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PROPERTIES OF PSEUDO-ORTHOGONAL CHEBYSHEV-LIKE POLYNOMIALS

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Abstract. Chebyshev polynomials are one of the classes of rationally generated polynomials that are orthogonal in the interval $[-1, 1]$. In this paper, we introduce a new class of Chebyshev-like polynomials denoted by $R_n(x)$ and satisfying the recurrence relation $R_n(x) = 2xR_{n-1}(x) - R_{n-2}(x)$, with initial conditions $R_0(x) = 1$ and $R_1(x) = 3x$. We show that these polynomials are rationally generated and prove connections to the classical Chebyshev polynomials of the first and of the second kind. We then prove that they are pseudo-orthogonal in the interval $[-1, 1]$ and have all their zeros in this interval. Lastly, we give identities for resultants involving these polynomials.

1. Introduction

Chebyshev polynomials have been studied extensively. Among them are the Chebyshev polynomials of the first and of the second kind, commonly referred to as classical Chebyshev polynomials, which are sequences of orthogonal polynomials related to sine and cosine functions. In [8, eq.1.3a], Chebyshev polynomials of the first kind are defined recursively by

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \tag{1}$$

for $n \geq 2$, with initial conditions $T_0(x) = 1$ and $T_1(x) = x$. On the other hand, the recurrence relation for the Chebyshev polynomials of the second kind is

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x) \tag{2}$$

for $n \geq 2$ and the initial conditions are $U_0(x) = 1$ and $U_1(x) = 2x$.

The Chebyshev polynomials of the first kind and of the second kind are respectively defined by the rational generating functions

$$\sum_{n=0}^{\infty} T_n(x)t^n = \frac{1 - xt}{1 - 2xt + t^2} \quad \text{and} \quad \sum_{n=0}^{\infty} U_n(x)t^n = \frac{1}{1 - 2xt + t^2}, \tag{3}$$

see [8, p15] for details. From the generating functions (3), connections between the first and second kind Chebyshev polynomials can be drawn, i.e.,

$$U_{n+1}(x) = xU_n(x) + T_{n-1}(x),$$

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and

$$T_n(x) = \frac{1}{2}U_n(x) - \frac{1}{2}U_{n-2}(x), \tag{4}$$

[8]. The two polynomials are also related by the sum formula

$$U_n(x) = \begin{cases} 2\sum_{\text{odd } j}^{n} T_j(x), & \text{for odd } n, \\ 2\sum_{\text{even } j}^{n} T_j(x) - 1, & \text{for even } n. \end{cases} \tag{5}$$

Explicit formula for Chebyshev polynomials of the first kind, and of the second kind are obtained by solving the recurrence relations (1) and (2) respectively. The explicit formulas are

$$T_n(x) = \frac{1}{2}\Big((x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n\Big),$$

and

$$U_n(x) = \frac{1}{2}\Big(\frac{(x + \sqrt{x^2-1})^{n+1} - (x - \sqrt{x^2-1})^{n+1}}{\sqrt{x^2-1}}\Big).$$

The derivatives of Chebyshev polynomials of the first kind, and of the second kind are respectively given by;

$$\frac{\mathrm{d}}{\mathrm{d}x}T_n(x) = nxU_{n-2}(x) + nT_n(x) = nU_{n-1}(x),$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}U_n(x) = \frac{(n+1)T_{n+1}(x) - xU_n(x)}{x^2-1}.$$

We now introduce a new class of pseudo-orthogonal Chebyshev-like polynomials denoted by $R_n(x)$ and give their properties.

Definition 1.1. *Let $n \geq 0$ and define the $(n+1)^{th}$ term of a sequence of polynomials as*

$$R_n(x) = 2xR_{n-1}(x) - R_{n-2}(x), \tag{6}$$

for $n \geq 0$, with initial conditions $R_0(x) = 1$ and $R_1(x) = 3x$.

The first 11 polynomials of this sequence are given in Table 1.

n	The corresponding term of the sequence
0	1
1	$3x$
2	$-1 + 6x^2$
3	$-5x + 12x^3$
4	$1 - 16x^2 + 24x^4$
5	$7x - 44x^3 + 48x^5$
6	$-1 + 30x^2 - 112x^4 + 96x^6$
7	$-9x + 104x^3 - 272x^5 + 192x^7$
8	$1 - 48x^2 + 320x^4 - 640x^6 + 384x^8$
9	$11x - 200x^3 + 912x^5 - 1472x^7 + 768x^9$
10	$-1 + 70x^2 - 720x^4 + 2464x^6 - 3328x^8 + 1536x^{10}$

TABLE 1. The $R_n(x)$ polynomials for $0 \leq n \leq 10$.

The graph of the first five $R_n(x)$ polynomials is given in Figure 1 for the range $-1 \leq x \leq 1$.

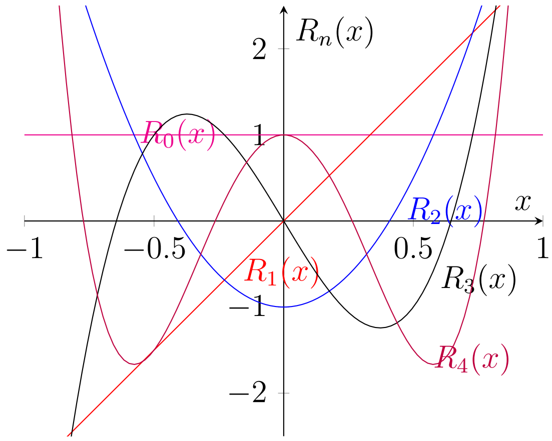

FIGURE 1. Graph of the first five $R_n(x)$ polynomials.

2. GENERATING FUNCTION AND EXPLICIT FORMULA

By solving the recurrence relation (6) with the initial conditions $R_0(x) = 1$ and $R_1(x) = 3x$, we obtain Lemma 2.1.

Lemma 2.1. *The Chebyshev-like polynomials $R_n(x)$ are defined by the rational generating function:*

$$\sum_{n=0}^{\infty} R_n(x)t^n = \frac{1+xt}{1-2xt+t^2}. \tag{7}$$

We now draw the connection between the $R_n(x)$ polynomials and the classical Chebyshev polynomials of the first and of the second kind. These connections follow from the generating functions (3) and (7).

Lemma 2.2. *The Chebyshev-like polynomials $R_n(x)$ are connected to the classical Chebyshev polynomials by the following identities:*

$$R_n(x) = U_n(x) + xU_{n-1}(x) = \frac{3}{2}U_n(x) + \frac{1}{2}U_{n-2}(x), \tag{8}$$

and

$$R_n(x) = 2U_n(x) - T_n(x) = 2xU_{n-1}(x) + T_n(x). \tag{9}$$

Proof. The first equality in (8) follows directly by comparing the rational generating function (7) of the $R_n(x)$ polynomials and the generating function of the Chebyshev polynomials of the second kind $U_n(x)$ given in (3). That is,

$$R_n(x) = U_n(x) + xU_{n-1}(x). \tag{10}$$

Now, from the recurrence relation (2), we have that

$$xU_{n-1}(x) = \frac{1}{2}U_n(x) + \frac{1}{2}U_{n-2}(x). \tag{11}$$

Now, using equation (10) and (11), we obtain

$$R_n(x) = \frac{3}{2}U_n(x) + \frac{1}{2}U_{n-2}(x). \tag{12}$$

From the relations (4) and (12), we can write

$$R_n(x) = \frac{3}{2}U_n(x) + \frac{1}{2}U_n(x) - T_n(x) = 2U_n(x) - T_n(x). \tag{13}$$

Lastly, from the generating function (3), we observe that

$$T_n(x) = U_n(x) - xU_{n-1}(x). \tag{14}$$

Substituting equation (14) in equation (10), we obtain

$$R_n(x) = 2xU_{n-1}(x) + T_n(x)$$

which proves (9). □

The following theorem gives the explicit formula of $R_n(x)$ polynomials.

Theorem 2.3. *Let $n \geq 1$, then the $R_n(x)$ polynomials are explicitly defined by*

$$R_n(x) = \frac{1}{2}\left(\frac{(x+\sqrt{x^2-1})^n(2x+\sqrt{x^2-1})}{\sqrt{x^2-1}} - \frac{(x-\sqrt{x^2-1})^n(2x-\sqrt{x^2-1})}{\sqrt{x^2-1}}\right).$$

The explicit formula can easily be obtained by solving the recurrence relation (6) using the initial conditions $R_0(x) = 1$ and $R_1(x) = 3x$.

3. The fundamental recurrence formula and orthogonality of $R_n(x)$ polynomials

Favard [1] showed that any polynomial sequence which satisfies the three term recurrence given by $P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x),\ n = 1, 2, 3 \ldots,$ where $P_{-1}(x) = 0$ and $P_0(x) = -1$, is orthogonal. The sequence of $R_n(x)$ polynomials satisfies the recurrence relation relation (6), which meets the conditions of the Favard's Theorem. This implies that

Theorem 3.1. *There exists a weight function $w(x)$ for which the Chebyshev-like polynomials $R_n(x)$ are an orthogonal polynomial sequence in the interval $[-1, 1]$.*

Now, we give pseudo-orthogonality identities involving the $R_n(x)$ polynomials. In [8], it was showed that Chebyshev polynomials of the first kind, and of the second kind are orthogonal in the interval $[-1, 1]$ with respect to the weight functions $\frac{1}{\sqrt{1-x^2}}$ and $\sqrt{1-x^2}$ respectively, and are given by the identities;

$$\int_{-1}^{1} T_n(x)T_m(x)\frac{\mathrm{d}x}{\sqrt{1-x^2}} = \begin{cases} 0, \text{if } n \neq m, \\ \pi, \text{if } n = m = 0, \\ \frac{\pi}{2}, \text{if } n = m \neq 0, \end{cases} \tag{15}$$

and

$$\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}\mathrm{d}x = \begin{cases} 0, \text{if } n \neq m, \\ \frac{\pi}{2}, \text{if } n = m. \end{cases} \tag{16}$$

Theorem 3.2. *The integrals of the Chebyshev-like polynomials $R_n(x)$ with respect to the weight function $w_1(x) = \sqrt{1-x^2}$ on the interval $[-1, 1]$ is given by:*

$$\int_{-1}^{1} R_n(x)R_m(x)w_1(x)\mathrm{d}x = \begin{cases} \frac{3\pi}{8}, \textit{ if } n = m+2 \textit{ or } m = n+2, \\ \frac{\pi}{2}, \textit{ if } n = m = 0, \\ \frac{9\pi}{8}, \textit{ if } n = m = 1, \\ \frac{5\pi}{4}, \textit{ if } n = m \geq 2 \\ 0, \ \textit{ else.} \end{cases}$$

Proof. Since $R_n(x) = \frac{3}{2}U_n(x) + \frac{1}{2}U_{n-2}(x)$, we have that

$$\int_{-1}^{1} R_n(x)R_m(x)w_1(x)\mathrm{d}x = \frac{9}{4}\int_{-1}^{1} U_n(x)U_m(x)w_1(x)\mathrm{d}x + \frac{3}{4}\int_{-1}^{1} U_n(x)U_{m-2}(x)w_1(x)\mathrm{d}x$$
$$+\frac{3}{4}\int_{-1}^{1} U_m(x)U_{n-2}(x)w_1(x)\mathrm{d}x + \frac{1}{4}\int_{-1}^{1} U_{n-2}(x)U_{m-2}(x)w_1(x)\mathrm{d}x. \tag{17}$$

When $n \neq m$, then by (16), the only non-zero contribution is when $m = n+2$ or $n = m+2$, thus,

$$\int_{-1}^{1} R_n(x)R_m(x)w_1(x)\mathrm{d}x = \frac{3\pi}{8}.$$

On the other hand, when $n = m = 0$, and the fact that $U_{-1}(x) = 0$, $U_0(x) = 1$ and $U_{-2}(x) = -1$, equation (17) reduces to

$$\int_{-1}^{1} R_n(x)R_m(x)w_1(x)\mathrm{d}x = \int_{-1}^{1} \sqrt{1-x^2}\mathrm{d}x = \frac{\pi}{2}.$$

Now when $n = m = 1$, equation (17) reduces to

$$\int_{-1}^{1} R_n(x)R_m(x)w_1(x)\mathrm{d}x = \frac{9}{4}\int_{-1}^{1} U_1(x)U_1(x)w_1(x)\mathrm{d}x,$$

and by (16), we have

$$\int_{-1}^{1} R_n(x)R_m(x)w_1(x)\mathrm{d}x = \frac{9\pi}{8}.$$

Similarly, when $n = m \geq 2$, and by applying (16), we have

$$\int_{-1}^{1} R_n(x)R_m(x)w_1(x)\mathrm{d}x = \frac{5\pi}{4}.$$

If $n \neq m$ (and $n \neq m+2$ or $m \neq n+2$), then by the identity (16), we have

$$\int_{-1}^{1} R_n(x)R_m(x)w_1(x)\mathrm{d}x = 0.$$

□

Next, we prove the identities of $R_n(x)$ polynomials with respect to the weight function $w_2(x) = \frac{1}{\sqrt{1-x^2}}$ in the interval $[-1, 1]$.

Theorem 3.3. *The integrals of the Chebyshev-like polynomials $R_n(x)$ with respect to the weight function $w_2(x) = \frac{1}{\sqrt{1-x^2}}$ on the interval $[-1, 1]$ are given by:*

$$\int_{-1}^{1} R_n(x)R_m(x)w_2(x)\mathrm{d}x = \begin{cases} \pi, & \text{if} \quad n = m = 0, \\ \frac{(8n+1)}{2}\pi, & \text{if} \quad n = m \geqslant 1, \\ 2(2n-3)\pi, & \text{if} \quad n \text{ and } m \text{ are both even, and } m < n, \\ 2(n-1)\pi, & \text{if} \quad n \text{ and } m \text{ are both odd, and } m < n, \\ 0, & \text{if} \quad n \text{ is odd and } m \text{ is even.} \end{cases}$$

Proof. Recall that $R_n(x) = 2U_n(x) - T_n(x)$, so that

$$\begin{aligned} \int_{-1}^{1} R_n(x)R_m(x)w_2(x)\mathrm{d}x &= \int_{-1}^{1} 4U_n(x)U_m(x)w_2(x)\mathrm{d}x - \int_{-1}^{1} 2U_n(x)T_m(x)w_2(x)\mathrm{d}x \\ &\quad - \int_{-1}^{1} 2U_m(x)T_n(x)w_2(x)\mathrm{d}x + \int_{-1}^{1} T_n(x)T_m(x)w_2(x)\mathrm{d}x. \end{aligned} \tag{18}$$

First, when $n = m = 0$, with $T_0(x) = U_0(x) = 1$, the integral (18) evaluates to

$$\int_{-1}^{1} R_n(x)R_m(x)w_2(x)\mathrm{d}x = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\mathrm{d}x = \pi.$$

Now, when $n = m \geq 1$ and n is odd, substituting (5) into (18) gives

$$\begin{aligned} \int_{-1}^{1} R_n(x)R_m(x)w_2(x)\mathrm{d}x &= 16 \sum_{\substack{l=2 \\ l \text{ even}}}^{2n} \sum_{\substack{j=1 \\ j \text{ odd}}}^{l} \int_{-1}^{1} T_j(x)T_{l-j}(x)w_2(x)\mathrm{d}x \\ &\quad - 8 \sum_{\substack{j=1 \\ j \text{ odd}}}^{n} \int_{-1}^{1} T_j(x)T_n(x)w_2(x)\mathrm{d}x + \int_{-1}^{1} T_n(x)T_n(x)w_2(x)\mathrm{d}x. \end{aligned} \tag{19}$$

By applying (15), the only non-zero integrals in (19) are when $j = l - j$ or $l = 2j$ for the first integral and when $j = n$ for the second the integral of (19), so that

$$\int_{-1}^{1} R_n(x)R_m(x)w_2(x)\mathrm{d}x = 16 \sum_{\substack{l=2 \\ 4|l}}^{2n} \int_{-1}^{1} T_{l/2}(x)T_{l/2}(x)w_2(x)\mathrm{d}x - 4\pi + \frac{\pi}{2}.$$

Since, there are $\frac{n+1}{2}$ numbers between 2 and $2n$ that are divisible by 4, then by (15), we obtain

$$\int_{-1}^{1} R_n(x)R_m(x)w_2(x)\mathrm{d}x = 16\left(\frac{n+1}{2}\right)\frac{\pi}{2} - 4\pi + \frac{\pi}{2} = \frac{(8n+1)\pi}{2}.$$

When $n = m \geq 1$ and n is even, then substituting (5) into (18), we obtain upon simplification,

$$\begin{aligned} \int_{-1}^{1} R_n(x)R_m(x)w_2(x)\mathrm{d}x &= 16 \sum_{\substack{l=0 \\ l \text{ even}}}^{2n} \sum_{\substack{j=0 \\ j \text{ even}}}^{l} \int_{-1}^{1} T_jT_{l-j}(x)(x)w_2(x)\mathrm{d}x \\ &\quad - 8 \sum_{\substack{j=0 \\ j \text{ even}}}^{n} \int_{-1}^{1} T_j(x)T_0(x)w_2(x)\mathrm{d}x - 8 \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \int_{-1}^{1} T_k(x)T_0(x)w_2(x)\mathrm{d}x \\ &\quad + 4\int_{-1}^{1} T_0(x)T_0(x)w_2(x)\mathrm{d}x - 8 \sum_{\substack{j=0 \\ j \text{ even}}}^{n} \int_{-1}^{1} T_j(x)T_n(x)w_2(x)\mathrm{d}x \end{aligned}$$

$$+4\int_{-1}^{1} T_0(x)T_n(x)w_2(x)\mathrm{d}x + \int_{-1}^{1} T_n(x)T_n(x)w_2(x)\mathrm{d}x.$$

This evaluates to;

$$\int_{-1}^{1} R_n(x)R_m(x)w_2(x)\mathrm{d}x = 16\pi + 16\left(\frac{n}{2}\right)\frac{\pi}{2} - 8\pi - 8\pi + 4\pi - 8\left(\frac{\pi}{2}\right) + 4(0) + \frac{\pi}{2}$$
$$= \frac{(8n+1)\pi}{2}.$$

If n is odd and m is even (the case when n is even and m is odd is similar), then again by substituting (5) into (18) and simplifying, we obtain

$$\begin{aligned}\int_{-1}^{1} R_n(x)R_m(x)w_2(x)\mathrm{d}x &= 16\sum_{\substack{j=1\\ j\text{ odd}}}^{n}\sum_{\substack{k=0\\ k\text{ even}}}^{m}\int_{-1}^{1} T_j(x)T_k(x)w_2(x)\mathrm{d}x\\ &\quad - 8\sum_{\substack{j=1\\ j\text{ odd}}}^{n}\int_{-1}^{1} T_j(x)T_0(x)w_2(x)\mathrm{d}x - 4\sum_{\substack{j=1\\ j\text{ odd}}}^{n}\int_{-1}^{1} T_j(x)T_m(x)w_2(x)\mathrm{d}x\\ &\quad - 4\sum_{\substack{k=0\\ k\text{ even}}}^{m}\int_{-1}^{1} T_k(x)T_n(x)w_2(x)\mathrm{d}x + 2\int_{-1}^{1} T_0(x)T_n(x)w_2(x)\mathrm{d}x\\ &\quad + \int_{-1}^{1} T_n(x)T_m(x)w_2(x)\mathrm{d}x. \end{aligned} \tag{20}$$

Since j is odd and k is even, n is odd and m is even, this implies that $j \neq k$, and $n \neq m$, thus by identity (15), the integral (20) is equal to zero.

If n and m are both even, and $m < n$ (and thus the maximum value of m is $n-2$) then by using (12), we have:

$$\begin{aligned}\int_{-1}^{1} R_n(x)R_m(x)w_2(x)\mathrm{d}x &= \frac{9}{4}\int_{-1}^{1} U_n(x)U_m(x)w_2(x)\mathrm{d}x + \frac{3}{4}\int_{-1}^{1} U_n(x)U_{m-2}(x)w_2(x)\mathrm{d}x\\ &\quad + \frac{3}{4}\int_{-1}^{1} U_m(x)U_{n-2}(x)w_2(x)\mathrm{d}x + \frac{1}{4}\int_{-1}^{1} U_{n-2}(x)U_{m-2}(x)w_2(x)\mathrm{d}x.\end{aligned}$$

Now, using identity (5), we have that:

$$\begin{aligned}\int_{-1}^{1} U_n(x)U_m(x)w_2(x)\mathrm{d}x &= 4\sum_{\substack{l=2\\ l\text{ even}}}^{2n-2}\sum_{\substack{j=2\\ j\text{ even}}}^{l}\int_{-1}^{1} T_j(x)T_{l-j}(x)w_2(x)\mathrm{d}x\\ &\quad - 2\sum_{\substack{j=2\\ j\text{ even}}}^{n}\int_{-1}^{1} T_j(x)T_0(x)w_2(x)\mathrm{d}x - 2\sum_{\substack{k=2\\ k\text{ even}}}^{n-2}\int_{-1}^{1} T_k(x)T_0(x)w_2(x)\mathrm{d}x\\ &\quad + \int_{-1}^{1} T_0(x)T_0(x)w_2(x)\mathrm{d}x.\end{aligned}$$

There are $n-1$ even numbers between 1 and $2n-2$, of which $(n-2)/2$ are divisible by 4. We only get a contribution from the first integral on the right hand side of the equality sign when $j = l - j$ or $l = 2j$, that is, l must me a multiple of 4. Therefore by identity (15), we have

$$\int_{-1}^{1} U_n(x)U_m(x)w_2(x)\mathrm{d}x = 4\left(\frac{n-2}{2}\right)\frac{\pi}{2} - 0 - 0 + \pi = (n-1)\pi. \tag{21}$$

Similarly,

$$\begin{aligned}\int_{-1}^{1} U_n(x)U_{m-2}(x)w_2(x)\mathrm{d}x = 4\sum_{\substack{l=2\\ l \text{ even}}}^{2n-4}\sum_{\substack{j=2\\ j \text{ even}}}^{l}\int_{-1}^{1} T_j(x)T_{l-j}(x)w_2(x)\mathrm{d}x\\ - 2\sum_{\substack{j=2\\ j \text{ even}}}^{n}\int_{-1}^{1} T_j(x)T_0(x)w_2(x)\mathrm{d}x - 2\sum_{\substack{k=2\\ k \text{ even}}}^{n-4}\int_{-1}^{1} T_k(x)T_0(x)w_2(x)\mathrm{d}x\\ + \int_{-1}^{1} T_0(x)T_0(x)w_2(x)\mathrm{d}x.\end{aligned}$$

There are $(n-2)$ even numbers between 2 and $2n-4$ of which $(n-4)/2$ are divisible by 4, so we obtain

$$\int_{-1}^{1} U_n(x)U_{m-2}(x)w_2(x)\mathrm{d}x = 4\left(\frac{n-4}{2}\right)\frac{\pi}{2} - 0 - 0 + \pi = (n-3)\pi. \tag{22}$$

By similar argument, we have that

$$\int_{-1}^{1} U_m(x)U_{n-2}(x)w_2(x)\mathrm{d}x = (n-1)\pi, \tag{23}$$

and

$$\int_{-1}^{1} U_{n-2}(x)U_{m-2}(x)w_2(x)\mathrm{d}x = (n-3)\pi. \tag{24}$$

Substituting (21), (22), (23) and (24) in (17), we obtain

$$\begin{aligned}\int_{-1}^{1} R_n(x)R_m(x)w_2(x)\mathrm{d}x &= \frac{9}{4}(n-1)\pi + \frac{3}{4}(n-3)\pi + \frac{3}{4}(n-1)\pi + \frac{1}{4}(n-3)\pi\\ &= 2(2n-3)\pi.\end{aligned}$$

If n and m are odd, and $m < n$, then from the integral (18), we have

$$4\int_{-1}^{1} U_n(x)U_m(x)w_2(x)\mathrm{d}x = 16\sum_{\substack{l=2\\ l \text{ even}}}^{2n-2}\sum_{\substack{j=1\\ j \text{ odd}}}^{l}\int_{-1}^{1} T_j(x)T_{l-j}(x)w_2(x)\mathrm{d}x.$$

We only get a contribution if $j = l - j$ or $l = 2j$, that is, l is divisible by 4. This is given as

$$4\int_{-1}^{1} U_n(x)U_m(x)w_2(x)\mathrm{d}x = 16\left(\frac{n-1}{2}\right)\frac{\pi}{2} = 4(n-1)\pi. \tag{25}$$

Similarly,

$$2\int_{-1}^{1} U_n(x)T_m(x)w_2(x)\mathrm{d}x = \sum_{\substack{j=1\\ j \text{ odd}}}^{n}\int_{-1}^{1} T_j(x)T_{n-2}(x)w_2(x)\mathrm{d}x = 4\left(\frac{n-1}{2}\right)\frac{\pi}{2} = (n-1)\pi. \tag{26}$$

Also,

$$2\int_{-1}^{1} U_m(x)T_n(x)w_2(x)\mathrm{d}x = 4\sum_{\substack{j=1\\ j \text{ odd}}}^{n-2}\int_{-1}^{1} T_j(x)T_n(x)w_2(x)\mathrm{d}x = 4\left(\frac{n-1}{2}\right)\frac{\pi}{2} = (n-1)\pi. \tag{27}$$

Since $m < n$, the last integral in (18) is zero.

Substituting (25), (26) and (27) in (18), we obtain

$$\int_{-1}^{1} R_n(x)R_m(x)w_2(x)\mathrm{d}x = 4(n-1)\pi - (n-1)\pi - (n-1)\pi = 2(n-1)\pi.$$

This completes the proof. □

4. Zeros and resultants of $R_n(x)$ polynomials

In this section, we describe the properties of the zeros of the $R_n(x)$ polynomials. These include the geometry of the zeros and identities for resultants involving $R_n(x)$ polynomials, $T_n(x)$ polynomials and $U_n(x)$ polynomials. First, we describe the geometry of the zeros.

From a well known theorem by Szego [10], the zeros of orthogonal polynomials are real and distinct, and are located in the interior of the interval of orthogonality. A similar theorem was given by Jordaan [5] in which he proved that a sequence of orthogonal polynomials with degree n has exactly n real simple zeros in the interval of orthogonality, and that the zeros of any two consecutive polynomials in the sequence also interlace with each other, i.e. if $P_n(x)$ and $P_{n+1}(x)$ are consecutive orthogonal polynomials in a sequence, then the zeros of $P_n(x)$ and $P_{n+1}(x)$ separate each other. Lucas [7] proved the interlacing property of the zeros of an orthogonal polynomials with the zeros of its derivative. Further, he proved that the zeros of the derivative of an orthogonal polynomial all lie within the convex hull of the zeros of the polynomial.

We thus have the following theorem:

Theorem 4.1. *The zeros of $R_n(x)$ are real, simple, and lie in the interval of orthogonality [-1,1]. The zeros of $R_n(x)$ also interlace with the zeros of $R_n'(x)$, in the interval [-1,1].*

Figure 2 illustrates the interlacing property of the $R_{10}(x)$ polynomial and $R_{10}'(x)$ polynomial. From the graph of $R_{10}(x)$ polynomial in blue and the $R_{10}'(x)$ polynomial in red, it can be seen that the zeros are in $[-1,1]$ and interlace in this interval.

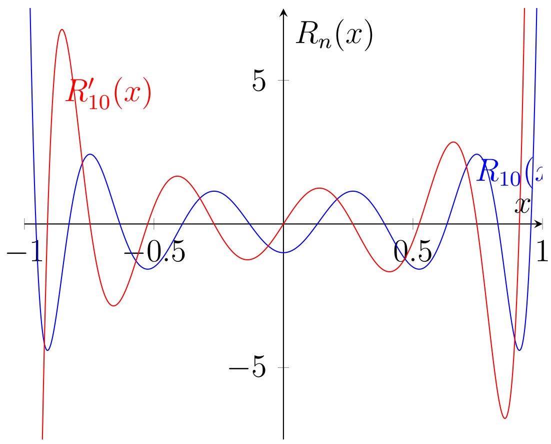

FIGURE 2. Graph of $R_{10}(x)$ and $R_{10}'(x)$ polynomial showing the interlacing property of the polynomial with its derivative.

Definition 4.2 ([4]). *Let polynomials $f(x)$ and $g(x)$ of degrees m and n respectively be expressed as a product of monomials involving their roots as:*

$$f(x) = a_m \prod_{k=1}^{m} (x - \alpha_k),$$

with a_m as the leading coefficient, and α_k the zero of $f(x)$, and

$$g(x) = b_n \prod_{j=1}^{n} (x - \beta_j),$$

with b_n as the leading coefficient, and β_j the zero of and $g(x)$, then the resultant is given by

$$res(f(x), g(x)) = a_m^n \prod_{k=1}^{m} g(\alpha_k) = b_n^m \prod_{j=1}^{n} f(\beta_j)$$

or

$$res(f(x), g(x)) = a_m^n b_n^m \prod_{k=1}^{m} \prod_{j=1}^{n} (\alpha_k - \beta_j).$$

In this section, we prove identities for resultants involving $R_n(x)$ polynomials. The following results, proved in [2–4, 9, 11], are necessary in this study.

(i) Consider the polynomials $f(x), g(x)$ and $h(x)$ such that the degrees of $f(x)$ and $g(x)$ are m and n respectively. Then,

$$res(f(x), g(x)) = (-1)^{mn} res(g(x), f(x)) \tag{28}$$

and

$$res(f(x), g(x)h(x)) = res(f(x), g(x)) res(f(x), h(x)). \tag{29}$$

(ii) If a is a constant and $f(x)$ is a polynomial of degree m, then

$$res(f(x), a) = res(a, f(x)) = a^m. \tag{30}$$

(iii) Let $f(x), g(x), q(x)$ and $r(x)$ be polynomials such that $f(x) = q(x)g(x) + r(x)$ and degrees of $f(x)$ and $r(x)$ are m and δ respectively, then

$$res(g(x), f(x)) = b_n^{m-\delta} res(g(x), r(x))$$

where b_n is the leading coefficient of $g(x)$.

(iv) Let $f(x)$ and $g(x)$ be polynomials such that $deg(q(x)f(x) + g(x)) = deg(g(x))$ for some polynomial $q(x)$, then

$$res(f(x), q(x)f(x) + g(x)) = res(f(x), g(x)). \tag{31}$$

(v) If m and n are odd integers, and $T_n(x)$ is the Chebyshev polynomial of the first kind, then

$$res(T_m(x), T_n(x)) = 0.$$

Generally, if $\gcd(m, n)$ is the greatest common divisor of m and n, then

$$res(T_m(x), T_n(x)) = \begin{cases} 0, & \text{if } nm \text{ is odd}, \\ (-1)^{\frac{mn}{2}} 2^{(m-1)(n-1)+\gcd(m,n)-1}, & \text{otherwise} \end{cases}$$

and for the Chebyshev polynomial of the second kind,

$$res(U_m(x), U_n(x)) = \begin{cases} 0, & \text{if } \gcd(m+1, n+1) \neq 1, \\ (-1)^{\frac{mn}{2}} 2^{mn}, & \text{otherwise.} \end{cases} \tag{32}$$

Since $\gcd(n, n+1) = 1$, then by (32), we have

$$res(U_n(x), U_{n-1}(x)) = (-1)^{\frac{n(n-1)}{2}} 2^{n(n-1)}. \tag{33}$$

By (31) and (33), we get

$$res(U_n(x) + kU_{n-1}(x), U_{n-1}(x)) = (-1)^{\frac{n(n-1)}{2}} 2^{n(n-1)}$$

where k is a real number.

The resultant of the classical Chebyshev polynomials is given as

$$res(T_n(x), U_m(x)) = \begin{cases} 2^{(m-1)(n-1)+\gcd(m,n)-1}, & \text{if } n \text{ and } m \text{ are both even,} \\ 0, & \text{otherwise.} \end{cases} \tag{34}$$

The following fundamental theorem of resultants is also necessary.

Theorem 4.3 ([11, Lemma 2.2])**.** *The resultant of f(x) and g(x) is equal to zero if and only if the two polynomials have a common root or a common divisor of positive degree.*

We now give the identities of resultants involving the $R_n(x)$ polynomials.

Theorem 4.4. *Let n be a positive integer, then*

$$res(R_n(x), T_n(x)) = \begin{cases} 2^{n^2}, & \text{if } n \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. If n is odd, then both $R_n(x)$ and $T_n(x)$ are odd functions with a common divisor (note that $x = 0$ is a zero for both $R_n(x)$ and $T_n(x)$) and by fundamental theorem of resultants,

$$res(R_n(x), T_n(x)) = 0.$$

Now, when n even and using the identity $R_n(x) = 2U_n(x) - T_n(x)$, we have that

$$res(R_n(x), T_n(x)) = res(2U_n(x) - T_n(x), T_n(x)) \tag{35}$$

Using (31), equation (35) becomes

$$res(R_n(x), T_n(x)) = res(2U_n(x), T_n(x))$$

Using property (29) of the resultants, we obtain

$$res(R_n(x), T_n(x)) = res(2, T_n(x))res(U_n(x), T_n(x)). \tag{36}$$

By property (30), we have

$$res(2, T_n(x)) = 2^n. \tag{37}$$

To evaluate $res(U_n(x), T_n(x))$, we use (34). Since the $\gcd(n, n) = n$, then this implies that

$$res(T_n(x), U_n(x)) = 2^{(n-1)(n-1)+n-1} = 2^{n(n-1)}. \tag{38}$$

Using property (28) of the resultants, we get

$$res(U_n(x), T_n(x)) = (-1)^{n^2} res(T_n(x), U_n(x)) = 2^{n(n-1)}, \tag{39}$$

since $(-1)^{n^2} = 1$, when n is even. Now, plugging (37) and (39) in (36), we obtain

$$res(R_n(x), T_n(x)) = 2^{n^2}.$$

This completes the proof. □

We remark that

$$res(R_n(x), T_n(x)) = res(T_n(x), R_n(x))$$

for all $n \geq 1$.

Theorem 4.5. *Let n be natural number, then*

$$res(R_n(x), U_{n-1}(x)) = (-1)^{\frac{n(n-1)}{2}} 2^{n(n-1)}.$$

Proof. Recall that

$$R_n(x) = U_n(x) + xU_{n-1}(x),$$

so that

$$res(R_n(x), U_{n-1}(x)) = res(xU_{n-1}(x) + U_n(x), U_{n-1}(x)).$$

Now, using property (31), we get

$$res(R_n(x), U_{n-1}(x)) = res(U_n(x), U_{n-1}(x)).$$

By property (33), we obtain

$$res(R_n(x), U_{n-1}(x)) = (-1)^{\frac{n(n-1)}{2}} 2^{n(n-1)}.$$

□

Theorem 4.6. *If n is a positive integer, then*

$$res(R_n(x), U_n(x)) = \begin{cases} 2^{n(n-1)}, & \text{if } n \text{ is even,} \\ 0, otherwise. \end{cases}$$

Proof. When n is odd, then $R_n(x)$ and $U_n(x)$ are both odd functions with a common factor x. The two polynomials therefore have a common zero, and by the fundamental theorem of resultants, we have

$$res(R_n(x), U_n(x)) = 0.$$

When n is even, we have from equation (13), that:

$$R_n(x) = 2U_n(x) - T_n(x).$$

So,

$$res(R_n(x), U_n(x)) = res(2U_n(x) - T_n(x), U_n(x)).$$

Using equation (31), we arrive at,

$$res(R_n(x), U_n(x)) = res(-T_n(x), U_n(x)).$$

By (29), we have

$$res(R_n(x), U_n(x)) = res(-1, U_n(x)) res(T_n(x), U_n(x)). \tag{40}$$

Using (30), we obtain

$$res(-1, U_n(x)) = (-1)^n \tag{41}$$

and from equation (38),

$$res(T_n(x), U_n(x)) = 2^{n(n-1)}. \tag{42}$$

Substituting equations (41) and (42) into (40), we obtain

$$res(R_n(x), U_n(x)) = (-1)^n \cdot 2^{n(n-1)}.$$

Since n is even, then $(-1)^n = 1$. We therefore have,

$$res(R_n(x), U_n(x)) = 2^{n(n-1)},$$

when n is even. □

5. Conclusion and further work

In this paper, we have introduced a new class of Chebyshev-like polynomials which we denoted by $R_n(x)$. We then used their generating functions, obtained in Section 2, to connect them to the classical Chebyshev polynomials. In Section 3, we showed that the new class of Chebyshev-like polynomials are orthogonal with respect to weight functions $\frac{1}{\sqrt{1-x^2}}$ and $\sqrt{1-x^2}$. We have also proved identities for the integrals of $R_n(x)$ polynomials in relation to the classical Chebyshev polynomials. Lastly, in Section 4, we showed that the zeros of $R_n(x)$ interlace with the zeros of its derivative, and lie in the interval $[-1, 1]$. Moreover, we have obtained expressions for the resultants of $R_n(x)$ polynomials in relation to the classical Chebyshev polynomials. It would be interesting to investigate discriminants of this class of Chebyshev-like polynomials as well as generalizations of this new class.

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