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RESULTS OF SEMIGROUP OF LINEAR OPERATORS GENERATED BY FRACTIONAL POWERS OF CLOSED OPERATORS

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Abstract. This paper presents the results of ω-order reversing partial contraction mapping generated by fractional powers of closed operators. We consider the fractional powers of closed operators by defining the fractional power of the negative of an infinitesimal generator of a C_0-semigroup. We obtained results of fractional powers generated by $-A$ and showed that the operator is linear, closed, and convergent. Furthermore, we established that the operator in injective, bounded in $\mathcal{L}(X)$ and $D(A^\alpha)$ is dense in X.

1. Introduction

Let us recall that for each $\alpha \in (0, +\infty)$, the improper integral

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt \tag{1}$$

is convergent, and its values are the so-called Euler Γ-function. Assume $a > 0$ and $\alpha > 0$. The change of variable $ta = \tau$ shows that

$$a^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-ta} dt. \tag{2}$$

Then if $A : D(A) \subseteq X \to X$ is a linear operator with $-A \in \omega - ORCP_n$, the infinitesimal generator of a semigroup of class $C_0\{T(t); t \geq 0\}$. It is quite natural to consider that, substituting a with A and e^{-ta} with $T(t)$ in the relation in (2), we obtain an operator $A^{-\alpha}$ and recall that $T(t)$ can be interpreted as $T(t) = e^{-tA}$. Clearly $D(A^{-\alpha})$ is a linear subspace in X and $A^{-\alpha}$ is a linear, closed operator. Moreover for each $0 \leq \alpha \leq \beta$, we have

$$\|T\|_{\mathcal{L}(X)} \leq Me^{-\omega t} \tag{3}$$

then for each $\alpha \geq 0$, $D(A^{-\alpha}) = X$. Indeed, this follows that

$$\int_0^\infty t^{\alpha-1} \|T(t)\|_{\mathcal{L}(X)} dt < +\infty. \tag{4}$$

In addition, since $\int_0^\infty t^{\alpha-1} T(t) dt$ is convergent in the uniform operator norm, it follows that for each $\alpha \geq 0$, $A^{-\alpha}$ is a linear bounded operator.

Assume X is a Banach space, $X_n \subseteq X$ is a finite set, $\omega - ORCP_n$ the ω-order reversing partial contraction mapping, M_m be a matrix, $\mathcal{L}(X)$ be a bounded linear operator on X, P_n

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a partial transformation semigroup, $\rho(A)$ a resolvent set, $\sigma(A)$ a spectrum of A. This paper consist of results of semigroup of linear operators generated by fractional powers of closed operators. In [1] and [2], Akinyele *et al.* established differentiable and analytical conclusions on ω-order preserving partial contraction mapping in semigroup of linear operator. They also made a description of ω-order reversing partial contraction mapping as a compact semigroup of linear operator. An operator calculus for infinitesimal semigroup generators was showed by Balakrishnan [3]. Banach [4] introduced and first proposed the idea of Banach spaces. The nonlinear Schrödinger evolution equation was established by Brezis and Gallouet [5]. A resolvent method to the stability operator semigroup was generated by Chill and Tomilov [6]. Davies [7] discovered the spectrum of linear operators. For equations of linear evolution, Engel and Nagel presented the one-parameter semigroup in their paper [8]. Omosowon *et al.* [9] produced some analytical results of semigroup of linear operator with dynamic boundary conditions, as well as introducing dual properties of ω-order reversing partial contraction mapping in semigroup of linear operator in [10]. In their study, Omosowon *et al.* [11] derived the outcomes of a semigroup of linear equations that produced a wave equation. Rauf and Akinyele [12] created ω-order preserving partial contraction mapping and acquired its qualities. Also in [13], Rauf *et al.* established some results of stability and spectra properties on semigroup of linear operator. Vrabie [14] demonstrated a few applications of the C_0-semigroup's findings. Yosida [15] derived several conclusions on the differentiability and representation of a linear operator one-parameter semigroup.

2. Preliminaries

Definition 2.1 (C_0-Semigroup) [14]
A C_0-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.
Definition 2.2 (ω-$ORCP_n$) [12]
A transformation $\alpha \in P_n$ is called ω-order preserving partial contraction mapping if $\forall x, y \in \mathrm{Dom}\alpha$: $x \leq y \implies \alpha x \geq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.
Definition 2.3 (closed Operator) [3]
A closed operator is an operator$A$ such that if $\{x_n\} \subset D(A)$ converges to $x \in X$ and $\{Ax_n\}$ converges to $y \in X$, then $x \in D(A)$ and $Ax = y$.

Definition 2.4 (Analytic Semigroup) [14]
We say that the C_0-semigroup $\{T(t);\ t \geq 0\}$ is analytic if there exists $0 < \varphi \leq \pi$, and a mapping $\widetilde{T} : \overline{\mathbb{C}}_\varphi \to \mathcal{L}(X)$ such that:

(i) $T(t) = \widetilde{T}(t)$ for each $t \geq 0$;
(ii) $\widetilde{T}(z+w) = \widetilde{T}(z)\widetilde{T}(w)$ for each $z, w \in \overline{\mathbb{C}}_\varphi$;
(iii) $\lim_{z \in \overline{\mathbb{C}}_\varphi, z \to 0} \widetilde{T}(z)x = x$ for each $x \in X$; and
(iv) the mapping $z \mapsto \widetilde{T}(z)$ is analytic from $\mathbb{C}_\varphi$ to $\mathcal{L}(X)$. If in addition for each $0 < \delta < \varphi$, the mapping $z \mapsto \widetilde{T}(z)$ is bounded from $\mathbb{C}_\delta$ to $\mathcal{L}(X)$, then the C_0-semigroup $\{T(t); t \geq 0\}$ is called analytic and uniformly bounded.

Definition 2.5 (Power of Exponent) [14]
The operator $A^{-\alpha} : D(A^{-\alpha}) \subseteq X \to X$ defined by

$$A_x^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t) x dt \tag{5}$$

for each $x \in D(A^{-\alpha})$, where

$$D(A^{-\alpha}) = \left\{ x \in X; \int_0^\infty t^{\alpha-1} T(t) x dt \text{ is convergent} \right\},$$

is called the power of exponent $-\alpha$ of the operator A. By definition, $D(A^0) = X$ and $A^0 = I$.
Definition 2.6 [14]
Let $A : D(A) \subseteq X \to X$ be a linear operator, with $-A$ the infinitesimal genrator of a C_0-semigroup of type $(M, -\omega)$, with $M \geq 1$ and $\omega > 0$ and let $\alpha > 0$. Then we have

$$A^\alpha = (A^{-\alpha})^{-1}. \tag{6}$$

Example 1
For every 2×2 matrix in $[M_m(\mathbb{R}^n)]$.
Suppose

$$A = \begin{pmatrix} 2 & 0 \\ \Delta & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then we have

$$e^{tA} = \begin{pmatrix} e^{2t} & I \\ e^{\Delta t} & e^{2t} \end{pmatrix}.$$

Example 2
For every 3×3 matrix in $[M_m(\mathbb{C})]$, we have
for each $\lambda > 0$ such that $\lambda \in \rho(A)$ where $\rho(A)$ is a resolvent set on X.
Suppose we have

$$A = \begin{pmatrix} 2 & 2 & I \\ 2 & 2 & 2 \\ \Delta & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA_\lambda}$, then we have

$$e^{tA_\lambda} = \begin{pmatrix} e^{2t\lambda} & e^{2t\lambda} & I \\ e^{2t\lambda} & e^{2t\lambda} & e^{2t\lambda} \\ e^{\Delta t\lambda} & e^{2t\lambda} & e^{2t\lambda} \end{pmatrix}.$$

Example 3
Let $X = C_{ub}(\mathbb{N} \cup \{0\})$ be the space of all bounded and uniformly continuous function from $\mathbb{N} \cup \{0\}$ to $\mathbb{R}$, endowed with the sup-norm $\| \cdot \|_\infty$ and let $\{T(t); t \in \mathbb{R}_+\} \subseteq L(X)$ be defined by

$$[T(t)f](s) = f(t+s)$$

For each $f \in X$ and each $t, s \in \mathbb{R}_+$, one may easily verify that $\{T(t); t \in \mathbb{R}_+\}$ satisfies Examples 1 and 2 above.

Example 4 Let H be a real Hilbert space, whose inner product is denoted by $\langle \cdot, \cdot \rangle$ and let $A : D(A) \subseteq H \to H$ where $A \in \omega - ORCP_n$ such that $-A$ generates a C_0-semigroup of contractions on H. Assume that A is self-adjoint and invertible with compact inverse A^{-1}.

then by a theorem of Hilbert, there exists a sequence of positive numbers $\mu_k > 0$, $\mu_{k+1} \leq \mu_k$ for $k \in \mathbb{N}$ and an orthonormal basis $\{e_k; k \in \mathbb{N}\}$ of H such that

$$A^{-1}e_k = \mu_k e_k \tag{7}$$

for $k \in \mathbb{N}$ and $A \in \omega - ORCP_n$. Let $\lambda_k = \mu_k^{-1}$, and observe that $e_k \in D(A)$ and

$$Ae_k = \lambda_k e_k$$

for $k \in \mathbb{N}$ and $A \in \omega - ORCP_n$.

We also have that $\lim_{k\to\infty} \lambda_k = +\infty$. Assume $\alpha > 0$ and defined $A_\alpha : D(A\alpha) \subseteq H \to H$ by

$$D(A_\alpha) = \left\{ u \in H; u = \sum_{k=1}^{\infty} u_k e_k,\ \sum_{k=1}^{\infty} \lambda_k^{2\alpha} |u_k|^2 < +\infty \right\}$$

$$A_\alpha u = \sum_{k=1}^{\infty} \lambda_k^{\alpha} u_k e \quad for \quad u \in D(A_\alpha) \quad and \quad A_\alpha \in \omega - ORCP_n.$$

A simple calculation shows that $A_\alpha = A^\alpha$. We notice that $D(A^\alpha)$, endowed with the natural inner product $\langle \cdot, \cdot \rangle_\alpha : D(A^\alpha) \times D(A^\alpha) \to \mathbb{R}$, defined by

$$\langle u, v \rangle_\alpha = \sum_{k=1}^{\infty} \lambda_k^{2\alpha} \langle u_k, v_k \rangle$$

for each $u, v \in D(A^\alpha)$, $u = \sum_{k=1}^{\infty} u_k e_k$, $v = \sum_{k=1}^{\infty} v_k e_k$ and $A \in \omega - ORCP_n$ is a real Hilbert space. In addition, with respect to this inner product, the family $\{\lambda_k^{-\alpha} e_k; k \in \mathbb{N}\}$ is an orthonormal basis in $D(A^\alpha)$.

Theorem 2.1 [Hille][15]

Let $A : D(A) \subseteq X \to Y$ be a linear closed operator and let $x : \Omega \to D(A)$. Then

$$\int_\Omega Ax(w) d\mu(w) = A \int_\Omega x(w) d\mu(w)$$

wherever both sides of the above equality are well-defined.

3. Main Results

This section present results of analytic semigroup operators generated by fractional powers of closed operators using ω-$ORCP_n$:

Theorem 3.1

Suppose the C_0-semigroup $\{T(t); t \geq 0\}$, generated by $-A$ is of type $(M, -\omega)$, with $M \geq 1$, $\omega > 0$ and $A \in \omega - ORCP_n$, then for each $\alpha, \beta \in [0, +\infty)$, we have

$$A^{-(\alpha+\beta)} = A^{-\alpha} A^{-\beta}. \tag{8}$$

Proof:
Let us observe that

$$
\begin{aligned}
A^{-\alpha}A^{-\beta} &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty\int_0^\infty t^{\alpha-1}s^{\beta-1}T(t)T(s)dtds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\beta t^{\alpha-1}\int_0^\infty (u-t)^{\beta-1}dtT(u)du \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty \left(\int_0^u t^{\alpha-1}(u-t)^{\beta-1}dt\right)T(u)du \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 v^{\alpha-1}(1-v)^{\beta-1}dv\int_0^\infty u^{\alpha+\beta-1}T(u)du. \qquad (9)
\end{aligned}
$$

Since

$$
\int_0^1 v^{\alpha-1}(1-v)^{\beta-1}dv = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad (10)
$$

From (10), we deduce

$$
A^{-\alpha}A^{-\beta} = \frac{1}{\Gamma(\alpha+\beta)}\int_0^\infty u^{\alpha+\beta-1}T(u)du = A^{-(\alpha+\beta)}
$$

and this achieves the proof.

Theorem 3.2
Assume the C_0-semigroup $\{T(t); t \geq 0\}$, generated by $-A$, is of type $(M, -\omega)$, with $M \geq 1$, $A \in \omega - ORCP_n$ and $\omega > 0$, then for each $\alpha \in (0,1)$, we have

$$
A^{-\alpha} = \frac{\sin \pi\alpha}{\pi}\int_0^\infty \lambda^{-\alpha}(\lambda I + A)^{-1}d\lambda. \qquad (11)
$$

Proof:
Since $-A$ is the infinitesimal generator of a C_0-semigroup of type (M, ω), then we have that operator A is a linear operator satisfying both $(0, +\infty) \subseteq e(A)$ and

$$
\|\lambda^n R(\lambda; A)^n\|_{\mathcal{L}(X)} \leq M \qquad (12)
$$

for each $n \in \mathbb{N}$, $\lambda > 0$ and $A \in \omega - ORCP_n$. Then there exists a norm $|\cdot|$ on X such that

$$
\|x\| \leq |x| \leq M\|x\| \qquad (13)
$$

and

$$
|\lambda R(\lambda; A)x| \leq |x| \qquad (14)
$$

for each $x \in X$, $\lambda > 0$ and $A \in \omega - ORCP_n$. Then we have

$$
\|(\lambda I + A)^{-1}\| \leq \frac{M}{\lambda + \omega}
$$

for $\lambda > -\omega$, and therefore the integral on the right-hand side in (11) is convergent. More so, as

$$
(\lambda I + A)^{-1} = \int_0^\infty e^{-\lambda t}T(t)dt,
$$

we have

$$\begin{aligned}\frac{\sin\pi\alpha}{\pi}\int_0^\infty \lambda^{-\alpha}(\lambda I + A)^{-1}d\lambda &= \frac{\sin\pi\alpha}{\pi}\int_0^\infty \lambda^{-\alpha}e^{-\lambda t}d\lambda\int_0^\infty T(t)dt\\ &= \frac{\sin\pi\alpha}{\pi}\int_0^\infty v^{-\alpha}e^{-v}dv\int_0^\infty t^{\alpha-1}T(t)dt\\ &= \frac{\sin\pi\alpha}{\pi}\Gamma(\alpha)\Gamma(1-\alpha)A^{-\alpha}\end{aligned}$$

since

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin\pi\alpha}.$$

Hence the proof is completed.

Theorem 3.3
Assume the C_0-semigroup $\{T(t); t \geq 0\}$, generated by $-A$, is of type $(M, -\omega)$, with $M \geq 1$, $\omega > 0$ and $A \in \omega - ORCP_n$, then there exists $C > 0$ such that for each $\alpha \in [0, 1]$, we have

$$\|A^{-\alpha}\|_{\mathcal{L}(X)} \leq C.$$

Proof:
It is sufficient to prove the inequality only for $a \in (0, 1)$. From (9), we have

$$\begin{aligned}\|A^{-\alpha}\|_{\mathcal{L}(X)} \leq &\left\|\frac{\sin\pi\alpha}{\pi}\int_0^1 \lambda^{-\alpha}(\lambda I + A)^{-1}d\lambda\right\|_{\mathcal{L}(X)}\\ &+\left\|\frac{\sin\pi\alpha}{\pi}\int_1^\infty \lambda^{-\alpha}(\lambda I + A)^{-1}d\lambda\right\|_{\mathcal{L}(X)}.\end{aligned} \tag{15}$$

Since

$$\|(\lambda I + A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{M}{\lambda + \omega} \tag{16}$$

with $\omega > 0$, there exits $C_0 > 0$ and $C_1 > 0$ such that

$$\|(\lambda I + A)^{-1}_{\mathcal{L}(X)}\| \leq C_0$$

for each $\lambda \in [0, 1]$, $A \in \omega - ORCP_n$ and

$$\|\lambda(\lambda I + A)^{-1}\|_{\mathcal{L}(X)} \leq C_1$$

for each $\lambda \geq 1$ and $A \in \omega - ORCP_n$. Therefore, we have

$$\|A^{-\alpha}\|_{\mathcal{L}(X)} \leq C_0\left|\frac{\sin\pi(1-\alpha)}{\pi(1-\alpha)}\right| + C_1\left|\frac{\sin\alpha}{\pi\alpha}\right| \leq C,$$

where $C = C_0 + C_1$. Hence the proof is completed.

Theorem 3.4
If the C_0-semigroup $\{T(t); t \geq 0\}$, generated by $-A$, is of type $(M, -\omega)$, with $M \geq 1$, $\omega > 0$ and $A \in \omega - ORCP_n$, then for each $x \in X$, we have

$$\lim_{\alpha\to 0} A^{-\alpha}x = x. \tag{17}$$

Proof:

Assume $x \in D(A)$. Since $(-\omega, +\infty) \subset \rho(-A)$, we have $0 \in \rho(A)$ and therefore there exists $y \in X$, $A \in \omega - ORCP_n$ with $x = A^{-1}y$. We then have

$$A^{-\alpha}x - x = A^{-(\alpha+1)}y - A^{-1}y = \int_0^{\infty} \left(\frac{t^{\alpha}}{\Gamma(1+\alpha)} - 1 \right) T(t) y dt.$$

From the growth condition $\|T(t)\|_{\mathcal{L}(X)} \le Me^{-\omega t}$, we deduce

$$\|A^{-\alpha}x - x\| \le M\|y\| \int_0^{\infty} \left| \frac{t^{\alpha}}{\Gamma(1+\alpha)} - 1 \right| e^{-\omega t} dt. \tag{18}$$

On the other hand, there exists $C > 0$, such that for each $\alpha \in [0,1]$ and each $t \ge 1$, we have

$$\left| \frac{t^{\alpha}}{\Gamma(1+\alpha)} - 1 \right| \le Ct.$$

It then follows that for each $k \ge 1$, we have

$$\|A^{-\alpha}x - x\| \le M\|y\| \int_0^{k} \left| \frac{t^{\alpha}}{\Gamma(1+\alpha)} - 1 \right| e^{-\omega t} dt + CM\|y\| \int_0^{\infty} te^{-\omega t} dt. \tag{19}$$

Let $\varepsilon > 0$. Let us fix a sufficiently large $k \ge 1$, such that the second term on the right-hand side in (19) be less than $\varepsilon/2$. We observe that there exits $\delta(\varepsilon) > 0$ such that for each $\alpha \in (0, \delta(\varepsilon))$, then the first term on the right-hand side in (19) is less that $\varepsilon/2$. Then for each $x \in D(A)$ and $A \in \omega - ORCP_n$, we have

$$\lim_{\alpha \to 0} A^{-\alpha}\alpha = x.$$

Since $D(A)$ is dense in X and by virtue of Theorem 3.3, the family of operators $\{A^{-\alpha}; \alpha \in [0,1]\}$ is bounded in $\mathcal{L}(X)$, and this achieved the proof.

Theorem 3.5

Suppose $A : D(A) \subseteq X \to X$, where A is the infinitesimal generator of a C_0-semigroup of type $(M, -\omega)$, with $M \ge 1$, $\omega > 0$, $A \in \omega - ORCP_n$ and let $\alpha \in (0,1)$. Then there exists $C > 0$ such that for each $x \in D(A)$ and each $\rho > 0$ we have

$$\|A^{\alpha}x\| \le C(\rho^{\alpha}\|x\| + \rho^{\alpha-1}\|Ax\|) \tag{20}$$

and

$$\|A^{\alpha}x\| \le 2C\|x\|^{1-\alpha}\|Ax\|^{\alpha}. \tag{21}$$

Proof:

By Theorem 3.4, we have that since

$$(\lambda I + A)^{-1}Ax = x - \lambda(\lambda I + A)^{-1}x,$$

we deduce

$$\begin{aligned}
\|A^{\alpha}x\| \le& \left|\frac{\sin \pi\alpha}{\pi}\right| \int_0^{\rho} \lambda^{\alpha-1} \|A(\lambda I + A)^{-1}\|_{\mathcal{L}(X)} \|x\| d\lambda \\
&+ \left|\frac{\sin \pi\alpha}{\pi}\right| \int_{\rho}^{\infty} \lambda^{\alpha-1} \|(\lambda I + A)^{-1}\|_{\mathcal{L}(X)} \|Ax\| d\lambda \\
\le& \left|\frac{\sin \pi\alpha}{\pi}\right| (1+M)\rho^{\alpha}\|x\| + \left|\frac{\sin \pi(1-\alpha)}{\pi(1-\alpha)}\right| M\rho^{\alpha-1}\|Ax\| \\
\le& C(\rho^{\alpha}\|x\| + \rho^{\alpha-1}\|Ax\|),
\end{aligned}$$

where

$$C = (1+M)_\alpha \sup_{\alpha\in(0,1)} \left\{ \left|\frac{\sin\pi\alpha}{\pi}\right| + \left|\frac{\sin\pi(1-\alpha)}{\pi(1-\alpha)}\right| \right\}$$

so that (20) holds. Since (21) is obviously true for $x = 0$ and follows from (20), putting $\rho = \|Ax\|/\|x\|$, for $x \neq 0$, and this achieved the proof.

Conclusion

It has been demonstrated in this study that results of semigroup of linear operators generated by fractional powers of closed operators by using partial contraction mapping with ω-order reversing has been established.

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