1}W(x)^{2\ell-2k+d+2}W(x)^{k-1} = x^{\ell+d+1}W(x)^{2\ell-k+d+1}.$$

We remark that both (19) and (20) are not in the right form to use Lagrange inversion formula given in Theorem 1.1. To get the right form for $W(x)$ to use Lagrange inversion formula, let $F(x) = xW(x)$ so that

$$F(x) = \frac{x}{(1 - F)^2}.$$

Applying Lagrange Inversion Formula [18], we have

$$\begin{aligned}[x^n]x^{\ell+d+1}W(x)^{2\ell-k+d+1} &= [x^{n-\ell-d-1}]x^{-2\ell+k-d-1}F(x)^{2\ell-k+d+1} = [x^{n+\ell-k}]F(x)^{2\ell-k+d+1}\\
&= \frac{2\ell-k+d+1}{n+\ell-k}[f^{n-\ell-d-1}](1-f)^{-2(n+\ell-k)}\\
&= \frac{2\ell-k+d+1}{n+\ell-k}[f^{n-\ell-d-1}]\sum_{i\geq 0}\binom{-2(n+\ell-k)}{i}(-1)^i f^i\\
&= \frac{2\ell-k+d+1}{n+\ell-k}[f^{n-\ell-d-1}]\sum_{i\geq 0}\binom{2n+2\ell-2k+i-1}{i}f^i\\
&= \frac{2\ell-k+d+1}{n+\ell-k}\binom{2n+2\ell-2k+n-\ell-d-1-1}{n-\ell-d-1}\\
&= \frac{2\ell-k+d+1}{n+\ell-k}\binom{3n+\ell-2k-d-2}{n-\ell-d-1}.\end{aligned}$$

Thus, the required formula is

$$\frac{2\ell-k+d+1}{n+\ell-k}\binom{3n+\ell-2k-d-2}{n-\ell-d-1}\binom{\ell-k-1}{k-1}.$$

□

Summing over all values of d and k in (18), we find that there are

$$\sum_{k=1}^{\lfloor\frac{\ell+1}{2}\rfloor}\frac{4\ell-2k+3}{n-\ell-1}\binom{3n+\ell-2k-1}{n-\ell-2}\binom{\ell-k-1}{k-1} \tag{23}$$

black vertices that reside at level $\ell>0$ in 2-plane trees on n vertices with white roots. Further, setting $\ell=1$ in (23), we get that there are a total of

$$\frac{5}{n-2}\binom{3n-2}{n-3} \tag{24}$$

black children of the white root in 2-plane trees on n vertices.

Dividing (24) by (5) and tending n to infinity, we find that on average there are $5/4$ black children of a white root in a random 2-plane tree. The following result follows by setting $d=0$ in (18) and summing over all values of k.

Corollary 4.4. *There are*

$$\sum_{k=1}^{\lfloor\frac{\ell+1}{2}\rfloor}\frac{2\ell-k+1}{n+\ell-k}\binom{3n+\ell-2k-2}{n-\ell-1}\binom{\ell-k-1}{k-1} \tag{25}$$

black leaves that reside at level $\ell>0$ in 2-plane trees on n vertices with white roots.

So, by summing over all values of ℓ in (25), we find that there are

$$\sum_{\ell=1}^{n-1}\sum_{k=1}^{\lfloor\frac{\ell+1}{2}\rfloor}\frac{2\ell-k+1}{n+\ell-k}\binom{3n+\ell-2k-2}{n-\ell-1}\binom{\ell-k-1}{k-1}$$

black leaves in 2-plane trees on n vertices and having white roots. Setting $\ell=1$ in (25), we get the total number of black leaves which are children of the root in 2-plane trees on n vertices with white roots as

$$\frac{1}{n}\binom{3n-3}{n-2}. \tag{26}$$

Again, dividing (26) by (5) and tending n to infinity, we find that on average there are 1/3 black leaves which are children of a white root in a random 2-plane tree.

Proposition 4.5. *The number of white vertices that reside at level $\ell > 0$ in 2-plane trees on n vertices with white roots such that there are k black vertices on a path from the white root to the the white vertex at level ℓ is given by*

$$\frac{2\ell-k+1}{n+\ell-k}\binom{3n+\ell-2k-2}{n-\ell-1}\binom{\ell-k-1}{k}. \tag{27}$$

Proof. Let $B(x)$ and $W(x)$ be the generating functions for 2-plane trees with black and white roots respectively, where x marks a vertex. Let the number of black vertices on the path from the root to the white vertex at level ℓ be k. By (17), there are

$$\binom{\ell-k-1}{k}$$

such paths, i.e., paths starting at a white vertex and ending at a white vertex such that the path length is ℓ and the number of black vertices is k. We decompose the trees as shown in Figure 9. The generating function is thus

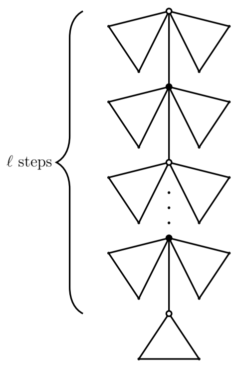

FIGURE 9. Decomposition of 2-plane tree with a white root and a white vertex at level ℓ.

$$(xW(x)^2)^{\ell-k}(xB(x)^2)^k xW(x) = x^{\ell+1}W(x)^{2\ell-2k+1}(B(x)^2)^k,$$

where k is the number of black vertices on the path from the root to the white vertex at level ℓ. By (22), the generating function is $x^{\ell+1}W(x)^{2\ell-k+1}$. From (21), we know that

$$W(x) = \frac{1}{(1-xW)^2}.$$

This generating function is not in the form we can apply Lagrange inversion formula (Theorem 1.1). We set $F(x) = xW(x)$ to get

$$F(x) = \frac{x}{(1-F)^2},$$

which is in the form we can apply Lagrange inversion to extract the coefficient of x^n in the generating function.

$$\begin{aligned}[x^n]x^{\ell+1}W(x)^{2\ell-k+1} &= [x^{n-\ell-1}]x^{-2\ell+k-1}F(x)^{2\ell-k+1} = [x^{n+\ell-k}]F(x)^{2\ell-k+1}\\
&= \frac{2\ell-k+1}{n+\ell-k}[f^{n-\ell-1}](1-f)^{-2(n+\ell-k)}\\
&= \frac{2\ell-k+1}{n+\ell-k}[f^{n-\ell-1}]\sum_{i\geq 0}\binom{-2(n+\ell-k)}{i}(-f)^i\\
&= \frac{2\ell-k+1}{n+\ell-k}[f^{n-\ell-1}]\sum_{i\geq 0}\binom{2n+2\ell-2k+i-1}{i}f^i\\
&= \frac{2\ell-k+1}{n+\ell-k}\binom{2n+2\ell-2k+n-\ell-1-1}{n-\ell-1}\\
&= \frac{2\ell-k+1}{n+\ell-k}\binom{3n+\ell-2k-2}{n-\ell-1}.\end{aligned}$$

The total number of white vertices at level ℓ in these trees is therefore

$$\frac{2\ell-k+1}{n+\ell-k}\binom{3n+\ell-2k-2}{n-\ell-1}\binom{\ell-k-1}{k}.$$

□

Setting $\ell = 0$ and $k = 0$ in (27) we recover (5) for the number of 2-plane trees on n vertices with white roots. Also setting $\ell = 1$ and $k = 0$ we find that there are

(28) $$\frac{3}{n+1}\binom{3n-1}{n-2}$$

white children of the root in 2-plane trees on n vertices with white roots. Upon division of (28) by (5), we find that the average number of white children in 2-plane trees with white roots is

(29) $$\frac{9n^2-12n+3}{2n^2+3n+1}.$$

So tending n to infinity in (29), we get that on average there are $9/2$ white children of the root in a random 2-plane tree with a white root.

Proposition 4.6. *The number of white leaves that reside at level $\ell > 0$ in 2-plane trees on n vertices with white roots such that there are k black vertices on a path from the white root to the white leaf is given by*

(30) $$\frac{2\ell-k}{n+\ell-k-1}\binom{3n+\ell-2k-4}{n-\ell-1}\binom{\ell-k-1}{k}.$$

Proof. Let $B(x)$ and $W(x)$ be the generating functions for 2-plane trees with black and white roots respectively, where x marks a vertex. Let the number of black vertices on the path from the root to the black vertex at level ℓ be k. By (17) there are

$$\binom{\ell-k-1}{k}$$

such paths, i.e., paths starting at a white vertex and ending at a white vertex such that the path length is ℓ and the number of black vertices is k. According to the decomposition of the 2-plane trees with white roots and white leaf at level ℓ as shown in Figure 10, we have that the

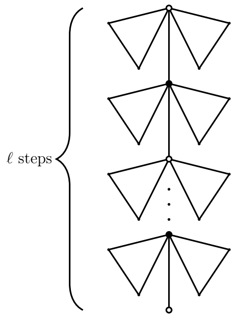

FIGURE 10. Decomposition of 2-plane tree with a white root and a white leaf at level ℓ.

generating function for these trees is

$$(xW(x)^2)^{\ell-k}(xB(x)^2)^k x = x^{\ell+1}W(x)^{2\ell-2k}(B(x)^2)^k,$$

where k is the number of black vertices on the path from the root to the leaf.

By (22), the generating function is $x^{\ell+1}W(x)^{2\ell-k}$. We know that

$$W(x) = \frac{1}{(1-xW)^2}.$$

As before, let $F(x) = xW(x)$ so that

$$F(x) = \frac{x}{(1-F)^2}.$$

Now,

$$\begin{aligned}
[x^n]x^{\ell+1}W(x)^{2\ell-k} &= [x^{n-\ell-1}]x^{-2\ell+k}F(x)^{2\ell-k} = [x^{n+\ell-k-1}]F(x)^{2\ell-k} \\
&= \frac{2\ell-k}{n+\ell-k-1}[f^{n-\ell-1}](1-f)^{-2(n+\ell-k-1)} \\
&= \frac{2\ell-k}{n+\ell-k-1}[f^{n-\ell-1}]\sum_{i\geq 0}\binom{-2(n+\ell-k-1)}{i}(-f)^i \\
&= \frac{2\ell-k}{n+\ell-k-1}[f^{n-\ell-1}]\sum_{i\geq 0}\binom{2n+2\ell-2k-2+i-1}{i}f^i \\
&= \frac{2\ell-k}{n+\ell-k-1}\binom{2n+2\ell-2k+n-\ell-1-3}{n-\ell-1} \\
&= \frac{2\ell-k}{n+\ell-k-1}\binom{3n+\ell-2k-4}{n-\ell-1}.
\end{aligned}$$

So, the total number of white leaves at level ℓ in these trees is

$$\frac{2\ell-k}{n+\ell-k-1}\binom{3n+\ell-2k-4}{n-\ell-1}\binom{\ell-k-1}{k}.$$

This completes the proof. □

Setting $\ell = 1$ and $k = 0$ in (30), we find that there are a total

$$\frac{2}{n}\binom{3n-3}{n-2} \tag{31}$$

white leaves which are children of the root in 2-plane trees on n vertices with white roots.

Dividing (31) by (5) and tending n to infinity, we find that on average there are 2/3 white leaves which are children of a white root in a random 2-plane tree.

4.2. Trees with black roots.

Theorem 4.7. *The number of black vertices of degree d that reside at level $\ell > 1$ in 2-plane trees on n vertices with black roots such that there are $k \geq 1$ black vertices on a path from the root to the black vertex of degree d is given by*

$$\frac{2\ell - k + d + 1}{n + \ell - k}\binom{3n + \ell - 2k - d - 2}{n - \ell - d - 1}\binom{\ell - k - 1}{k - 2}. \tag{32}$$

Proof. Let $B(x)$ and $W(x)$ be the generating functions 2-plane trees with black and white roots respectively, where x marks a vertex. Let the root of the trees under consideration be black and let a black vertex of degree d reside at level $\ell > 1$. Also, let the number of black vertices on the path from the root to the black vertex at level ℓ be k. By (17) there are

$$\binom{\ell - k - 1}{k - 2}$$

such paths, i.e., paths starting at a black vertex and ending at a black vertex such that the path length is $\ell > 1$ and the number of black vertices is k. By the decomposition of the trees in Figure 11, we have that the generating function for 2-plane trees in the statement of the

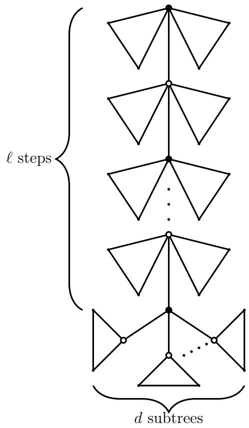

FIGURE 11. Decomposition of 2-plane tree with a black root such that a black vertex of degree d is at level $\ell > 1$.

theorem is given by:

$$(x(W(x)^2))^{\ell-k+1}(xB(x)^2)^{k-1}x(xW(x))^d = x^{\ell+d+1}W(x)^{2\ell-2k+d+2}(B(x)^2)^{k-1}.$$

By (22), the generating function is

$$x^{\ell+d+1}W(x)^{2\ell-2k+d+2}W(x)^{k-1} = x^{\ell+d+1}W(x)^{2\ell-k+d+1} = x^{\ell+d+1}B(x)^{4\ell-2k+2d+2}.$$

From equations (19) and (20), we have

$$B(x) = \frac{1}{1 - xB(x)^2}.$$

Let $F(x) = \sqrt{x}B(x)$ so that

$$F(x) = \frac{\sqrt{x}}{1 - F^2}.$$

By Lagrange Inversion Formula [18], we have

$$\begin{aligned}
[x^n]x^{\ell+d+1}B(x)^{4\ell-2k+2d+2} &= [x^{n-\ell-d-1}]x^{-2\ell+k-d-1}F(x)^{4\ell-2k+2d+2} = [x^{\frac{2n+2\ell-2k}{2}}]F(x)^{4\ell-2k+2d+2} \\
&= \frac{4\ell-2k+2d+2}{2n+2\ell-2k}[f^{2n-2\ell-2d-2}](1-f^2)^{-(2n+2\ell-2k)} \\
&= \frac{2\ell-k+d+1}{n+\ell-k}[f^{2n-2\ell-2d-2}]\sum_{i\geq 0}\binom{-(2n+2\ell-2k)}{i}(-1)^i f^{2i} \\
&= \frac{2\ell-k+d+1}{n+\ell-k}[f^{2(n-\ell-d-1)}]\sum_{i\geq 0}\binom{2n+2\ell-2k+i-1}{i}f^{2i} \\
&= \frac{2\ell-k+d+1}{n+\ell-k}\binom{2n+2\ell-2k+n-\ell-d-1-1}{n-\ell-d-1} \\
&= \frac{2\ell-k+d+1}{n+\ell-k}\binom{3n+\ell-2k-d-2}{n-\ell-d-1}.
\end{aligned}$$

Thus, the required formula is

$$\frac{2\ell-k+d+1}{n+\ell-k}\binom{3n+\ell-2k-d-2}{n-\ell-d-1}\binom{\ell-k-1}{k-2}.$$

□

Summing over all values of d and k in (32), we get

$$\sum_{k=2}^{\lfloor\frac{\ell+2}{2}\rfloor}\frac{4\ell-2k+3}{n-\ell-1}\binom{3n+\ell-2k-1}{n-\ell-2}\binom{\ell-k-1}{k-2} \tag{33}$$

as the formula for the number of black vertices that reside at level $\ell > 1$ in 2-plane trees on n vertices with black roots.

By setting $d = 0$ in (32) and summing over all values of k, we get the following result:

Corollary 4.8. *There are*

$$\sum_{k=2}^{\lfloor\frac{\ell+2}{2}\rfloor}\frac{2\ell-k+1}{n+\ell-k}\binom{3n+\ell-2k-2}{n-\ell-1}\binom{\ell-k-1}{k-2} \tag{34}$$

black leaves that reside at level $\ell > 1$ in 2-plane trees on n vertices with black roots.

By summing over all values of ℓ in (34), it follows that there are

$$\sum_{\ell=2}^{n-1}\sum_{k=2}^{\lfloor\frac{\ell+2}{2}\rfloor}\frac{2\ell-k+1}{n+\ell-k}\binom{3n+\ell-2k-2}{n-\ell-1}\binom{\ell-k-1}{k-2}$$

black leaves in 2-plane trees on n vertices with black roots.

Proposition 4.9. *The number of white vertices that reside at level $\ell > 0$ in 2-plane trees on n vertices with black roots such that there are $k \geq 1$ black vertices on a path from the root to the white vertex at level ℓ is given by*

$$\frac{2\ell-k+1}{n+\ell-k}\binom{3n+\ell-2k-2}{n-\ell-1}\binom{\ell-k-1}{k-1}. \tag{35}$$

Proof. Let $B(x)$ and $W(x)$ be the generating functions 2-plane trees with black and white roots respectively, where x marks a vertex. Let the root of the trees under consideration be black and let a white vertex reside at level $\ell > 0$. Let the number of black vertices on the path from the root to the white vertex at level ℓ be k. By (17), there are

$$\binom{\ell-k-1}{k-1}$$

such paths, i.e., paths starting at a black vertex and ending at a white vertex such that the path length is $\ell > 0$ and the number of black vertices is k. By the decomposition of the trees in Figure 12, we have that the generating function for 2-plane trees in the statement of the

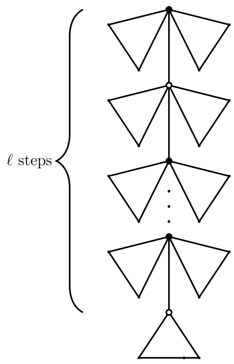

FIGURE 12. Decomposition of 2-plane tree with a black root such that a white vertex is at level $\ell > 0$.

theorem is given by:

$$(xB(x)^2)^k(x(W(x)^2))^{\ell-k}xW(x) = x^{\ell+1}(B(x)^2)^kW(x)^{2\ell-2k+1}.$$

By (22), the generating function is

$$x^{\ell+1}(B(x)^2)^kW(x)^{2\ell-2k+1} = x^{\ell+1}B(x)^{4\ell-2k+2}.$$

From (19) and (20), we have

$$B(x) = \frac{1}{1-xB(x)^2}.$$

Let $F(x) = \sqrt{x}B(x)$ so that

$$F(x) = \frac{\sqrt{x}}{1-F^2}.$$

By Lagrange Inversion Formula [18], we have

$$\begin{aligned}
[x^n]x^{\ell+1}B(x)^{4\ell-2k+2} &= [x^{n-\ell-1}]x^{-2\ell+k-1}F(x)^{4\ell-2k+2} = [x^{\frac{2n+2\ell-2k}{2}}]F(x)^{4\ell-2k+2}\\
&= \frac{4\ell-2k+2}{2n+2\ell-2k}[f^{2n-2\ell-2}](1-f^2)^{-(2n+2\ell-2k)}\\
&= \frac{2\ell-k+1}{n+\ell-k}[f^{2n-2\ell-2}]\sum_{i\geq 0}\binom{-(2n+2\ell-2k)}{i}(-1)^if^{2i}\\
&= \frac{2\ell-k+1}{n+\ell-k}[f^{2(n-\ell-1)}]\sum_{i\geq 0}\binom{2n+2\ell-2k+i-1}{i}f^{2i}\\
&= \frac{2\ell-k+1}{n+\ell-k}\binom{2n+2\ell-2k+n-\ell-1-1}{n-\ell-1}
\end{aligned}$$

$$= \frac{2\ell - k + 1}{n + \ell - k}\binom{3n + \ell - 2k - 2}{n - \ell - 1}.$$

The formula for the number of the trees is therefore

$$\frac{2\ell - k + 1}{n + \ell - k}\binom{3n + \ell - 2k - 2}{n - \ell - 1}\binom{\ell - k - 1}{k - 1}.$$

□

Summing over all values of k in (35), we get

$$\sum_{k=1}^{\lfloor \frac{\ell+1}{2} \rfloor} \frac{2\ell - k + 1}{n + \ell - k}\binom{3n + \ell - 2k - 2}{n - \ell - 1}\binom{\ell - k - 1}{k - 1} \tag{36}$$

as the formula for the number of white vertices that reside at level $\ell > 0$ in 2-plane trees on n vertices with black roots.

Also, summing over all values of ℓ in (36), it follows that there are a total of

$$\sum_{\ell=1}^{n-1} \sum_{k=1}^{\lfloor \frac{\ell+1}{2} \rfloor} \frac{2\ell - k + 1}{n + \ell - k}\binom{3n + \ell - 2k - 2}{n - \ell - 1}\binom{\ell - k - 1}{k - 1}$$

white vertices in 2-plane trees on n vertices with black roots.

Setting $\ell = 1$ in (36), we get that the total number of children of the root in 2-plane trees with black roots is

$$\frac{2}{n}\binom{3n - 3}{n - 2}.$$

Proposition 4.10. *The number of white leaves that reside at level $\ell > 0$ in 2-plane trees on n vertices with black roots such that there are $k \geq 1$ black vertices on a path from the root to the leaf at level ℓ is given by*

$$\frac{2\ell - k + 1}{n + \ell - k}\binom{3n + \ell - 2k - 2}{n - \ell - 1}\binom{\ell - k - 1}{k - 1}. \tag{37}$$

Proof. Let $B(x)$ and $W(x)$ be the generating functions 2-plane trees with black and white roots respectively, where x marks the number of vertices. Let the root of the trees under consideration be black and let a white leaf reside at level $\ell > 0$. Let the number of black vertices on the path from the root to the white leaf at level ℓ be k. By (17), there are

$$\binom{\ell - k - 1}{k - 1}$$

such paths, i.e., paths starting at a black vertex and ending at a white leaf such that the path length is $\ell > 0$ and the number of black vertices is k.

The decomposition of the trees in shown in Figure 13.

The generating function for these 2-plane trees is thus

$$(xB(x)^2)^k(x(W(x)^2))^{\ell-k}x = x^{\ell+1}(B(x)^2)^k W(x)^{2\ell-2k}.$$

By (22), we have the generating function for the trees under enumeration as

$$x^{\ell+1}(B(x)^2)^k W(x)^{2\ell-2k} = x^{\ell+1}B(x)^{4\ell-2k}.$$

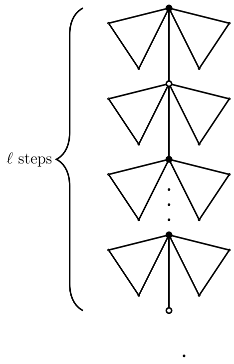

FIGURE 13. Decomposition of 2-plane tree with a black root such that a white leaf is at level $\ell > 0$.

From (19) and (20) we have

$$B(x) = \frac{1}{1 - xB(x)^2}.$$

Let $F(x) = \sqrt{x}B(x)$. Then

$$F(x) = \frac{\sqrt{x}}{1 - F^2},$$

which is in a form we can apply Lagrange inversion to obtain

$$\begin{aligned}
[x^n]x^{\ell+1}B(x)^{4\ell-2k} &= [x^{n-\ell-1}]x^{-2\ell+k}F(x)^{4\ell-2k} = [x^{\frac{2n+2\ell-2k-2}{2}}]F(x)^{4\ell-2k} \\
&= \frac{4\ell-2k}{2n+2\ell-2k-2}[f^{2n-2\ell-2}](1-f^2)^{-(2n+2\ell-2k-2)} \\
&= \frac{2\ell-k}{n+\ell-k-1}[f^{2n-2\ell-2}]\sum_{i\geq 0}\binom{-(2n+2\ell-2k-2)}{i}(-1)^i f^{2i} \\
&= \frac{2\ell-k}{n+\ell-k-1}[f^{2(n-\ell-1)}]\sum_{i\geq 0}\binom{2n+2\ell-2k-2+i-1}{i} f^{2i} \\
&= \frac{2\ell-k}{n+\ell-k-1}\binom{2n+2\ell-2k-2+n-\ell-1-1}{n-\ell-1} \\
&= \frac{2\ell-k}{n+\ell-k-1}\binom{3n+\ell-2k-4}{n-\ell-1}.
\end{aligned}$$

It follows that there are a total of

$$\frac{2\ell-k}{n+\ell-k-1}\binom{3n+\ell-2k-4}{n-\ell-1}\binom{\ell-k-1}{k-1}$$

white leaves that reside at level $\ell > 0$ in 2-plane trees on n vertices with black roots such that there are k black vertices on the path from the root to the leaves. □

Summing over all values of k in (35), we get

$$\sum_{k=1}^{\lfloor\frac{\ell+1}{2}\rfloor}\frac{2\ell-k}{n+\ell-k-1}\binom{3n+\ell-2k-4}{n-\ell-1}\binom{\ell-k-1}{k-1} \tag{38}$$

as the formula for the number of white leaves that reside at level $\ell > 0$ in 2-plane trees on n vertices with black roots.

Moreover, summing over all values of ℓ in (38), it follows that there are a total of

$$\sum_{\ell=1}^{n-1}\sum_{k=1}^{\lfloor\frac{\ell+1}{2}\rfloor}\frac{2\ell-k}{n+\ell-k-1}\binom{3n+\ell-2k-4}{n-\ell-1}\binom{\ell-k-1}{k-1}$$

white leaves in 2-plane trees on n vertices and having black roots.

Setting $\ell=1$ in (38), we get that the total number of leaves which are also children of the root in 2-plane trees with black roots is

$$\frac{1}{n-1}\binom{3n-5}{n-2}.$$

5. Conclusion

In this paper, we have enumerated 2-plane trees according to the colour of the first child of the root, root degree and number of vertices that reside at level ℓ. It will be of importance to obtain counting formulas for the number of 2-plane trees according to degree sequences, length of a leftmost path and number of leaves. Enumeration of forests of 2-plane trees based on number of components and vertices was recently done by Nyariaro and Okoth in [8].

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