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BOOTSTRAP CONFIDENCE INTERVAL FOR FRACTIONAL DIFFUSIONS AND AMERICAN OPTIONS

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ABSTRACT. The paper obtains bootstrap confidence interval for the drift parameter in fractional diffusion processes. It also obtains bootstrap stochastic gradient descent algorithm for American option. It connects maximum likelihood estimation with pricing American options.

1. Introduction and Preliminaries

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a stochastic basis. Let $\{W_t^H\}_{t\geq 0}$ be a persistent (Hurst parameter $H > 0.5$) fractional Brownian motion with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Recall that a fractional Brownian motion (fBM) has the covariance

$$\widetilde{C}_H(s,t) = \frac{1}{2}\left[s^{2H} + t^{2H} - |s-t|^{2H}\right], \quad s,t > 0. \tag{1.1}$$

For $H > 0.5$ the process has long range dependence or long memory. For $H \neq 0.5$, the process is neither a Markov process nor a semimartingale. For $H = 0.5$, the process reduces to standard Brownian motion.

Recently Ichiba *et al.* [17, 18] studied generalized fractional Brownian motion (GFBM). A generalized fractional Brownian motion is a Gaussian self-similar process whose increments are not necessarily stationary. It appears in the scaling limit of a shot-noise process with a power law shape function and non-stationary noises with a power law variance function. They studied semimartingale properties of the mixed process made up of an independent Brownian motion and a GFBM for the persistent Hurst parameter.

Define

$$\begin{aligned} \kappa_H &:= 2H\Gamma(3/2-H)\Gamma(H+1/2), \quad k_H(t,s) := \kappa_H^{-1}(s(t-s))^{\frac{1}{2}-H} \\ \lambda_H &:= \frac{2H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(3/2-H)}, \quad v_t \equiv v_t^H := \lambda_H^{-1} t^{2-2H}, \quad M_t^H := \int_0^t k_H(t,s)dW_s^H. \end{aligned} \tag{1.2}$$

Davydov (1970) obtained an AR(1) approximation of the fractional Brownian motion:

$$y_j = \rho y_{j-1} + v_j, (1-L)^{H-1/2} v_j = \epsilon_j, \quad \rho = 1, \ y_0 = 0, \ j = 1, 2, \ldots, n \tag{1.3}$$

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where L is the lag operator, $\epsilon_j \sim$ i.i.d. $(0, \sigma^2)$ with $E(\epsilon_j^4) < \infty$, whereas $\{v_j\}$ is a stationary long-memory process generated by

$$v_j = (1-L)^{H-1/2}\epsilon_j = \sum_{k=0}^{\infty} \frac{\Gamma(k+H-1/2)}{\Gamma(H-1/2)\Gamma(k+1)}\epsilon_{j-k}. \tag{1.4}$$

Davydov (1970) proved that

$$\frac{\lambda_H^{1/2}}{\sigma n^H} y_{[nt]} \to^{\mathcal{D}} W_t^H \quad \text{as} \quad n \to \infty. \tag{1.5}$$

On the stochastic basis the fractional Ornstein-Uhlenbeck process X_t is defined and satisfying the Itô stochastic differential equation

$$dX_t = \theta X_t dt + dW_t^H, \quad t \geq 0, \quad X_0 = \xi \tag{1.6}$$

where $\{W_t^H\}$ is a fractional Brownian motion with $H > 1/2$ with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and $\theta < 0$ is the unknown parameter.

We have the following martingale approximation to the nonsemimartingale fBm: From Norros *et al.* [21] it is well known that M_t^H is a Gaussian martingale, called the *fundamental martingale* whose variance function $\langle M^H\rangle_t$ is v_t^H. The natural filtration of the martingale M^H coincides with the natural filtration of the fBm W^H since

$$W_t^H := \int_0^t K_H(t,s) dM_s^H \tag{1.7}$$

holds for $H \in (1/2, 1)$ where $K_H(t,s) := H(2H-1)\int_s^t r^{h-\frac{1}{2}}(r-s)^{H-\frac{3}{2}}, \quad 0 \leq s \leq t$ and for $H = 1/2$, the convention $K_{1/2} \equiv 1$ is used. Observe that the increments of M_t^H are independent of W_t^H and

$$\text{Cov}(M_t^H, M_t^H) = \lambda_H^{-1/2}(s \wedge t)^{2-2H}, \quad \text{Cov}(W_t^H, M_t^H) = \lambda_H^{1/2}, \tag{1.8}$$

$$M_t^H =^{\mathcal{D}} b_H \int_0^t u^{1/2-H} dW_u, \quad b_H = \sqrt{2-2H}\lambda_H. \tag{1.9}$$

Define

$$G_t := \frac{d}{dv_t}\int_0^t k_H(t,s) X_s ds. \tag{1.10}$$

It is easy to see that

$$G_t = \frac{\lambda_H}{2(2-2H)}\left\{t^{2H-1}Z_t + \int_0^t r^{2H-1} dZ_s\right\}. \tag{1.11}$$

Define the process $Z = (Z_t, t \in [0,T])$ by $Z_t := \int_0^t k_H(t,s) dX_s$. The process X admits the representation

$$X_t = \int_0^t K_H(t,s) dZ_s. \tag{1.12}$$

The natural filtration generated by the fundamental semimartingale process

$$Z_t = \theta \int_0^t G_s dv_s + M_t^H \tag{1.13}$$

and the process X coincide, see Kleptsyna and Le Breton [19]. The available information for X and Z are strictly equivalent.

Let the realization $\{X_t, 0 \leq t \leq T\}$ or equivalently $\{Z_t, 0 \leq t \leq T\}$ be denoted by Z_0^T. Let P_θ^T be the measure generated on the space (C_T, B_T) of continuous functions on $[0,T]$ with the associated Borel σ-algebra B_T generated under the supremum norm by the process X_0^T and

P_0^T be the standard Wiener measure. Applying fractional Girsanov formula, when θ is the true value of the parameter, P_θ^T is absolutely continuous with respect to P_0^T and the Radon-Nikodym derivative (likelihood) of P_θ^T with respect to P_0^T based on Z_0^T is given by

$$L_T(\theta) := \frac{dP_\theta^T}{dP_0^T}(Z_0^T) = \exp\left\{\theta \int_0^T G_t dZ_t - \frac{\theta^2}{2}\int_0^T G_t^2 dv_t\right\}. \tag{1.14}$$

Consider the score function, the derivative of the log-likelihood function, which is given by

$$l_T(\theta) := \int_0^T G_t dZ_t - \theta \int_0^T G_t^2 dv_t. \tag{1.15}$$

2. Drift Estimation

Now consider the fractional SDE

$$dX_t = f(\theta, X_t, t)dt + dW_t^H, t \in [0,T] \tag{2.1}$$

where W^H is the fractional Brownian motion with Hurst parameter $H > 0.5$.

Then by Proposition 5.7 of Buchmann and Kluppelberg [8], we have $X_t = g(Y_t)$ where

$$dY_t = \theta Y_t dt + dW_t^H, \quad Y_0 = g^{-1}(X_0), \quad t \in [0,T] \tag{2.2}$$

and g is the state space transform (SST).

Let $\widetilde{Z}$ is the fundamental semimartingale associated with the process X. Let the collection of continuous time martingales $\{G(\theta,t), \mathcal{G}_t, t \geq 0\}_{\theta\in\mathbb{R}}$ where for each (θ,t), $G(\theta,t) = \int_0^t f(\theta, \widetilde{Z}_s, s)dW_s$ is an Itô integral whose corresponding increasing process is $\langle G(\theta,t)\rangle_t = \int_0^t f^2(\theta, \widetilde{Z}_s, s)ds$.

Recall that by Girsanov theorem, the likelihood function of θ based on the observations $\{X_s, 0 \leq s \leq t\}$ is given by

$$L_t(\theta) = \exp\left\{\int_0^t f(\theta, X_s, s)dX_s - H(2H-1)\int_0^t f^2(\theta, X_s, s)\left(\int_0^s (s-r)^{2H-2}dr\right)ds\right\}. \tag{2.3}$$

Let

$$l_t(\theta) = \log L_t(\theta). \tag{2.4}$$

The maximum likelihood estimator (MLE) is defined as

$$\theta_t := \arg\sup_{\theta\in\mathbb{R}} l_t(\theta), \text{ i.e., } l_t(\theta_t) = \sup_{\theta\in\mathbb{R}} l_t(\theta).$$

Let the pivot be defined by

$$I_t = \int_0^t f_\theta^2(\theta_0, X_s, s)ds. \tag{2.5}$$

Strong consistency and asymptotic normality of the MLE in the standard Brownian diffusion was studied in Levanony *et al.* [20]. See also Bishwal [4,5].

Due to the fundamental semimartingale representation $\widetilde{Z}$ of fractional diffusions along with state-space transform, main tools are Taylor expansion of the derivative of the log-likelihood $U_t(\theta)$ along with martingale SLLN and martingale CLT and delta method (see Bishwal [6] for details), we obtain the strong consistency and asymptotic normality of the MLE for the fractional diffusion:

Theorem 2.1

$$\text{a)} \quad \theta_t \to \theta_0 \quad \text{a.s.} \quad \text{as } t \to \infty,$$

$$\text{b)} \quad I_t^{1/2}(\theta_t - \theta_0) \to^{\mathcal{D}} \mathcal{N}(0,\ 1) \ \ \text{as } t \to \infty.$$

Redefine the MLE as

$$\theta_t = \lim_{n\to\infty} \inf \arg \max_{|\theta|\le n} l_t(\theta), \ \text{i.e.,} \ \ l_t(\theta_t) = \sup_{\theta\in\mathbb{R}} l_t(\theta).$$

An $\mathcal{F}_t$-adapted MLE exists. We derive the evolution equation for the trajectories of the MLE using the fractional Itô formula. Assume that our candidate for the MLE is a continuous Dirichlet process of the form

$$d\theta_t = a_t\ dt + b_t\ dX_t, \quad t \ge t_0. \tag{2.6}$$

The first derivative (with respect to θ) of the log-likelihood denoted as $U_t(\theta)$ is a continuous Dirichlet process. Also $U_t(\cdot) \in C^2$ for all $t \ge 0$ a.s. and together with its derivatives is jointly (θ, t) continuous. Hence by fractional Itô formula

$$\begin{aligned} dU_t(\theta_t) \ &= \ f_\theta(\theta_t, X_t, t)[dX_t - f(\theta_t, X_t, t)dt] + R_t(\theta_t)d\theta_t \\ &\quad + H(2H-1)Q_t(\theta_t)b_t^2 dt + f_{\theta\theta}(\theta_t, X_t, t)b_t dt, \quad t \ge t_0 \end{aligned} \tag{2.7}$$

where $R_t(\theta)$ and $Q_t(\theta)$ are the second derivative and the third derivative of the log-likelihood w.r.t. θ respectively. Assuming that $R_t(\theta_t) < 0$ for all $t \ge t_0$, the MLE which solves $U_t(\theta) = 0 \ \ \forall t > 0$, is a solution of the equation

$$\begin{aligned} d\theta_t \ &= \ -R_t^{-1}(\theta_t)\{f_\theta(\theta_t, X_t, t)[dX_t - f(\theta_t, X_t, t)dt] \\ &\quad + [H(2H-1)Q_t(\theta_t)b_t^2 + f_{\theta\theta}(\theta_t, X_t, t)b_t]dt\}, \quad t \ge t_0 \end{aligned} \tag{2.8}$$

which after equating with (2.6) yields the MLE equation

$$\begin{aligned} d\theta_t \ &= \ -R_t^{-1}(\theta_t)\{f_\theta(\theta_t, X_t, t)[dX_t - f(\theta_t, X_t, t)dt] \\ &\quad + [H(2H-1)Q_t(\theta_t)R_t^{-2}(\theta_t)f_\theta^2(\theta_t, X_t, t) \\ &\quad - R_t^{-1}(\theta_t)f_\theta(\theta_t, X_t, t)f_{\theta\theta}(\theta_t, X_t, t)]dt\} \end{aligned} \tag{2.9}$$

with initial conditions: $|\theta_{t_0}| < \infty, \ U_t(\theta_{t_0}) = 0, \ R_t(\theta_{t_0}) < 0.$

Newton-type Algorithm: Newton type algorithms are approximation of the MLE evolution equation (2.9). However, (2.9) is not suitable for recursive estimation, it is valid for large t, and moreover, it requires the knowledge of exact MLE at the initial time.

Newton type algorithms are insensitive to initial conditions and implementable for all $t_0 > 0$. The algorithm makes the estimator θ_t follow the gradient when $U \neq 0$ until it enters the neighborhood of a local maximum and then keeps θ_t in this neighborhood as long as possible, i.e., as long as singularity does not arise (where afterwards the process repeats itself). This switching policy is needed in order to maintain the necessary flexibility which prevents the estimator for being 'trapped' in a no solution situation (i.e, when $R = 0$ in (2.9)).

Fix $\alpha > 0$ and some small ϵ, δ, define the set

$$A(t) := \{\theta : |U_t(\theta)| \le \delta, R_t(\theta) \le -\epsilon\}. \tag{2.10}$$

A *simplified version of the Newton Algorithm* is

$$\begin{aligned} d\theta_t &= -R_t^{-1}(\theta_t)\{f_\theta(\theta_t, X_t, t)[dX_t - f(\theta_t, X_t, t)dt] \\ &\quad +[H(2H-1)Q_t(\theta_t)R_t^{-2}(\theta_t)f_\theta^2(\theta_t, X_t, t) \\ &\quad -R_t^{-1}(\theta_t)f_\theta(\theta_t, X_t, t)f_{\theta\theta}(\theta_t, X_t, t) \\ &\quad +\alpha \;\; U_t(\theta_t)]dt\}I_{\{\theta_0^t \in A(t)\}} + t^{-\nu}U_t(\theta_t)dtI_{\{\theta_0^t \notin A(t)\}} \end{aligned} \tag{2.11}$$

with initial condition $\theta_{t_0}, t_0 > 0$.

When $\theta_t \in A(t)$, the algorithm follows the likelihood equation (with a decay term), where as when $\theta_t \in A^c(t)$, it follows the gradient towards a local maximum. The main problem with (2.10) is the fact that this scheme could result in infinitely many switchings in the bounded time intervals (or even uncountably many switchings). This prevents (2.10) from being an implementable algorithm.

Choose continuous $0 < \delta_t \downarrow 0$ and $0 < \epsilon_t \downarrow 0$ where δ_t satisfies

$$\int_{t_0}^{\infty} \delta_t dt = \infty, \quad (s/t)^\nu < \delta_t/\delta_s \; \forall t_0 \le s < t. \tag{2.12}$$

For example $\delta = t^{-\beta}, 0 < \beta < 1 \wedge \nu$ will do.

Redefine the set $A(t)$,

$$A(t) := \{\theta : |U_t(\theta)| \le \delta_t t^\nu, R_t(\theta) \le -\epsilon_t\}. \tag{2.13}$$

Let $\mathcal{A}(t) := \{\phi_0^t \in C[0,t] : \exists\, s \le t \text{ such that } R_s(\phi_s) \le -2\epsilon_s \text{ and } \phi_r \in A(r) \forall r \in [s,t]\}\}.$ (2.14)

$\mathcal{A}(t)$ sets for R the 'entrance level' $-2\epsilon_t$ into $A(t)$ and 'exit level' $-\epsilon_t$ (into and from $A(t)$ respectively).

The changes in (2.12) are in the definition of good event and the normalizing of the second term. The proposed *algorithm* is given by

$$\begin{aligned} d\theta_t &= -R_t^{-1}(\theta_t)\{f_\theta(\theta_t, X_t, t)[dX_t - f(\theta_t, X_t, t)dt] \\ &\quad +[H(2H-1)Q_t(\theta_t)R_t^{-2}(\theta_t)f_\theta^2(\theta_t, X_t, t) \\ &\quad -R_t^{-1}(\theta_t)f_\theta(\theta_t, X_t, t)f_{\theta\theta}(\theta_t, X_t, t) + \alpha\; U_t(\theta_t)]dt\}\; I_{\{\theta_0^t \in \mathcal{A}(t)\}} \\ &\quad +t^{-\nu}U_t(\theta_t)dt\; I_{\{\theta_0^t \notin \mathcal{A}(t)\}} \end{aligned} \tag{2.15}$$

which holds in $[t_0, \tau)$ (where τ is the explosion time), with any initial condition $\theta_{t_0}, t_0 > 0$ (where $\theta_t = \theta_{t_0} \forall t \in [0, t_0]$).

3. Bootstrap Confidence Interval

The bootstrap belongs to the family of modern statistical techniques which exploit the Monte Carlo method in order to obtain precise estimators and powerful statistical tests for complex models. The EDF of X_i given by $\mathbb{P}_n = \frac{1}{n}\sum_{i=1}^n \delta_{X_i}$ converges to the CDF $\mathbb{P}$ as $n \to \infty$. Sequential Importance Sampling with Resampling (SISR) uses bootstrap. The resampling idea is to get rid in a principled way particles with small weight and multiply the particles with large weight.

Bootstrap confidence intervals was studied in Hall [15, 16] and DiCiccio and Efron [12]. Corradi and Swanson [9] studied bootstrap specification test for diffusion processes. Confidence interval estimation for SGDCT estimators has remained largely unexplored.

We have the system of fractional SDEs given by

$$dX_t = f(\theta_t, X_t, t)dt + dW_t^H, \tag{3.1}$$

$$\begin{aligned} d\theta_t \;\; = \;\; & -R_t^{-1}(\theta_t)\{f_\theta(\theta_t, X_t, t)[dX_t - f(\theta_t, X_t, t)dt] \\ & +[H(2H-1)Q_t(\theta_t)R_t^{-2}(\theta_t)f_\theta^2(\theta_t, X_t, t) \\ & -R_t^{-1}(\theta_t)f_\theta(\theta_t, X_t, t)f_{\theta\theta}(\theta_t, X_t, t) + \alpha\, U_t(\theta_t)]dt\}\; I_{\{\theta_0^t \in \mathcal{A}(t)\}} \\ & +t^{-\nu}U_t(\theta_t)dt\; I_{\{\theta_0^t \notin \mathcal{A}(t)\}}. \end{aligned} \tag{3.2}$$

Continuous Observation

We need the following proposition from Pardoux and Veretennikov [22] in the sequel.

Proposition 3.1 *Let $\mathcal{L}_x$ be the infinitesimal generator of the X process. Let $F(x,\theta) \in C^{\alpha,2}(\mathcal{X}, \mathbb{R}^n)$ which satisfies $\int_{\mathcal{X}} F(x,\theta)\pi(dx) = 0$. and for some positive constants M and q, and*

$$|F(x,\theta)| + \left|\frac{\partial}{\partial\theta}F(x,\theta)\right| + \left|\frac{\partial^2}{\partial\theta^2}F(x,\theta)\right| \leq M(1+|x|^q).$$

Then the Poisson equation $\mathcal{L}_x u(x,\theta) = F(x,\theta)$, $\int_{\mathcal{X}} u(x,\theta)\pi(dx) = 0$ has a unique solution that satisfies $u(x,\cdot) \in C^2$ for every $x \in \mathcal{X}, \partial_\theta^2 u \in C(\mathcal{X} \times \mathbb{R}^n)$ and there exist positive constants K and p such that

$$|u(x,\theta)| + \left|\frac{\partial}{\partial\theta}u(x,\theta)\right| + \left|\frac{\partial^2}{\partial\theta^2}u(x,\theta)\right| \leq K(1+|x|^p).$$

Let the parametric model be given by

$$dX_t = \theta b(X_t)dt + \sigma(X_t)dW_t^H,\;\; t \geq 0. \tag{3.3}$$

The following lemma follows from Theorem 7 in Yoshida [27].

Lemma 3.1 Let $M_t := \frac{1}{\sqrt{tI(\theta_0)}} \int_0^t f(\theta_0, X_s)dW_s$. Then

$$\sup_{x\in\mathbb{R}} |P_{\theta_0}\{M_t \leq x\} - \Phi(x)| \leq Ct^{-1/2}. \tag{3.4}$$

MLE is given by

$$\theta_t = \frac{\int_0^t \frac{b(X_s)}{\sigma^2(X_s)}dX_s}{\int_0^t \frac{b^2(X_s)}{\sigma^2(X_s)}ds}. \tag{3.5}$$

The empirical model is given by

$$dY_t = \theta_t b(Y_t)dt + \sigma(Y_t)dB_t^H \tag{3.6}$$

where B^H is a fractional Brownian motion independent of W^H and θ_t satisfies the SDE

$$\begin{aligned} d\theta_t \;\; = \;\; & -R_t^{-1}(\theta_t)\{g_\theta(\theta_t, Y_t, t)[dY_t - g(\theta_t, Y_t)dt] \\ & +[H(2H-1)Q_t(\theta_t)R_t^{-2}(\theta_t)g_\theta^2(\theta_t, Y_t) \\ & -R_t^{-1}(\theta_t)g_\theta(\theta_t, Y_t)g_{\theta\theta}(\theta_t, Y_t) + \alpha\, U_t(\theta_t)]dt\}\; I_{\{\theta_0^t \in \mathcal{A}(t)\}} \\ & +t^{-\nu}U_t(\theta_t)dt\; I_{\{\theta_0^t \notin \mathcal{A}(t)\}} \end{aligned} \tag{3.7}$$

where $g(\theta, y) := \theta y$.

The Bootstrap MLE is given by

$$\widehat{\theta}_t = \frac{\int_0^T \frac{b(Y_s)}{\sigma^2(Y_s)} dY_s}{\int_0^T \frac{b^2(Y_s)}{\sigma^2(Y_s)} ds} \tag{3.8}$$

and satisfies the SDE

$$\begin{aligned} d\widehat{\theta}_t \;=\; & -R_t^{-1}(\widehat{\theta}_t)\{g_\theta(\widehat{\theta}_t, Y_t, t)[dY_t - g(\theta_t, Y_t)dt] \\ & +[H(2H-1)Q_t(\widehat{\theta}_t)R_t^{-2}(\widehat{\theta}_t)g_\theta^2(\widehat{\theta}_t, Y_t) \\ & -R_t^{-1}(\widehat{\theta}_t)g_\theta(\widehat{\theta}_t, Y_t)g_{\theta\theta}(\widehat{\theta}_t, Y_t) + \alpha\ U_t(\theta_t)]dt\}\ I_{\{\widehat{\theta}_0^t \in \mathcal{A}(t)\}} \\ & +t^{-\nu}U_t(\theta_t)dt\ I_{\{\widehat{\theta}_0^t \notin \mathcal{A}(t)\}} \end{aligned} \tag{3.9}$$

where $g(\theta, y) := \theta y$.

Let

$$S_t := \sqrt{\mu\left(\frac{b^2}{\sigma^2}\right) t}\,(\theta_t - \theta_0)\,.$$

If the coefficients of the polynomial h were explicitly known the coverage level of the confidence interval could be easily corrected using the fact from Yoshida [27] that

$$P\left(S_t \le y - \frac{1}{\sqrt{t}} h(y)\right) = \Phi(y) + o\left(\frac{1}{\sqrt{t}}\right).$$

uniformly in $y \in I$, an interval in $\mathbb{R}$.

The confidence interval is given by

$$I_t(\alpha) := \left[\theta_t - \frac{q_\alpha}{\sqrt{\mu\left(\frac{b^2}{\sigma^2}\right) t}},\ \infty\right).$$

where q_α is the α-th quantile of the normal distribution. We have

$$P\left(\theta \in I_t(\alpha)\right) - \alpha = -\frac{h_\theta(q_\alpha)}{\sqrt{t}} + o\left(\frac{1}{\sqrt{t}}\right)$$

where

$$h_\theta(q_\alpha) := \frac{\varphi(q_\alpha)}{\sqrt{\mu\left(\frac{b^2}{\sigma^2}\right)}}\left(\frac{\mu\left(bF''_{\frac{b^2}{\sigma^2}}\right) h_2(q_\alpha)}{2\mu\left(\frac{b^2}{\sigma^2}\right)}\right) + \frac{\varphi(q_\alpha)}{\sqrt{\mu\left(\frac{b^2}{\sigma^2}\right)}}\left(\frac{\mu\left(bF''_{\frac{b^2}{\sigma^2}}\right)}{\mu\left(\frac{b^2}{\sigma^2}\right)}\right),$$

$$\mu\left(\frac{b^2}{\sigma^2}\right) := \int_0^\infty \frac{b^2(x)}{\sigma^2(x)} d\mu(x), \tag{3.10}$$

F is defined in Proposition 3.1, μ is the invariant measure of the diffusion and h_2 is second order Hermite polynomial, i.e., $h_2(y) = y^2 - 1$.

The α-th quantile of S_t is given by $\omega_\alpha(t) := \inf_{y\in\mathbb{R}} \left\{P\left(\sqrt{\mu\left(\frac{b^2}{\sigma^2}\right) t}\,(\theta_t - \theta_0) \le y\right) \ge \alpha\right\}$.

The empirical estimate of α-th quantile is given by

$\widetilde{\omega}_\alpha(t) := \inf_{y\in\mathbb{R}} \left\{P\left(\sqrt{\mu\left(\frac{b^2}{\sigma^2}\right) t}\left(\widehat{\theta}_t - \theta_t\right) \le y\right) \ge \alpha\right\}$.

The quantile $\widetilde{\omega}_\alpha(t)$ can be estimated by the Monte Carlo method.

The bootstrap confidence interval is given by

$$I_t^{boot}(\alpha) := \left[\theta_t - \frac{\widetilde{\omega}_\alpha(t)}{\sqrt{\mu\left(\frac{b^2}{\sigma^2}\right) t}},\ \infty\right). \tag{3.11}$$

Conditional on $\widehat{\theta}_t$ the quantile $\widetilde{\omega}$ can be expanded as

$$\widetilde{\omega}_\alpha(t) = q_\alpha + \frac{1}{\sqrt{t}} h_{\widehat{\theta}_t}(q_\alpha) + o\left(\frac{1}{\sqrt{t}}\right), \tag{3.12}$$

$$\omega_\alpha(t) = q_\alpha + \frac{1}{\sqrt{t}} h_\theta(q_\alpha) + o\left(\frac{1}{\sqrt{t}}\right).$$

Since

$$h_{\widehat{\theta}_t}(q_\alpha) - h_\theta(q_\alpha) \to 0 \text{ a.s. as } t \to \infty,$$

we have

$$\widetilde{\omega}_\alpha(t) = \omega_\alpha(t) + o\left(\frac{1}{\sqrt{t}}\right).$$

It only remains to proceed to a Taylor expansion in the first term of the Edgeworth development of S_t in Yoshida (1997) to ensure that

$$P\left(\theta \in I_t^{boot}(\alpha)\right) - \alpha = o\left(\frac{1}{\sqrt{t}}\right). \tag{3.13}$$

We therefore corrected the confidence interval coverage error.

Discretization

Based on discrete observations $X_{t_i}, 0 \le i \le n$, the parametric Euler model is given by

$$X_{t_i} - X_{t_{i-1}} = \theta b(X_{t_{i-1}})(t_i - t_{i-1}) + \sigma(X_{t_{i-1}})(W_{t_i}^H - W_{t_{i-1}}^H). \tag{3.14}$$

The AMLE is given by

$$\theta_{n,T} := \frac{\sum_{i=1}^n \frac{b(X_{t_{i-1}})}{\sigma^2(X_{t_{i-1}})}(X_{t_i} - X_{t_{i-1}})}{\sum_{i=1}^n \frac{b^2(X_{t_{i-1}})}{\sigma^2(X_{t_{i-1}})}(t_i - t_{i-1})}. \tag{3.15}$$

Empirical Euler model is given by

$$Y_{t_i} - Y_{t_{i-1}} = \theta_{n,T} b(Y_{t_{i-1}})(t_i - t_{i-1}) + \sigma(Y_{t_{i-1}})(B_{t_i}^H - B_{t_{i-1}}^H). \tag{3.16}$$

where B^H is a fractional Brownian motion independent of W^H.
The Bootstrap AMLE is given by

$$\widehat{\theta}_{n,T} := \frac{\sum_{i=1}^n \frac{b(Y_{t_{i-1}})}{\sigma^2(Y_{t_{i-1}})}(Y_{t_i} - Y_{t_{i-1}})}{\sum_{i=1}^n \frac{b^2(Y_{t_{i-1}})}{\sigma^2(Y_{t_{i-1}})}(t_i - t_{i-1})}. \tag{3.17}$$

The α-th quantile is given by

$$\widetilde{\omega}_\alpha(n,T) := \inf_{y\in\mathbb{R}} \left\{ P\left(\sqrt{\mu\left(\frac{b^2}{\sigma^2}\right) T} \left(\widehat{\theta}_{n,T} - \theta_{n,T}\right) \le y \right) \ge \alpha \right\}. \tag{3.18}$$

The bootstrap confidence interval is given by

$$I_{n,T}^{boot}(\alpha) := \left[\theta_{n,T} - \frac{\widetilde{\omega}_\alpha(n,T)}{\sqrt{\mu\left(\frac{b^2}{\sigma^2}\right) T}},\ \infty \right). \tag{3.19}$$

Conditional on $\theta_{n,T}$, the quantile $\widetilde{\omega}$ can be expanded as

$$\widetilde{\omega}_\alpha(n,T) = q_\alpha + \frac{1}{\sqrt{T}} h_{\widehat{\theta}_{n,T}}(q_\alpha) + o\left(\frac{1}{\sqrt{T}} \bigvee \frac{T}{\sqrt{n}}\right). \tag{3.20}$$

$$P(\theta \in I_{n,T}^{boot}(\alpha)) - \alpha = o\left(\frac{1}{\sqrt{T}} \bigvee \frac{T}{\sqrt{n}}\right). \tag{3.21}$$

Edgeworth Expansion for MCE

Podloskij, Veliyev and Yoshida [23] studied Edgeworth expansion for Euler approximation of continuous diffusion processes. Consider the Ornstein-Uhlenbeck process

$$dX_t = \theta X_t dt + dW_t, t \geq 0,\ \theta < 0. \tag{3.22}$$

It is well known that the minimum contrast estimator (MCE) $\widehat{\theta}_T$ is consistent as $T \to \infty$:

P-$\lim_{T\to\infty}\widehat{\theta}_T =$ P-$\lim_{T\to\infty}\dfrac{-T}{2\int_0^T X_t^2 dt} = \theta$ since $E(\int_0^T X_t^2 dt) = \frac{T}{2\theta}$, see Bishwal (2008).

We have the following Edgeworth expansion:

Theorem 3.2

$$P_\nu[\sqrt{T}(\widehat{\theta}_T - \theta)/\sigma \leq x] = \Phi(x) + T^{-1/2}\phi(x)(a_1 + a_2(1 - x^2)) + O(T^{-1})$$

uniformly in $x \in \mathbb{R}$ *where*

$$\sigma^2 = \frac{E(\bar{F}_0)^2}{E(L_0)},\ \bar{F}_0 = F_0 - \theta L_0,\ \alpha = E(L_0),$$

$$a_1 = E_\mu(\bar{F}_0) - E_\nu(\bar{F}_0),\ a_2 = \frac{\kappa - 3\rho\sigma^2}{6\alpha\sigma^3},\ \kappa = E_x(\bar{F}_0^3),\ \rho = E_x(\bar{F}_0 L_0).$$

Regenerative Method

The regenerative method consists, in the case when the chain possesses an accessible atom (regeneration point), in dividing the trajectory of the Markov process into i.i.d. blocks of observations (namely, regenerative cycles) corresponding to the successive visits to the atom, see Datta and McCormick [11].

A cadlag process X is called a *regenerative process* if there exists an increasing sequence of finite random times $\{\tau_j\}_{j\geq 1}$ such that

$$\{X_t\}_{0\leq t\leq \tau_1}, \{X_t\}_{\tau_1\leq t\leq \tau_2}, \ldots, \{X_t\}_{\tau_j\leq t\leq \tau_{j+1}}, \ldots$$

are independent and

$$\{X_t\}_{\tau_1\leq t\leq \tau_2}, \ldots, \{X_t\}_{\tau_j\leq t\leq \tau_{j+1}}, \ldots$$

are identically distributed. The random times τ_j is called a j-th regenerative epoch.

Let E be the state space of the regenerative strong Markov process X. Let E_x stand for E_{δ_x} for $x \in E$. Let

$$\tau_{j+1} = \inf\{t > \tau_j : X_t = x,\ \text{there exists } s \in (\tau_j, t) \text{ such that } X_s \in \widehat{x}\}$$

where $\tau_0 = 0$ and $x, \widehat{x}$ are a point and a closed set of E respectively such that $x \notin \widehat{x}$. In particular, $\{\tau_j\}$ is a sequence of stopping times with respect to the canonical filtration of X and $X_{\tau_j} = x$ for all $j \geq 1$. Here $\widehat{x}$ was introduced to ensure that $\tau_{j+1} > \tau_j$ a.s.

The asymptotic variance σ is practically unknown in the nonparametric context. Hence we have to construct an estimator when constructing confidence interval for instance. We propose the following estimator.

Let $L_j := \tau_{j+1} - \tau_j, j \geq 1$, $M_T := \max\{j : \tau_{j+1} \leq T\}$. Observe that $\sum_{j=1}^{M_T} L_j = \tau_N - \tau_1$.

Define the estimators of θ and σ as

$$\check{\theta}_T := \frac{-\sum_{j=1}^{M_T} L_j}{2\sum_{j=1}^{M_T} F_j}, \quad \widehat{\sigma}_T^2 := \frac{\sum_{j=1}^{M_T} |L_j - \check{\theta}_T F_j|^2}{2\sum_{j=1}^{M_T} F_j} \tag{3.23}$$

where $F_j := \int_{\tau_j}^{\tau_{j+1}} X_t^2 dt,\ j \geq 0$ is a sequence of i.i.d. random variables.

The sequence $L_j := \tau_{j+1} - \tau_j,\ j \geq 1$ is a sequence of i.i.d. random variables and $\tau_j \to \infty$ as $j \to \infty$. It holds that $\tau_j \to \infty$ a.s. as $j \to \infty$. The primary use of regenerative method appears in the proof of consistency of the MCE as follows. By the law of large numbers for i.i.d. sequences,

$$\text{P-lim}_{T\to\infty} \frac{-T}{2\int_0^T X_t^2 dt} = \text{P-lim}_{N\to\infty} \frac{\tau_1 - \tau_N}{2\sum_{j=0}^{N-1} \int_{\tau_j}^{\tau_{j+1}} X_t^2 dt} = \theta \tag{3.24}$$

since $\theta = E(L_0)/2E(F_0)$. Thus $P - \lim \check{\theta}_T = \theta$ and $P - \lim \widehat{\sigma}_T^2 = \sigma^2$ as $T \to \infty$.

This section is inspired by Bertail and Clemencon [1–3] and Fukasawa [13,14]. The studentized statistic admits the following Edgeworth expansion.

Theorem 3.3

$$P_\mu[\sqrt{T}(\check{\theta}_T - \theta)/\widehat{\sigma}_T \leq x] = \Phi(x) + T^{-1/2}\phi(x)(a_1 + a_2(1 - x^2)) + O(T^{-1})$$

uniformly in $x \in \mathbb{R}$ where

$$\sigma^2 = E_x(\bar{F}_0^2)/\alpha,\ \bar{F}_0 = F_0 - \theta L_0,\ \alpha = E_x(L_0),\ a_1 = E_\mu(\bar{F}_0) - E_\nu(\bar{F}_0),\ a_2 = \frac{\kappa - 3\rho\sigma^2}{6\alpha\sigma^3},$$

$$\kappa = E_x(\bar{F}_0^3),\ \rho = E_x(\bar{F}_0 L_0).$$

Since X is stationary, $\mu = \nu$. Hence $a_1 = 0$.

Theorem 3.4

$$P_\mu[\sqrt{T}(\check{\theta}_T - \theta)/\widehat{\sigma}_T \leq x] = \Phi(x) + T^{-1/2}\phi(x)(\widehat{a}_2(2x^2 + 1)) + O_P(T^{-1})$$

uniformly in $x \in \mathbb{R}$ where $\widehat{a}_2$ is an estimator of a_2 with $\sqrt{T}(\widehat{a}_2 - a_2) = O(1)$. We can use for instance

$$\widehat{\sigma}^2 = \frac{1}{M_T}\sum_{j=1}^{M_T} \frac{\check{F}_j^2}{\widehat{\alpha}},\ \check{F}_j = F_j - \check{\theta} L_j,\ \widehat{\alpha} = \frac{1}{M_T}\sum_{j=1}^{M_T} L_j,$$

$$\widehat{a}_2 = \frac{\widehat{\kappa} - 3\widehat{\rho}\widehat{\sigma}^2}{6\widehat{\alpha}\widehat{\sigma}^3},\ \widehat{\kappa} = \frac{1}{M_T}\sum_{j=1}^{M_T} \check{F}_j^3,\ \widehat{\rho} = \frac{1}{M_T}\sum_{j=1}^{M_T} \check{F}_j L_j, \quad \check{F}_j = F_j - \check{\theta}_T L_j.$$

In fact, it is easy to show that $P_\mu[|T-\alpha M_T| \geq \delta T] = O(T^{-1})$ for $\delta \in (0,1/2)$. Using Kolmogorov's inequality, we have

$$\sup_{T>0} P_\mu\left[\frac{\sqrt{T}}{M_T}\left|\sum_{j=1}^{M_T}\{(F_j,L_j)^n - E_\mu(F_j,L_j)^n\}\right| > K\right] \to 0$$

as $K \to \infty$. Hence the above expansion formula is practically of use to obtain second order correct confidence intervals for instance by means of Cornish-Fisher expansion. For the same purpose, it is then natural to expect that there corresponds a bootstrap method.

Let $\mathcal{F}_T = \{(F_j,L_j)\}, j=1,2,\ldots,M_T$ be the set of the observed regenerative blocks. To bootstrap the sampling distribution of $\check{\theta}_T$, we resample the cycles $\{(F_j,L_j)\}, j=1,2,\ldots,M_T$ following the sample random sampling with replacement. Let $(F_j^*,L_j^*), j=1,2,\ldots,M_T$ denote the selected cycles.

Let $(F_j^*,L_j^*), j=1,2,\ldots,M_T$ be an i.i.d. sequence and each (F_j^*,L_j^*) be uniformly distributed on $\mathcal{F}_T$. Here M_T and $\mathcal{F}_T$ are fixed conditionally on the observation $\{X_t, 0\leq t\leq T\}$. Put $\sum_{j=1}^{M_T} L_j^* = T^*$.

Define the bootstrap statistics:

$$\check{\theta}_T^* = \frac{-\sum_{j=1}^{M_T} L_j^*}{2\sum_{j=1}^{M_T} F_j^*} = \frac{-T^*}{2\sum_{j=1}^{M_T} F_j^*}, \quad \widehat{\sigma}_T^{*2} = \frac{\sum_{j=1}^{M_T}|L_j^* - \check{\theta}_T^* F_j^*|^2}{2\sum_{j=1}^{M_T} F_j^*}. \tag{3.25}$$

Theorem 3.5

$$P_\mu^*[\sqrt{T^*}(\check{\theta}_T^* - \check{\theta}_T)/\widehat{\sigma}_T^* \leq x] = \Phi(x) + T^{-1/2}\phi(x)(\widehat{b} + \widehat{a}_2(2x^2+1)) + O_P(T^{-1})$$

uniformly in $x\in\mathbb{R}$ where P_μ^* is the conditional bootstrap probability given $\{X_t, 0\leq t\leq T\}$ and $\widehat{b} = \widehat{\rho}/(2\widehat{\alpha}\widehat{\sigma})$. In particular,

$$P_\mu[\sqrt{T}(\check{\theta}_T - \theta)/\widehat{\sigma}_T \leq x] = P_\mu^*[\sqrt{T^*}(\check{\theta}_T^* - \check{\theta}_T)/\widehat{\sigma}_T^* - \widehat{b}T^{-1/2} \leq x] + O_P(T^{-1})$$

uniformly in $x\in\mathbb{R}$.

The approximate minimum contrast estimator (AMCE) which is an Euler discretization of the MCE is defined as

$$\widehat{\theta}_{n,T} = \frac{-T}{2\sum_{i=1}^n X_{t_{i-1}}^2(t_i - t_{i-1})}. \tag{3.26}$$

We define another symmetric AMCE based on trapezoidal rule as

$$\widetilde{\theta}_{n,T} := \frac{-T}{\sum_{i=1}^n (X_{t_{i-1}}^2 + X_{t_i}^2)(t_i - t_{i-1})}. \tag{3.27}$$

Define the symmetric bootstrap statistic:

$$\widetilde{\theta}_T^* := \frac{-\sum_{j=1}^{M_T} L_j^*}{\sum_{j=1}^{M_T}(F_j^* + F_{j+1}^*)} = \frac{-T^*}{\sum_{j=1}^{M_T}(F_j^* + F_{j+1}^*)}. \tag{3.28}$$

We have the following Edgeworth expansion of the AMCEs:

Theorem 3.6 *As $T\to\infty, n\to\infty, T/\sqrt{n}\to 0$,*

$$P_\nu[\sqrt{T}(\widehat{\theta}_{n,T} - \widehat{\theta}_T)/\widehat{\sigma} \leq x] = \Phi(x) + T^{-1/2}\phi(x)(\widehat{a}_1 + \widehat{a}_2(1-x^2)) + O((T^{-1}\log T)\vee((T/\sqrt{n})^8(\log T)^{-2}))$$

uniformly in $x\in\mathbb{R}$ where $\widehat{a}_1$ and $\widehat{a}_2$ are estimators of a_1 and a_2 in Theorem 3.3.

Theorem 3.7 *As $T \to \infty, n \to \infty, T/n^{2/3} \to 0$,*

$$P_\nu[\sqrt{T}(\widetilde{\theta}_{n,T} - \widehat{\theta}_T)/\widehat{\sigma} \le x] = \Phi(x) + T^{-1/2}\phi(x)(\widehat{a}_1 + \widehat{a}_2(1-x^2)) + O(T^{-1} \vee (T^2/n^{4/3}))$$

uniformly in $x \in \mathbb{R}$ where $\widehat{a}_1$ and $\widehat{a}_2$ are estimators of a_1 and a_2 in Theorem 3.3.

Theorem 3.8

$$P^*_\mu[\sqrt{T^*}(\widetilde{\theta}^*_T - \check{\theta}_T)/\widehat{\sigma}^*_T \le x] = \Phi(x) + T^{-1/2}\phi(x)(\widehat{b} + \widehat{a}_2(2x^2+1)) + O_P(T^{-1})$$

uniformly in $x \in \mathbb{R}$ where P^*_μ is the conditional probability given $\{X_t, 0 \le t \le T\}$ and $\widehat{b} = \widehat{\rho}/(2\widehat{\alpha}\widehat{\sigma})$. In particular,

$$P_\mu[\sqrt{T}(\check{\theta}_T - \theta)/\widehat{\sigma}_T \le x] = P^*_\mu[\sqrt{T^*}(\widetilde{\theta}^*_T - \check{\theta}_T)/\widehat{\sigma}^*_T - \widehat{b}T^{-1/2} \le x] + O_P(T^{-1})$$

uniformly in $x \in \mathbb{R}$.

4. Bootstrap SGDCT Algorithm

Consider the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad t \ge 0. \tag{4.1}$$

Sirignano and Spiliopoulos [25] studied a deep learning algorithm for solving PDE. Sirignano and Spiliopoulos [24,26] studied stochastic gradient descent in continuous time (SGDCT). First, we recall the Q-learning algorithm: The Q-learning algorithm uses stochastic gradient descent to minimize an approximation to the discrete time Hamilton-Jacobi-Bellman (HJB) equation. Consider the Q-learning algorithm to estimate the value function

$$V(x) := E\left[\int_0^\infty e^{-\gamma t} r(X_t)dt \;\middle|\; X_0 = x\right], \quad X_t = x + W_t \tag{4.2}$$

where $\gamma > 0$ is a discount factor and $r(x)$ is a reward function. The function $Q(x,\theta)$ is an approximation for the value function $V(x)$. The traditional approach is to discretize the dynamics of $V(x)$ and apply a stochastic gradient descent update to the objective function:

$$E\left[\left(r(X_t)\Delta + e^{-\gamma\Delta}E[Q(X_{t+\Delta};\theta)|X_t] - Q(X_t;\theta)\right)^2\right]. \tag{4.3}$$

The result is the stochastic gradient descent algorithm:

$$\begin{aligned}\theta_{t+\Delta} = \theta_t - \frac{\alpha_t}{\Delta}&\left(e^{-\gamma\Delta}E[Q_\theta(X_{t+\Delta};\theta_t)|X_t] - Q_\theta(X_t;\theta_t))\right) \\ &\times \left(r(X_t)\Delta + e^{-\gamma\Delta}E[Q(X_{t+\Delta};\theta_t)|X_t] - Q(X_t;\theta_t)\right).\end{aligned} \tag{4.4}$$

The learning rate is Δ^{-1}. The Q-learning algorithm has a major computational issue. The expectation $E[Q_\theta(X_{t+\Delta};\theta_t)|X_t]$ is challenging to calculate if the process X_t is high dimensional. To circumvent this situation, Q-learning algorithm ignores the inner expectation leading to

$$\theta_{t+\Delta} = \theta_t - \frac{\alpha_t}{\Delta}(e^{-\gamma\Delta}Q_\theta(X_{t+\Delta};\theta_t) - Q_\theta(X_t;\theta_t)(r(X_t)\Delta + e^{-\gamma\Delta}Q(X_{t+\Delta};\theta_t) - Q(X_t;\theta_t)). \tag{4.5}$$

Although computationally efficient, the Q-learning algorithm is biased. The SGDCT algorithm can be directly derived by letting $\Delta \to 0$ and using Itô formula:

$$d\theta_t = -\alpha_t\left(\frac{1}{2}Q_{\theta xx}(X_t;\theta_t) - \gamma Q_\theta(X_t;\theta_t)\right)\left(r(X_t) + \frac{1}{2}Q_{xx}(X_t;\theta_t) - \gamma Q(X_t;\theta_t)\right)dt. \tag{4.6}$$

Furthermore, when $\Delta \to 0$, the Q-learning algorithm blows up.

Bootstrap SGDCT Algorithm for American Option

Let $X_t \in \mathbb{R}^d, d \geq 1$ be the prices of d stocks. The maturity time is T and the payoff function is $g(x) : \mathbb{R}^d \to \mathbb{R}$. The stock price dynamics and the value functions are given by

$$dX_t^i = \theta b(X_t^i)dt + \sigma(X_t^i)dW_t^i, \; i = 1, 2, \ldots, d \tag{4.7}$$

$$V_{t,x} := \sup_{\tau \geq t} E[e^{-r(\tau \wedge T)} g(X_{\tau \wedge T}) | X_t = x] \tag{4.8}$$

where $W_t \in \mathbb{R}^d$ is a Brownian motion. The distribution of W_t is specified by $\mathrm{Var}(W_t^i) = t, \;\; i = 1, 2, \ldots, d$ and $\mathrm{Corr}(W_t^i, W_t^j) = \rho_{i,j} dt$ for $i \neq j$. The price of the American option is $V_{0,x}$.

SGDCT for American option is given by

$$\begin{aligned} \theta_{t \wedge T}^{n+1} = \theta_0^n - \int_0^{\tau \wedge T} \alpha_t^{n+1} \left(\frac{\partial}{\partial t} Q_\theta(t, X_t; \theta_t^{n+1}) + \mathcal{L}_x Q_\theta(t, X_t; \theta_t^{n+1}) - r Q_\theta(t, X_t; \theta_t^{n+1}) \right) \\ \times \left(\frac{\partial}{\partial t} Q(t, X_t; \theta_t^{n+1}) + \mathcal{L}_x Q(t, X_t; \theta_t^{n+1}) - r Q(t, X_t; \theta_t^{n+1}) \right) dt \\ + \alpha_{\tau \wedge T}^{n+1} \, Q_\theta \left(\tau \wedge T, X_{\tau \wedge T}; \theta_{\tau \wedge T}^{n+1} \right) \left(g(X_{\tau \wedge T}) - Q(\tau \wedge T, X_{\tau \wedge T}; \theta_{\tau \wedge T}^{n+1}) \right), \end{aligned} \tag{4.9}$$

$$\tau := \inf\{t \geq 0 : Q(t, X_t; \theta_t^{n+1}) < g(X_t)\}, \quad X_0 \sim \nu(dx). \tag{4.10}$$

The function $Q(x, \theta)$ is an approximation of the value function. The parameter θ must be estimated. Here $\mathcal{L}_x$ is the infinitesimal generator of the X process. The algorithm is run for many iterations $n = 0, 1, 2, \ldots$ until convergence.

The empirical model is given by

$$dY_t^i = \theta_t b(Y_t^i)dt + \sigma(Y_t^i)dW_t^i, \quad i = 1, 2, \ldots, d \tag{4.11}$$

$$\widehat{V}_{t,x} := \sup_{\tau \geq t} E[e^{-r(\widehat{\tau} \wedge T)} g(Y_{\tau \wedge T}) | Y_t = y]. \tag{4.12}$$

Let $\widehat{\theta}$ be the bootstrap MLE of θ as defined in (3.8)-(3.9). The bootstrap SGDCT for American option is given by

$$\begin{aligned} \widehat{\theta}_{t \wedge T}^{n+1} = \widehat{\theta}_0^n - \int_0^{\tau \wedge T} \alpha_t^{n+1} \left(\frac{\partial}{\partial t} Q_\theta(t, Y_t; \widehat{\theta}_t^{n+1}) + \mathcal{L}_x Q_\theta(t, X_t; \widehat{\theta}_t^{n+1}) - r Q_\theta(t, Y_t; \widehat{\theta}_t^{n+1}) \right) \\ \times \left(\frac{\partial}{\partial t} Q(t, Y_t; \widehat{\theta}_t^{n+1}) + \mathcal{L}_x Q(t, Y_t; \widehat{\theta}_t^{n+1}) - r Q(t, Y_t; \widehat{\theta}_t^{n+1}) \right) dt \\ + \alpha_{\tau \wedge T}^{n+1} \, Q_\theta \left(\tau \wedge T, Y_{\tau \wedge T}; \widehat{\theta}_{\tau \wedge T}^{n+1} \right) \left(g(Y_{\tau \wedge T}) - Q(\tau \wedge T, Y_{\tau \wedge T}; \widehat{\theta}_{\tau \wedge T}^{n+1}) \right), \end{aligned} \tag{4.13}$$

$$\widehat{\tau} := \inf\{t \geq 0 : Q(t, Y_t; \widehat{\theta}_t^{n+1}) < g(Y_t)\}, \quad Y_0 \sim \widehat{\nu}(dy) \tag{4.14}$$

where $\widehat{\nu}$ is the empirical distribution.

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