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BANACH ALGEBRAS OF MEASURES WHICH HAVE DENSITY ON THE NONNEGATIVE HALFLINE

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Abstract. Convolution Banach algebras of measures are considered whose elements have absolutely continuous restrictions to the nonnegative halfline. We investigate various types of asymptotic behavior of the densities at infinity.

1. Introduction

Banach algebras of functions on the real line $\mathbb{R}$ have first been considered in [1, 2] and, independently, in another setting, in [3]. Later on, the subject has also been raised in [4] where some previous results have been generalized. This paper is a further development of [4]. Here we discuss convolution Banach algebras of measures such that their elements have absolutely continuous restrictions to the nonnegative halfline and their densities possess similar asymptotic behavior at infinity.

Denote by $\mathbb{R}_+$ the set of all nonnegative numbers and by $\mathbb{R}_- := \mathbb{R} \setminus \mathbb{R}_+$ the set of all negative numbers. The subsequent plan of the paper is as follows. Section ?? contains a formula for the absolutely continuous restriction to $\mathbb{R}_+$ of the convolution $\mu * \nu$ of two measures with absolutely continuous restrictions to $\mathbb{R}_+$. Besides, there is a brief description of the underlying convolution Banach algebra $S(\varphi)$ of measures which are finite with a submultiplicative weight function $\varphi(x)$; see Definition 1. In Section 3, we consider the Banach subalgebra $Z(\varphi)$ of $S(\varphi)$ whose elements have have absolutely continuous restrictions to $\mathbb{R}_+$. Then we introduce specific convolution Banach subalgebras of $Z(\varphi)$ with various types of common asymptotic behavior at infinity of the densities of their elements. Section 4 is devoted to the study of maximal ideals in Banach algebras under investigation. Section 5 deals with values of analytic functions at elements of our Banach algebras. Finally, Section 6 gives a simple probabilistic application of the theory.

In contrast to the previous works [1]– [4], the use of these new Banach algebras in applications allows us to obtains asymptotic results for functions in a more general setting, that is, without requiring that underlying measures be absolutely continuous on the *whole* line; see Remark 3 in Section 6.

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2. Preliminaries

Let ν and $\varkappa$ be finite measures on the σ-algebra $\mathscr{B}$ of Borel sets in $\mathbb{R}$. Their *convolution* is the measure

$$\nu * \varkappa(A) := \iint\limits_{\{x+y\in A\}} \nu(dx)\,\varkappa(dy) = \int_{\mathbb{R}} \nu(A-x)\,\varkappa(dx), \qquad A \in \mathscr{B};$$

here $A - x := \{y \in \mathbb{R} : x + y \in A\}$. If $f(x)$ and $g(x)$ are appropriate functions on $\mathbb{R}_+$, denote their convolution as

$$f * g(x) = \int_0^x f(x-y)g(y)\,dy, \qquad x \in \mathbb{R}_+.$$

Let μ and ν be complex-valued measures on $\mathscr{B}$ such that their restrictions $\mu|_{\mathbb{R}_+}$ and $\nu|_{\mathbb{R}_+}$ to $\mathbb{R}_+$ are absolutely continuous with respect to Lebesgue measure. Notice that $\mu|_{\mathbb{R}_+}(A) := \mu(A \cap \mathbb{R}_+)$, $A \in \mathscr{B}$. Suppose that convolution $\mu * \nu$ makes sense. Then the restriction $\mu * \nu|_{\mathbb{R}_+}$ is also absolutely continuous. To see this, denote $\mu_\pm := \mu|_{\mathbb{R}_\pm}$. We have

$$\mu * \nu = \mu_- * \nu_- + \mu_- * \nu_+ + \mu_+ * \nu_- + \mu_+ * \nu_+.$$

The measure $\mu_- * \nu_-$ is concentrated on $\mathbb{R}_-$, whereas the remaining summands on the right-hand side are absolutely continuous. It follows that the measure

$$(1) \qquad (\mu * \nu)|_{\mathbb{R}_+} = (\mu_- * \nu_+)|_{\mathbb{R}_+} + (\mu_+ * \nu_-)|_{\mathbb{R}_+} + \mu_+ * \nu_+$$

is obviously absolutely continuous.

For $c \in \mathbb{C}$, we assume that c/∞ is equal to zero. The relation $a(x) \sim cb(x)$ as $x \to \infty$ means that $a(x)/b(x) \to c$ as $x \to \infty$; if $c = 0$, then $a(x) = o[b(x)]$.

Definition 1. *A positive function $\varphi(x)$, $x \in \mathbb{R}(\mathbb{R}_+)$, is called submultiplicative if it is finite, Borel measurable and satisfies the conditions: $\varphi(0) = 1$, $\varphi(x + y) \le \varphi(x)\,\varphi(y)$, x, $y \in \mathbb{R}(\mathbb{R}_+)$.*

The following properties are valid for submultiplicative functions defined on the whole line [5, Theorem 7.6.2]:

$$\begin{aligned} -\infty < r_-(\varphi) &:= \lim_{x\to-\infty} \frac{\log\varphi(x)}{x} = \sup_{x<0} \frac{\log\varphi(x)}{x} \\ &\le \inf_{x>0} \frac{\log\varphi(x)}{x} = \lim_{x\to\infty} \frac{\log\varphi(x)}{x} =: r_+(\varphi) < \infty. \end{aligned} \qquad (2)$$

Here are some examples of submultiplicative function on $\mathbb{R}_+$: (i) $\varphi(x) = (x+1)^r$, $r > 0$; (ii) $\varphi(x) = \exp(cx^\beta)$, where $c > 0$ and $0 < \beta < 1$; (iii) $\varphi(x) = \exp(\gamma x)$, where $\gamma \in \mathbb{R}$. In (i) and (ii), $r_+(\varphi) = 0$, while in (iii), $r_+(\varphi) = \gamma$. The product of a finite number of submultiplicative function is again a submultiplicative function.

Consider the collection $S(\varphi)$ of all complex-valued measures $\varkappa$ on $\mathscr{B}$ such that

$$\|\varkappa\|_\varphi := \int_{\mathbb{R}} \varphi(x)\,|\varkappa|(dx) < \infty;$$

here $|\varkappa|$ stands for the total variation of $\varkappa$. The collection $S(\varphi)$ is a Banach algebra with norm $\|\cdot\|_\varphi$ by the usual operations of addition and scalar multiplication of measures, the product of two elements ν and $\varkappa$ of $S(\varphi)$ is defined as their convolution $\nu * \varkappa$ [5, Section 4.16]. The unit element of $S(\varphi)$ is the measure δ_0 of unit mass concentrated at zero. For arbitrary complex measure ν, define its *Laplace transform* as $\widehat{\nu}(s) = \int_{\mathbb{R}} e^{sx}\,\nu(dx)$ for those values of $s \in \mathbb{C}$ for which the integral absolutely converges with respect to the total variation $|\nu|$ of the measure ν. It follows from (2) that the Laplace transform of any $\nu \in S(\varphi)$ converges absolutely with respect to $|\nu|$ for all s in the strip

$$\Pi[r_-(\varphi), r_+(\varphi)] = \{s \in \mathbb{C} : r_-(\varphi) \le \Re s \le r_+(\varphi)\}.$$

3. Banach algebras

Denote by $\widetilde{Z}(\varphi)$ the subalgebra (without unity) of $S(\varphi)$ such that for all $\nu \in \widetilde{Z}(\varphi)$ the restriction $\nu|_{\mathbb{R}_+}$ is absolutely continuous. Adjoin the identity element δ_0 to $\widetilde{Z}(\varphi)$ and denote the resulting unital Banach algebra by $Z(\varphi)$:

$$Z(\varphi) := \{c\delta_0 + \nu_1 : c \in \mathbb{C}, \nu_1 \in \widetilde{Z}(\varphi)\}.$$

Measures in $Z(\varphi)$ will be denoted by small Greek letters and their densities on $\mathbb{R}_+$ by the same letters with arguments, e.g., ν and $\nu(x)$. Let $\tau(x)$, $x \in \mathbb{R}_+$, be a bounded Borel-measurable positive function such that

$$\lim_{x\to\infty} [\tau(x)]^{1/x} = 1, \tag{3}$$

$$\sup_{x\in\mathbb{R}_+, |y|\le 1} \frac{\tau(x)}{\tau(x-y)} = C_0 < \infty, \tag{4}$$

where $\tau(x) := \tau(0)$ for $x < 0$. Let $\nu \in Z(\varphi)$ with density $\nu(x)$, $x \in \mathbb{R}_+$. Put

$$P_\tau(\nu) = \operatorname*{ess\,sup}_{x\in\mathbb{R}_+} \frac{|\nu(x)|\varphi(x)}{\tau(x)}.$$

Denote

$$Z_\varphi(\tau) = \{\nu \in Z(\varphi) : P_\tau(\nu) < \infty\},$$

$$Z^0_\varphi(\tau) = \left\{\nu \in Z_\varphi(\tau) : \lim_{x\to\infty} \frac{\nu(x)\varphi(x)}{\tau(x)} = 0\right\}.$$

Let us stipulate that relations with densities are understood in the sense that in the classes of equivalent functions there are functions which satisfy the given relations. Assume that the function $\tau(x)$ is such that for all ν and $\mu \in Z_\varphi(\tau)$

$$P_\tau(\nu * \mu) \le C\big[\|\nu\|_\varphi P_\tau(\mu) + \|\mu\|_\varphi P_\tau(\nu) + P_\tau(\nu)P_\tau(\mu)\big], \tag{5}$$

where the constant $C \ge 1$ does not depend on ν and μ.

Definition 2. *Functions $\tau(x)$, $x \in \mathbb{R}_+$, satisfying the hypotheses (3)–(5) will be called norming.*

While dealing with the collection $Z^0_\varphi(\tau)$, we shall always assume that the following hypothesis is fulfilled:

$$\nu * \mu \in Z^0_\varphi(\tau) \qquad \text{for all } \nu, \mu \in Z^0_\varphi(\tau). \tag{6}$$

Concrete conditions enabling (5) and (6) will be given below. (See Theorem 3.)

For arbitrary $\nu \in Z_\varphi(\tau)$ set

$$\|\nu\|_{\varphi,\tau} := C[\|\nu\|_\varphi + P_\tau(\nu)].$$

Since $\|\nu * \mu\|_\varphi \le \|\nu\|_\varphi \|\mu\|_\varphi$, we have

$$\|\nu * \mu\|_{\varphi,\tau} \le \|\nu\|_{\varphi,\tau}\|\mu\|_{\varphi,\tau} \qquad \text{for all } \nu, \mu \in Z_\varphi(\tau). \tag{7}$$

Theorem 1. *Let φ and τ be submultiplicative and norming functions respectively. Then the collection $Z_\varphi(\tau)$ is a complex Banach with respect to the norm $\|\cdot\|_{\varphi,\tau}$, and the collection $Z^0_\varphi(\tau)$ is a Banach subalgebra of $Z_\varphi(\tau)$.*

Proof. By (5) and (7), the product $\nu*\mu \in Z_\varphi(\tau)$ for all $\nu,\mu \in Z_\varphi(\tau)$. We prove the completeness of $Z_\varphi(\tau)$. Let $\{\nu_n\}$ a fundamental sequence in $Z_\varphi(\tau)$. By the definition of the norm $\|\cdot\|_{\varphi,\tau}$, the sequence $\{\nu_n\}$ is fundamental in $S(\varphi)$ and, therefore, converges to some measure $\nu \in S(\varphi)$ with respect to the norm $\|\cdot\|_\varphi$. Let us show that $\nu \in Z_\varphi(\tau)$ and $P_\tau(\nu_n-\nu) \to 0$ as $n \to \infty$, which will prove the completeness of $Z_\varphi(\tau)$.

We have $P_\tau(\nu_m-\nu_n) \to 0$ as $m,n \to \infty$. Let $\{\varepsilon_k\}$ be a sequence of positive numbers such that $\varepsilon_k \downarrow 0$ as $k \to \infty$. There exists a sequence of positive numbers $\{N(k)\}_{k=1}^\infty$ (tending to infinity as $k \to \infty$) such that

$$P_\tau(\nu_m-\nu_n) \le \varepsilon_k \qquad \text{for all } m,n \ge N(k).$$

Consider the sets

$$A_{k,m,n} := \left\{x \in \mathbb{R}_+ : \frac{|\nu_m(x)-\nu_n(x)|\varphi(x)}{\tau(x)} > \varepsilon_k\right\}.$$

The Lebesgue measure of the union $A := \cup_{k=1}^\infty \cup_{m,n\ge N(k)} A_{k,m,n}$ is zero since all of $A_{k,m,n}$ for $m,n \ge N(k)$ are sets of Lebesgue measure zero. We have

$$\frac{|\nu_m(x)-\nu_n(x)|\varphi(x)}{\tau(x)} \le \varepsilon_k \qquad \text{for all } x \in A^C \text{ and all } m,n \ge N(k). \tag{8}$$

Thus, the sequence $\{\nu_n(x)\}$ is fundamental for every $x \in A^C$ and, therefore, converges to some value $\gamma(x)$, $x \in A^C$. On the other hand, the sequence $\{\nu_n(x)\varphi(x)\}$ converges to $\nu(x)\varphi(x)$ in $L_1(\mathbb{R}_+)$ as $n \to \infty$. It is well known that there exists a subsequence $\{\nu_{n_k}(x)\varphi(x)\}$ which tends to $\nu(x)\varphi(x)$ as $k \to \infty$ a.e. (almost everywhere) Obviously, $\nu(x) = \gamma(s)$ for $x \in A^C$, i.e., $\nu(x) = \gamma(s)$ a.e. Put $m = n_k$ in (8) and let k tend to ∞. We get

$$\frac{|\nu(x)-\nu_n(x)|\varphi(x)}{\tau(x)} \le \varepsilon_k \qquad \text{for all } x \in A^C \text{ and all } n \ge N(k). \tag{9}$$

It follows from (9) that $P_\tau(\nu) \le P_\tau(\nu_n) + \varepsilon_k < \infty$, i.e., $\nu \in Z_\varphi(\tau)$. It also follows from (9) that $P_\tau(\nu-\nu_n) \le \varepsilon_k$ and, therefore, $P_\tau(\nu-\nu_n) \to 0$ as $n \to \infty$. The completeness of $Z_\varphi(\tau)$ is proven.

To finish the proof of Theorem 1, it remains to show that $Z^0_\varphi(\tau)$ is a closed subalgebra of $Z_\varphi(\tau)$. By condition (6), $Z^0_\varphi(\tau)$ is a subalgebra of $Z_\varphi(\tau)$. Let us show that $Z^0_\varphi(\tau)$ is a closed subspace of $Z_\varphi(\tau)$. Let $\{\nu_n\}$ be a fundamental sequence in $Z^0_\varphi(\tau)$. Without loss of generality, we may assume that the elements ν_n of the equivalence classes are chosen in such a way that there exist ordinary limits

$$\lim_{x\to\infty} \frac{\nu_n(x)\varphi(x)}{\tau(x)} = 0.$$

Since $\{\nu_n\} \subset Z_\varphi(\tau)$ and $Z_\varphi(\tau)$ is complete, there exists an element $\nu \in Z_\varphi(\tau)$ such that $\|\nu_n-\nu\|_{\varphi,\tau} \to 0$ as $n \to \infty$. In particular, $P_\tau(\nu_n-\nu) \to 0$ as $n \to \infty$. Let us show that $\nu \in Z^0_\varphi(\tau)$. Given $\varepsilon > 0$, choose n_0 sufficiently large, so that $P_\tau(\nu_n-\nu) < \varepsilon$ for all $n \ge n_0$. Each of the sets

$$A_n := \left\{x \in \mathbb{R}_+ : \frac{|\nu_n(x)-\nu(x)|\varphi(x)}{\tau(x)} > P_\tau(\nu_n-\nu)\right\}$$

is of Lebesgue measure zero. We have

$$\frac{|\nu(x)|\varphi(x)}{\tau(x)} \le \frac{|\nu_n(x)|\varphi(x)}{\tau(x)} + \varepsilon \qquad \text{for all } n \ge n_0 \text{ and all } x \notin A_n. \tag{10}$$

Set $A := \cup_n A_n$. The Lebesgue measure of A is equal to zero. Passing to the limit in (10) as $x \to \infty$, $x \notin A$, we get

$$\limsup_{x\to\infty,\, x\notin A} \frac{|\nu(x)|\varphi(x)}{\tau(x)} \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\lim_{x\to\infty,\, x\notin A} \frac{\nu(x)\varphi(x)}{\tau(x)} = 0.$$

Redefine, if necessary, $\nu(x)$ on the set A by putting $\nu(x) := 0$ for $x \in A$. We get that

$$\lim_{x\to\infty} \frac{\nu(x)\varphi(x)}{\tau(x)} = 0.$$

This means that $\nu \in Z^0_\varphi(\tau)$. Thus, the completeness of $Z^0_\varphi(\tau)$ has been established. The proof of the theorem is complete. □

We now go over to the exact asymptotic behavior of densities in $Z(\varphi)$. Consider the collection

$$Z_\varphi(\tau, L) := \left\{ \nu \in Z_\varphi(\tau) : \text{there exists } \lim_{x\to\infty} \frac{\nu(x)\varphi(x)}{\tau(x)} =: L(\nu) \right\}.$$

In what follows, we shall assume that $Z_\varphi(\tau, L)$ is a Banach subalgebra of $Z_\varphi(\tau)$ and that the following relation holds:

$$L(\mu * \nu) = L(\mu)\hat{\nu}[r_+(\varphi)] + L(\nu)\,\hat{\mu}[r_+(\varphi)] \tag{11}$$

for all $\mu, \nu \in Z_\varphi(\tau, L)$. Sufficient conditions ensuring (11) are given in the following

Theorem 2. *Let $\varphi(x)$, $x \in \mathbb{R}$, and $\tau(x)$, $x \in \mathbb{R}_+$, be submultiplicative and norming functions, respectively. Suppose that the following hypotheses are fulfilled:*

(i) *for each $y \in \mathbb{R}$, the fraction $\dfrac{\varphi(x)\tau(x-y)}{\varphi(x-y)\tau(x)}$ tends to a finite limit as $x \to \infty$;*

(ii) *either $\displaystyle\lim_{n\to\infty} \sup_{x\ge 2n} \int_n^{x/2} \frac{\tau(x-y)\tau(y)}{\tau(x)}\,dy = 0$ or $\displaystyle K_1 := \sup_{x\ge 0} \sup_{x/2\le y\le x} \frac{\tau(y)}{\tau(x)} < \infty$;*

(iii) *$\displaystyle K_2 := \sup_{x\ge 0} \sup_{y\ge x} \frac{\tau(y)}{\tau(x)} < \infty$.*

Then

$$\lim_{x\to\infty} \frac{\varphi(x)\tau(x-y)}{\varphi(x-y)\tau(x)} = \exp[r_+(\varphi)y], \tag{12}$$

the collection $Z_\varphi(\tau, L)$ is a Banach subalgebra of $Z_\varphi(\tau)$ and relation (11) holds.

Proof. Denote the left-hand side of (12)) by $g(y)$. The function $g(y)$ is Borel measurable and satisfies the equation $g(u+v) = g(u)g(v)$ for all $u, v \in \mathbb{R}$. It follows that $g(y) = \exp(\alpha y)$ for some $\alpha \in \mathbb{R}$ [5, corollary of Theorem 4.17.3]. We show by contradiction that $\alpha = r_+(\varphi)$. Suppose, e.g., that $\alpha > r_+(\varphi)$. Take $\varepsilon > 0$ such that $r_+(\varphi) < \alpha - \varepsilon$. Choose x_0 so that

$$\frac{\varphi(x)\tau(x-1)}{\varphi(x-1)\tau(x)} \ge \exp(\alpha - \varepsilon) \qquad \text{for all } x \ge x_0.$$

Then

$$\frac{\varphi(x_0+n)\tau(x_0)}{\varphi(x_0)\tau(x_0+n)} \ge \exp[n(\alpha-\varepsilon)].$$

Take the logarithm and divide both sides of the resulting inequality by n:

$$\frac{\log\varphi(x_0+n)}{n}+\frac{\log\tau(x_0)}{n}-\frac{\log\tau(x_0+n)}{n}-\frac{\log\varphi(x_0)}{n}\geq\alpha-\varepsilon.$$

Passing to the limit as $n\to\infty$ and taking into account relations (2) and (3), we see that $r_+(\varphi)\geq\alpha-\varepsilon$. A contradiction. An analogous reasoning shows that the inequality $\alpha<r_+(\varphi)$ is impossible too. This proves (12).

We now go over to proving the remaining assertions. Let $\mu,\nu\in Z_\varphi(\tau,L)$. We show that $\mu*\nu\in Z_\varphi(\tau,L)$ and equality (11) holds. Let $x\in\mathbb{R}_+$. By (1),

$$\begin{aligned}\mu*\nu(x)&=\int_{-\infty}^{0}\nu(x-y)\,\mu(dy)+\int_{-\infty}^{0}\mu(x-y)\,\nu(dy)+\int_0^x\mu(x-y)\nu(y)\,dy\\&=\int_{-\infty}^{0}\nu(x-y)\,\mu(dy)+\int_{-\infty}^{0}\mu(x-y)\,\nu(dy)\\&+\int_0^{x/2}\mu(x-y)\nu(y)\,dy+\int_0^{x/2}\nu(x-y)\mu(y)\,dy=:\sum_{k=1}^{4}I_k(x).\end{aligned}\tag{13}$$

By symmetry reasons, it suffices to establish the asymptotic behavior of $I_1(x)$ and of $I_3(x)$. We have

$$\frac{\varphi(x)I_1(x)}{\tau(x)}=\int_{-\infty}^{0}\frac{\varphi(x)\tau(x-y)}{\varphi(x-y)\tau(x)}\frac{\varphi(x-y)\nu(x-y)}{\tau(x-y)}\,\mu(dy).$$

By Definition 1 and condition (iii), the integrand tends to $\exp[r_+(\varphi)y]L(\nu)$ as $x\to\infty$ and is majorized by the $|\mu|$-integrable function $\varphi(y)K_2P_\tau(\nu)$, $y\in\mathbb{R}_-$. By Lebesgue's bounded convergence theorem,

$$\lim_{x\to\infty}\frac{\varphi(x)I_1(x)}{\tau(x)}=L(\nu)\hat{\mu}_-[r_+(\varphi)].\tag{14}$$

Similarly,

$$\lim_{x\to\infty}\frac{\varphi(x)I_2(x)}{\tau(x)}=L(\mu)\hat{\nu}_-[r_+(\varphi)].\tag{15}$$

Let us analyze the behavior of $I_3(x)$. For simplicity, we restrict ourselves to the case $\mu(x)$, $\nu(x)\geq 0$. Suppose that the first relation of condition(ii) is fulfilled. We have, for $x>2n$, that

$$\frac{\varphi(x)I_3(x)}{\tau(x)}=\Big(\int_0^n+\int_n^{x/2}\Big)\frac{\varphi(x)\tau(x-y)}{\varphi(x-y)\tau(x)}\frac{\varphi(x-y)\mu(x-y)}{\tau(x-y)}\nu(y)\,dy=:J_1(x)+J_2(x),\tag{16}$$

By both (12) and $\mu\in Z_\varphi(\tau,L)$, the integrand tends to $\exp[r_+(\varphi)]L(\mu)\nu(y)$ as $x\to\infty$. Let $\varepsilon>0$ be arbitrary. Choose an integer $n=n(\varepsilon)$ such that

$$L(\mu)\int_n^\infty\exp[r_+(\varphi)y]\nu(y)\,dy<\varepsilon,\qquad P_\tau(\mu)P_\tau(\nu)\sup_{x\geq 2n}\int_n^{x/2}\frac{\tau(x-y)\tau(y)}{\tau(x)}\,dy<\varepsilon.$$

Further,

$$\sup_{y\in[0,n]}\frac{\varphi(x)\tau(x-y)}{\varphi(x-y)\tau(x)}\frac{\varphi(x-y)\mu(x-y)}{\tau(x-y)}\leq\sup_{y\in[0,n]}\varphi(y)C_0^n<\infty;$$

see Definition 1, [5, Theorem 7.4.1] and (4). By Lebesgue's bounded convergence theorem,

$$\lim_{x\to\infty}J_1(x)=L(\mu)\int_0^n\exp[r_+(\varphi)]\nu(y)\,dy.$$

Choose $n_1 > 0$ such that for all $x > n_1$

$$\left| J_1(x) - L(\mu) \int_0^n \exp[r_+(\varphi)]\nu(y)\, dy \right| < \varepsilon.$$

Then, for all $x > \max\{n, n_1\}$,

$$\left| J_1(x) - L(\mu) \int_0^\infty \exp[r_+(\varphi)y]\nu(y)\, dy \right| < 2\varepsilon.$$

Let us represent $J_2(x)$ in the form

$$J_2(x) = \int_n^{x/2} \frac{\varphi(x)}{\varphi(x-y)\varphi(y)} \frac{\tau(x-y)\tau(y)}{\tau(x)} \frac{\varphi(x-y)\mu(x-y)}{\tau(x-y)} \frac{\varphi(y)\nu(y)}{\tau(y)}\, dy. \tag{17}$$

Obviously,

$$J_2(x) \le P_\tau(\mu)P_\tau(\nu) \int_n^{x/2} \frac{\tau(x-y)\tau(y)}{\tau(x)}\, dy < \varepsilon$$

if $x > n$. Finally, for $x > \max\{n, n_1\}$,

$$\left| \frac{\varphi(x)I_3(x)}{\tau(x)} - L(\mu) \int_0^\infty \exp[r_+(\varphi)y]\nu(y)\, dy \right| \le$$
$$\left| J_1(x) - L(\mu) \int_0^\infty \exp[r_+(\varphi)y]\nu(y)\, dy \right| + J_2(x) < 3\varepsilon.$$

Thus,

$$\lim_{x\to\infty} \frac{\varphi(x)I_3(x)}{\tau(x)} = L(\mu)\widehat{\nu}_+[r_+(\varphi)]. \tag{18}$$

The case when the second relation of condition (ii) is fulfilled is even simpler to deal with. The integrand in (16) tends to $\exp[r_+(\varphi)y]L(\mu)$ as $x \to \infty$ and is majorized by the $|\nu|$-integrable function $\varphi(y)K_1P_\tau(\mu)$. Hence, by Lebesgue's bounded convergence theorem, relation (18) also holds true in this case. Notice that condition (iii) is superfluous in this case. Similarly,

$$\lim_{x\to\infty} \frac{\varphi(x)I_4(x)}{\tau(x)} = L(\nu)\hat{\mu}_+[r_+(\varphi)]. \tag{19}$$

Summing up relations (14), (15), (18) and (19), we see that and relation (11) holds. Thus, $\mu * \nu \in Z_\varphi(\tau, L)$. The proof of Theorem 2 is complete. □

In condition (ii) of Theorem2, the relation

$$\sup_{x\ge 2n} \int_n^{x/2} \frac{\tau(x-y)\tau(y)}{\tau(x)}\, dy \qquad \text{as } n \to \infty$$

seems, at first glance, rather intractable. So let us give the following more intuitive relations: for all $y \in \mathbb{R}$,

$$\frac{\tau(x+y)}{\tau(x)} \to 1 \quad \text{as } x \to \infty, \qquad \int_0^\infty \tau(x)\, dx = 1,$$
$$\tau(x) \downarrow \quad \text{as } x \uparrow, \qquad \lim_{x\to\infty} \int_0^x \frac{\tau(x-y)\tau(y)}{\tau(x)}\, dy = 2.$$

Write for $x \ge 2n$

$$\int_0^{x/2} \frac{\tau(x-y)\tau(y)}{\tau(x)}\, dy = \left(\int_0^n + \int_n^{x/2} \right) \frac{\tau(x-y)\tau(y)}{\tau(x)}\, dy =: I_n(x) + J_n(x).$$

Obviously, $\sup_{x\ge 2n} J_n(x)\downarrow$ as $n\uparrow$. Hence there exists $\lim_{n\to\infty}\sup_{x\ge 2n} J_n(x) =: \delta \ge 0$. Show that $\delta = 0$, which means that indicated relation in condition(ii) holds true. We argue by contradiction. Suppose $\delta > 0$. Then there exists a sequence $\{x_n\}$ such that $x_n \ge 2n$ and $J_n(x_n) \ge \delta/2$. Moreover,

$$\int_0^{x_n/2} \frac{\tau(x_n-y)\tau(y)}{\tau(x_n)}\,dy = I_n(x_n) + J_n(x_n) \to 1 \qquad \text{as } n\to\infty.$$

Evidently,

$$I_n(x_n) \ge \int_0^n \tau(x)\,dx \to 1 \qquad \text{as } n\to\infty.$$

It follows that

$$\lim_{n\to\infty}\big[I_n(x_n)+J_n(x_n)\big] \ge 1 + \liminf_{n\to\infty} J_n(x_n) \ge 1 + \frac{\delta}{2}.$$

A contradiction. Hence $\delta = 0$, which was to be proved.

Theorem 3. *Let $\tau(x)$, $x\in\mathbb{R}_+$, be a bounded Borel-measurable positive function such that relations (3) and (4) are fulfilled. Suppose that one of the folowing conditions is fulfilled:*

$$\sup_{x\ge 0}\Big[\frac{1}{\tau(x)}\int_0^{x/2}\tau(x-y)\tau(y)\,dy + \sup_{y\ge x}\frac{\tau(y)}{\tau(x)}\Big] =: C_1 < \infty, \tag{20}$$

$$\sup_{x\ge 0}\ \sup_{x/2\le y\le x}\frac{\tau(y)}{\tau(x)} =: K < \infty. \tag{21}$$

Then relations (5) and (6) hold true. In particular, $\tau(x)$ is a norming function.

Proof. Let $\nu,\mu\in Z_\varphi(\tau)$. Let us show that, for a suitable choice of the constant C, the ratio $A(x) := \varphi(x)|(\nu*\mu)|(x)/\tau(x)$ does not exceed the right-hand side of (5). Without loss of generality, we may assume that the measures μ and ν are nonnegative. We have, by (13),

$$A(x) = \sum_{k=1}^4 \frac{\varphi(x)}{\tau(x)} I_k(x) =: \sum_{k=1}^4 J_k(x).$$

By symmetry reasons, it suffices to estimate $J_1(x)$ and $J_3(x)$. We have

$$J_1(x) \le \int_{-\infty}^0 \frac{\tau(x-y)}{\tau(x)}\frac{\varphi(x-y)\nu(x-y)}{\tau(x-y)}\varphi(y)\,\mu(dy) \le C_1 P_\tau(\nu)\|\mu_-\|.$$

In order to estimate $J_3(x)$, we use the inequality

$$\frac{1}{\tau(x)}\int_0^{x/2}\tau(x-y)\tau(y)\,dy \le C_1$$

and proceed as follows:

$$J_3(x) \le \int_0^{x/2}\frac{\varphi(x-y)\nu(x-y)}{\tau(x-y)}\frac{\varphi(y)\nu(y)}{\tau(y)}\,dy \le C_1 P_\tau(\nu)P_\tau(\nu).$$

Combining the estimates for $J_1(x)$ and $J_3(x)$ and similar ones for $J_2(x)$ and $J_4(x)$, we get (5).

It remains to establish (6). Let $\nu,\mu\in Z^0_\varphi(\tau)$. First consider condition (20). Given $\varepsilon > 0$ choose $\Delta > 0$ such that $\varphi(x)[\nu(x)+\mu(x)]/\tau(x) < \varepsilon$ for all $x > \Delta$. Examine $J_1(x)$ and let

$x > \Delta$. We have

$$J_1(x) = \frac{\varphi(x)}{\tau(x)} \int_{-\infty}^{0} \nu(x-y)\,\mu(dy) \le \int_{-\infty}^{0} \frac{\tau(x-y)}{\tau(x)} \frac{\varphi(x-y)\nu(x-y)}{\tau(x-y)} \varphi(y)\,\mu(dy)$$
$$\le \varepsilon C_1 \int_{-\infty}^{0} \varphi(y)\,\mu(dy) \le \varepsilon C_1 \|\mu\|_\varphi.$$

Similarly, $J_2(x) \le \varepsilon C_1 \|\mu\|_\varphi$. Consider $J_3(x)$ and let $x > 2\Delta$. We have

$$J_3(x) = \int_0^{x/2} \frac{\varphi(x)}{\varphi(x-y)\varphi(y)} \frac{\tau(x-y)\tau(y)}{\tau(x)} \frac{\varphi(x-y)\mu(x-y)}{\tau(x-y)} \frac{\varphi(y)\nu(y)}{\tau(y)}\,dy$$
$$\le \varepsilon P_\tau(\nu) P_\tau(\mu) C_1.$$

Similarly, $J_4(x) \le \varepsilon P_\tau(\nu)P_\tau(\mu)C_1$. Combining these estimates, we get

$$A(x) \le \varepsilon C_1[\|\mu\|_\varphi + \|\nu\|_\varphi + 2P_\tau(\nu)P_\tau(\mu)],$$

that is, $A(x) \to 0$ as $x \to \infty$. Now consider condition (21). Obviously, for $x > 2\Delta$,

$$A(x) \le \int_{-\infty}^{x/2} \frac{\varphi(x-y)\nu(x-y)}{\tau(x-y)} \frac{\tau(x-y)}{\tau(x)} \varphi(y)\,\mu(dy)$$
$$+ \int_{-\infty}^{x/2} \frac{\varphi(x-y)\mu(x-y)}{\tau(x-y)} \frac{\tau(x-y)}{\tau(x)} \varphi(y)\,\nu(dy)$$
$$\le \varepsilon K[P_\tau(\nu)\|\mu\|_\varphi + P_\tau(\mu)\|\nu\|_\varphi],$$

that is, $A(x) \to 0$ as $x \to \infty$. This completes the proof of the theorem. □

4. Maximal ideals

In this section we describe the structure of maximal ideals in Banach algebras introduced in Section 3.

4.1. Algebras $Z_\varphi^0(\tau)$.

Theorem 4. *Each maximal ideal M in $Z_\varphi^0(\tau)$ is the intersection of a maximal ideal M_1 in $Z(\varphi)$ with $Z_\varphi^0(\tau)$:*

$$M = M_1 \cap Z_\varphi^0(\tau). \tag{22}$$

Vice versa, if M_1 is a maximal ideal in $Z(\varphi)$, then equation (22) defines a maximal ideal in $Z_\varphi^0(\tau)$.

Proof. The algebra $Z_\varphi^0(\tau)$ is dense in $Z(\varphi)$. In fact, let $\nu \in Z(\varphi)$ be arbitrary. Denote by ν_n the element in $Z_\varphi(\tau)$ such that $\nu_n(x) = \mathbf{1}_{[0,n]}(x)\nu(x)$ and $\nu_n|_{\mathbb{R}_- \cup \{0\}} = \nu|_{\mathbb{R}_- \cup \{0\}}$. Clearly, $\nu_n \in Z_\varphi^0(\tau)$ since $P_\tau(\nu_n) < \infty$ and $\nu_n(x) = 0$ for $x > n$. Moreover,

$$\|\nu - \nu_n\|_\varphi = \int_n^\infty \varphi(x)\,|\nu|(dx) \to 0 \qquad \text{as } n \to \infty.$$

Consider now the closure $\overline{M}$ of M in $Z(\varphi)$. For all $\mu \in M$, the convolution $\nu_n * \mu$ belongs to M because M is an ideal in $Z_\varphi^0(\tau)$. Besides, $\nu_n * \mu \to \nu * \mu$ as $n \to \infty$ and hence $\nu * \mu \in \overline{M}$. It follows that $\overline{M}$ is an ideal in $Z(\varphi)$. The ideal $\overline{M}$ is contained in some maximal ideal M_1 of the algebra $Z(\varphi)$ [5, Theorem 4.13.2]. Obviously, $M_1 \cap Z_\varphi^0(\tau)$ is an ideal in $Z_\varphi^0(\tau)$ and $M \subset M_1 \cap Z_\varphi^0(\tau)$. Since M is a maximal ideal in $Z_\varphi^0(\tau)$, we have $M = M_1 \cap Z_\varphi^0(\tau)$. Vice versa, let M_1 be a maximal ideal in $Z(\varphi)$. Let $h : Z_\varphi^0(\tau) \to \mathbb{C}$ be the homomorphism with kernel

M_1 [5, Theorem 4.14.3]. The restriction $h|_{Z^0_\varphi(\tau)}$ of h to $Z^0_\varphi(\tau)$ is obviously a homomorphism on $Z^0_\varphi(\tau)$ with kernel $M_1 \cap Z^0_\varphi(\tau)$. Hence $M_1 \cap Z^0_\varphi(\tau)$ is a maximal ideal [5, Theorem 4.14.4]. The proof of the theorem is complete. □

4.2. **Algebras $Z_\varphi(\tau)$.** In this subsection we additionally assume that the norming function $\tau(s)$, $x \in \mathbb{R}_+$, is such that the following relation holds true:

$$\lim_{n\to\infty} P_\tau(\nu_n^c * \nu_n^c) = 0 \qquad \text{for all } \nu \in Z^0_\varphi(\tau), \tag{23}$$

where $\nu_n^c(x) := \nu(x)\mathbf{1}_{(n,\infty)}(x)$, $x \in \mathbb{R}_+$, and $\nu_n^c|_{\mathbb{R}_-\cup\{0\}} = \nu|_{\mathbb{R}_-\cup\{0\}}$. Consider the following condition:

$$\lim_{n\to\infty} \sup_{x\geq 2n} \frac{1}{\tau(x)} \int_n^{x/2} \tau(x-y)\tau(y)\,dy = 0. \tag{24}$$

Theorem 5. *Suppose that a norming function $\tau(x)$, $x \in \mathbb{R}_+$, satisfies either condition (21) or condition (24). Then relation (23) holds true.*

Proof. Let $\nu \in Z_\varphi(\tau)$ be arbitrary. Without loss of generality, we can assume that $\nu(x) \geq 0$ a.e. Consider the fraction

$$A_n(x) := \frac{\nu_n^c * \nu_n^c(x)\varphi(x)}{\tau(x)}.$$

First, let condition (24) be fulfilled. We have

$$A_n(x) = 2\int_n^{x/2} \frac{\nu(x-y)\nu(y)\varphi(x)}{\tau(x)}\,dy.$$

Clearly, $A_n(x) = 0$ for $x \in [0, 2n]$ and

$$A_n(x) \leq 2P_\tau(f)^2 \int_n^{x/2} \frac{\tau(x-y)\tau(y)}{\tau(x)}\,dy. \tag{25}$$

for $x \geq 2n$. Let now condition (21) be fulfilled. Then

$$A_n(x) \leq KP_\tau(f)\int_n^{\infty} \nu(y)\varphi(y)\,dy. \tag{26}$$

It follows from (25) and (26) that

$$\lim_{n\to\infty} P_\tau(\nu_n^c * \nu_n^c) = \lim_{n\to\infty}\sup_{x\geq 0} A_n(x) = 0.$$

The proof of the theorem is complete. □

Theorem 6. *Let either condition (21) or condition (24) be fulfilled. Then each maximal ideal M in $Z_\varphi(\tau)$ is the intersection of a maximal ideal M_1 in $Z(\varphi)$ with $Z_\varphi(\tau)$:*

$$M = M_1 \cap Z_\varphi(\tau). \tag{27}$$

Vice versa, if M_1 is a maximal ideal in $Z(\varphi)$, then equation (22) defines a maximal ideal in $Z_\varphi(\tau)$.

Proof. Let M be a maximal ideal in $Z_\varphi(\tau)$. Then $M_2 = M \cap Z^0_\varphi(\tau)$ is obviously an ideal in $Z^0_\varphi(\tau)$. Moreover, M_2 is a maximal ideal in $Z^0_\varphi(\tau)$. To prove this, suppose the contrary: there exists a maximal ideal M_3 in $Z^0_\varphi(\tau)$ containing M_2 as a proper subset. By Theorem 4, there exists a maximal ideal M_4 in $Z(\varphi)$ such that $M_3 = M_4 \cap Z(\varphi)$. Consider the ideal $M_4 \cap Z_\varphi(\tau)$ in $Z_\varphi(\tau)$. It contains the ideal M as a proper subset. A contradiction. Hence M_2 is a maximal ideal in $Z_\varphi(\tau)$. It follows — by Theorem 4 — that $M_2 = M_1 \cap Z_\varphi^0(\tau)$, where M_1 is a maximal ideal in $Z(\varphi)$. Obviously, $M_1 \cap Z_\varphi(\tau)$ is an ideal in $Z_\varphi(\tau)$. To prove relation (27), it suffices to show that $M \subset M_1$. We argue by contradiction. Suppose that $M \not\subset M_1$, and let $h : Z_\varphi(\tau) \to \mathbb{C}$ and $h_1 : Z(\varphi) \to \mathbb{C}$ be the homomorphisms with kernels M and M_1, respectively. Put

$$g(\nu) := h(\nu) - h_1(\nu), \qquad \nu \in Z_\varphi(\tau). \tag{28}$$

The functional g on $Z_\varphi(\tau)$ is continuous [5, Section 4.14, Corollary]. Direct verification shows that

$$g(\nu * \mu) = h_1(\nu)g(\mu) + g(\nu)h_1(\mu) + g(\nu)g(\mu), \tag{29}$$

where ν, $\mu \in S_\varphi(\tau)$ are arbitrary and

$$g(\nu) = 0 \qquad \text{for all } \nu \in Z_\varphi^0(\tau). \tag{30}$$

We have assumed that there exists an element $\nu_0 \in M \setminus M_1$. This means that $h(\nu_0) = 0$ and $h_1(\nu_0) \neq 0$. Relations (28) and (29) imply

$$g(\nu_0 * \nu_0) = -[h_1(\nu_0)]^2 \neq 0.$$

Set $\nu_{0,n}(A) := \nu_0\{A \cap (\mathbb{R} \setminus [0, n])\}$, $A \in \mathscr{B}$. By (30),

$$g(\nu_{0,n}) = g(\nu_0), \qquad n = 1, 2, \ldots \tag{31}$$

In view of (29) and (31), we verify

$$g(\nu_{0,n} * \nu_{0,n}) \to [g(\nu_0)]^2 \qquad \text{as } n \to \infty. \tag{32}$$

$$\|\nu_{0,n} * \nu_{0,n}\|_\varphi \leq \|\nu_{0,n}\|_\varphi^2 \to 0 \qquad \text{as } n \to \infty. \tag{33}$$

According to the hypotheses of the theorem,

$$P_\tau(\nu_{0,n} * \nu_{0,n}) \to 0 \qquad \text{as } n \to \infty. \tag{34}$$

Summing up relations (32)–(34), we arrive at a contradiction with the continuity of the functional g. This proves (27). The converse assertion is obvious. The proof of the theorem is complete. □

4.3. **Algebras $Z_\varphi(\tau, L)$.**

Theorem 7. *Each maximal ideal M in $Z_\varphi(\tau, L)$ is the intersection of a maximal ideal M_1 in $Z(\varphi)$ with $Z_\varphi(\tau, L)$:*

$$M = M_1 \cap Z_\varphi(\tau, L). \tag{35}$$

Vice versa, if M_1 is a maximal ideal in $Z(\varphi)$, then equation (35) defines a maximal ideal in $Z_\varphi(\tau, L)$.

Proof. Obviously, $Z_\varphi^0(\tau) \subset Z_\varphi(\tau, L)$. Let M be a maximal ideal in $Z_\varphi(\tau, L)$ and let $h : Z_\varphi(\tau, L) \to \mathbb{C}$ be the homomorphism whose kernel is M. Then $M \cap Z_\varphi^0(\tau)$ is the kernel of the homomorphism $h_0 : Z_\varphi^0(\tau) \to \mathbb{C}$ which is the restriction of h onto $Z_\varphi^0(\tau)$. Therefore, $M \cap Z_\varphi^0(\tau)$ is a maximal ideal in $Z_\varphi^0(\tau)$. By Theorem 4, we have

$$M \cap Z_\varphi^0(\tau) = M_1 \cap Z_\varphi^0(\tau),$$

where M_1 is a maximal ideal in $Z(\varphi)$. Let $h_1 : Z(\varphi) \to \mathbb{C}$ be the homomorphism with kernel M_1. Then $h(\nu) = h_1(\nu)$ for all $\nu \in Z^0_\varphi(\tau)$. Show that $M \subset M_1$, which will imply equality (35) as follows. The set $M_1 \cap Z_\varphi(\tau, L)$ is the kernel of the restriction $h_1|_{Z_\varphi(\tau,L)}$ and hence $M_1 \cap Z_\varphi(\tau, L)$ is a maximal ideal in $Z_\varphi(\tau, L)$ containing M. Therefore, M must coincide with $M_1 \cap Z_\varphi(\tau, L)$ since M is maximal in $Z_\varphi(\tau, L)$. We return to the proof of $M \subset M_1$. We argue by contradiction. Suppose the contrary, that is, $M \not\subset M_1$. Then there exists an element $\nu_0 \in Z_\varphi(\tau, L)$ such that $h(\nu_0) = 0$, $h_1(\nu_0) \neq 0$ and $L(\nu_0) \neq 0$. For arbitrary $\nu \in Z_\varphi(\tau, L)$ the following representation holds:

$$\nu = \frac{L(\nu)}{L(\nu_0)}\nu_0 + \nu', \tag{36}$$

where $\nu' \in Z^0_\varphi(\tau)$. Setting in (36) $\nu = \nu_0^{k*}$, $k = 1, 2, \ldots$, we have, by ((11)),

$$\nu_0^{k*} = k\left\{\hat{\nu}[r_+(\varphi)]\right\}^{k-1}\nu_0 + (\nu_0^{k*})'. \tag{37}$$

Since $h(\nu_0) = 0$, equality (37) implies $h[(\nu_0^{k*})'] = h_1[(\nu_0^{k*})'] = 0$ and

$$[h_1(\nu_0)]^k = k\left\{\hat{\nu}[r_+(\varphi)]\right\}^{k-1} h_1(\nu_0),$$

or

$$[h_1(\nu_0)]^{k-1} = k\left\{\hat{\nu}[r_+(\varphi)]\right\}^{k-1}, \qquad k = 2, 3, \ldots,$$

which is impossible. Hence $M \subset M_1$. The converse assertion of Theorem 7 is obvious: $M = M_1 \cap Z^0_\varphi(\tau)$ is the kernel of the homomorphism h which is the restriction onto $Z_\varphi(\tau, L)$ of the homomorphism $h_1 : Z(\varphi) \to \mathbb{C}$ with kernel M_1, and hence M is a maximal ideal in $Z_\varphi(\tau, L)$. □

4.4. **Spectrum.**

Definition 3. *Let $\mathscr{A}$ be a commutative complex Banach algebra with unity e. The spectrum $\sigma(x)$ of an element $x \in \mathcal{A}$ is called the set of all $z \in \mathbb{C}$ such that the element $ze - x$ is not invertible.*

Theorem 8. *Let the hypotheses of Theorems 4 (6, 7). be fulfilled. Let $\nu \in Z^0_\varphi(\tau)$ ($Z_\varphi(\tau)$, $Z_\varphi(\tau, L)$). Then the spectrum of the element ν in $Z^0_\varphi(\tau)$ ($Z_\varphi(\tau)$, $Z_\varphi(\tau, L)$) coincides with its spectrum $\sigma(\nu)$ in the Banach algebra $Z(\varphi)$.*

Proof. Denote by $\sigma_0(\nu)$ the spectrum of $\nu \in Z^0_\varphi(\tau)$ in $Z^0_\varphi(\tau)$. Clearly, $\sigma(\nu) \subset \sigma_0(\nu)$, since the invertibility of $z\delta_0 - \nu$ in $Z^0_\varphi(\tau)$ automatically implies its invertibility in $Z(\varphi) \supset Z^0_\varphi(\tau)$. Let $z \notin \sigma(\nu)$. Then the element $z\delta_0 - \nu$ does not belong to any maximal ideal of the Banach algebra $Z(\varphi)$ and therefore, by Theorem 4, the element $z\delta_0 - \nu$ does not belong to any maximal ideal of the Banach algebra $Z^0_\varphi(\tau)$, that is, the element $z\delta_0 - \nu$ is invertible in $Z^0_\varphi(\tau)$. Hence $z \notin \sigma_0(\nu)$, that is, $\complement\sigma(\nu) \subset \complement\sigma_0(\nu)$, which proves the desired equality $\sigma(\nu) = \sigma_0(\nu)$. This proof is valid for the remaining cases $Z_\varphi(\tau)$ and $Z_\varphi(\tau, L)$: just replace $Z^0_\varphi(\tau)$ in the above with $Z_\varphi(\tau)$ or $Z_\varphi(\tau, L)$, respectively. □

5. Analytic functions

Let $\mathscr{A}$ be a commutative complex Banach algebra with unity e, and let $\Lambda(z)$ be an analytic function in a domain $\mathscr{D}$ containing the spectrum of an element $a \in \mathscr{A}$. Then there exists an element $\Lambda(a) \in \mathscr{A}$ such that $h[\Lambda(a)] = \Lambda[h(a)]$ for each homomorphism $h : \mathscr{A} \to \mathbb{C}$ [6, § 3]. The element $\Lambda(a)$ is called the value of the analytic function $\Lambda(z)$ at the element $a \in \mathscr{A}$.

Theorem 9. *Let $\mathscr{A}$ be a complex commutative Banach algebra with unity e and multiplication $*$. Suppose that $\mathscr{L} : \mathscr{A} \to \mathbb{C}$ is a continuous linear functional with the following properties:*

(i) $\mathscr{L}(e) = 0$,

(ii) $\mathscr{L}(x * y) = \mathscr{L}(x)h(y) + \mathscr{L}(y)h(x)$ *for all* $x, y \in \mathscr{A}$,

where $h : \mathscr{A} \to \mathbb{C}$ is some fixed homomorphism. Suppose that $\Lambda(z)$ is an analytic function in a domain $\mathscr{D}$ containing the spectrum of $x \in \mathscr{A}$, and $\Lambda(x)$ is the value of $\Lambda(z)$ at the element $x \in \mathscr{A}$. Then

$$\mathscr{L}[\Lambda(x)] = \Lambda'[h(x)]\mathscr{L}(x), \tag{38}$$

where $\Lambda'(z)$ is the derivative of the function $\Lambda(z)$.

Proof. We use the reasoning in the proof of [6, Theorem 3.1] about the existence of $\Lambda(x) \in \mathscr{A}$ and track the accompanying evolution of the functional $\mathscr{L}$. If $\Lambda(z) = z^n$ for integer $n \geq 1$, then induction on n yields

$$\mathscr{L}(x^n) = nh(x)^{n-1}\mathscr{L}(x) = \Lambda'[h(x)]\mathscr{L}(x).$$

Therefore, formula (38) is also valid for polynomials. Further, if $y \in \mathcal{A}$ and there exists $y^{-1} \in \mathcal{A}$, then $\mathcal{L}(y^{-1}) = -\mathcal{L}(y)/h(y)^2$. This follows from

$$0 = \mathcal{L}(e) = \mathcal{L}(y * y^{-1}) = \mathcal{L}(y)h(y^{-1}) + \mathcal{L}(y^{-1})h(y).$$

Let now $\Lambda(z) = P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are polynomials with $Q(z) \neq 0$ for all $z \in \sigma(x)$. Then there exists $Q(x)^{-1} \in \mathcal{A}$ and $\Lambda(x) = P(x)Q(x)^{-1}$. By the already proved,

$$\begin{aligned}
\mathcal{L}[\Lambda(x)] &= \mathcal{L}[P(x)]h[Q(x)^{-1}] + \mathcal{L}[Q(x)^{-1}]h[P(x)] \\
&= \frac{\mathcal{L}[P(x)]}{h[Q(x)]} - \frac{\mathcal{L}[Q(x)]h[P(x)]}{h[Q(x)]^2} \\
&= \left\{ \frac{P'[h(x)]}{Q[h(x)]} - \frac{Q'[h(x)]P[h(x)]}{h[Q(x)]^2} \right\} \mathcal{L}(x) = \Lambda'[h(x)]\mathcal{L}(x).
\end{aligned}$$

Due to Runge's theorem [6, Theorem 2.9], there exists a sequence of rational functions $\{\Lambda_n(z)\}$, analytic in $\mathcal{D}$, such that $\Lambda_n(z) \to \Lambda(z)$ as $n \to \infty$ uniformly on compact subsets of the domain $\mathcal{D}$ and, in particular, on the spectrum $\sigma(x)$. The sequence $\{\Lambda_n'(z)\}$ also tends to $\Lambda'(z)$ uniformly on compact subsets of the domain $\mathcal{D}$ [7, Chapter 3, § 4]. By Lemma 3.2 [6], the limit $\lim_{n\to\infty} \Lambda_n(x)$ exists in $\mathcal{A}$ and is equal to $\Lambda(x)$. By continuity of the functional $\mathcal{L}$, we have $\lim_{n\to\infty} \mathcal{L}[\Lambda_n(x)] = \mathcal{L}[\Lambda(x)]$; moreover,

$$\mathcal{L}[\Lambda_n(x)] = \Lambda_n'[h(x)]\mathcal{L}(x) \to \Lambda'[h(x)]\mathcal{L}(x) \qquad \text{as } n \to \infty.$$

The proof of the theorem is complete. □

Theorem 10. *Let $\Lambda(z)$ be an analytic function in a domain $\mathscr{D}$ containing the spectrum $\sigma(\nu)$ of an element $\nu \in Z(\varphi)$, and $\Lambda(\nu)$ be the value of $\Lambda(z)$ at ν. Then the following statements hold true.*

I. *If $\nu \in Z^0_\varphi(\tau)$, then $\Lambda(\nu) \in Z^0_\varphi(\tau)$.*

II. *If $\nu \in Z_\varphi(\tau)$ and condition (23)) is fulfilled, then $\Lambda(\nu) \in Z_\varphi(\tau)$.*

III. *If $Z_\varphi(\tau, L)$ is a Banach sub algebra of $Z_\varphi(\tau)$ and for all ν, $g \in Z_\varphi(\tau, L)$ equality ((11)) is verified, then $\nu \in Z_\varphi(\tau, L)$ implies $\Lambda(\nu) \in Z_\varphi(\tau, L)$ and*

$$L[\Lambda(\nu)] = \Lambda' \{\widehat{\nu}[r_+(\varphi)]\} L(\nu), \tag{39}$$

where $\Lambda'(z)$ is the derivative of $\Lambda(z)$.

Proof. Let us establish $\Lambda(\nu) \in Z^0_\varphi(\tau)$, and so on if ν is an element of other Banach algebras. The spectra of the element ν in the enumerated algebras coincide with $\sigma(\nu)$. Consequently, the values of $\Lambda(z)$ at ν exist in these algebras. In order to prove equality (39) it suffices to apply Theorem 9 with $\mathscr{A} := Z_\varphi(\tau, L)$, $x := \nu$, $\mathscr{L}(x) := L(\nu)$ and

$$h(\nu) := \widehat{\nu}[r_+(\varphi)], \qquad \nu \in Z_\varphi(\tau, L).$$

□

Remark 1. Equality (39) can be established using the proof of the similar equality (2) in [1, Theorem 1], which is based on the representation of $\Lambda(\nu)$ in the form of a contour integral:

$$\Lambda(\nu)\frac{1}{2\pi i}\oint_\Gamma (z\delta_0 - \nu)^{-1}\Lambda(z)\, dz;$$

here Γ is a contour such that the spectrum of $\nu \in Z_\varphi(\tau, L)$ lies inside Γ. However, we have given a proof of equality (39), which does not include arguments with contour integrals.

6. An application

In this section, we give a quite simple example of how these algebras can be used. Let $\{X_k\}_{k=1}^\infty$ be a sequence of independent, identically distributed random variables with a common nonarithmetic distribution F. Set $S_n = \sum_{k=1}^n X_k$, $n \ge 1$, $S_0 = 0$. We consider the distribution of the first ascending ladder height of the random walk $\{S_n\}$ and describe the asymptotic behavior of its density. Suppose the random walk $\{S_n\}$ drifts to $+\infty$, that is, with probability one $S_n \to +\infty$ as $n \to \infty$. Denote by F^{n*} the n-th convolution power of F:

$$F^{0*} := \delta_0, \quad F^{1*} := F, \quad F^{(n+1)*} := F^{n*} * F, \quad n \ge 1.$$

Put $\mathscr{T}_+ := \min\{n \ge 1 : S_n > 0\}$. The random variable $\mathscr{H}_+ := S_{\mathscr{T}_+}$ is called the *first ascending ladder height*. Similarly, $\mathscr{T}_- := \min\{n \ge 1 : S_n \le 0\}$ and $\mathscr{H}_- := S_{\mathscr{T}_-}$ is the *first weak descending ladder height*. We have the factorization identity (the symbol E stands for "expectation") [8, Section XVIII.3]

$$1 - \xi \mathsf{E}(e^{sX_1}) = \big[1 - \mathsf{E}\big(\xi^{\mathscr{T}_-} e^{s\mathscr{H}_-}\big)\big]\big[[1 - \mathsf{E}\big(\xi^{\mathscr{T}_+} e_+^{s\mathscr{H}}\big)\big], \quad |\xi| \le 1, \quad \Re s = 0. \tag{40}$$

Denote by $F_\pm$ the distributions of the random variables $\mathscr{H}_\pm$, respectively. It follows from the identity (40) that

$$\delta_0 - F = (\delta_0 - F_-) * (\delta_0 - F_+). \tag{41}$$

Let $U_- := \sum_{k=0}^\infty F_-^{k*}$ be the renewal measure generated by the distribution F_- and $\varphi(x)$, $x \in \mathbb{R}$, be a submultiplicative function such that $r_-(\varphi) \le 0 \le r_+(\varphi)$.

Theorem 11. *Suppose that $F \in Z^0_\varphi(\tau)$, where φ and τ satisfy the hypotheses of Theorem 1. Then $F_+ \in Z^0_\varphi(\tau)$. In particular,*

$$f_+(x) = o\Big[\frac{\tau(x)}{\varphi(x)}\Big] \qquad \textit{as } x \to \infty.$$

Proof. As mentioned in [8, Chapter XVIII, § 4, Example a)], drifting of $\{S_n\}$ to $+\infty$ takes place when the distribution F_- is defect: $F_-(\mathbb{R}) < 1$. Hence U_- is a finite measure and

$$\widehat{U}_-(s) = \sum_{n=0}^{\infty} \widehat{F}^n_-(s = \frac{1}{1 - \widehat{F}_-(s)}, \qquad \Re s \geq 0.$$

It follows from (41) that

$$1 - \widehat{F}_+(s) = \frac{1 - \widehat{F}(s)}{1 - \widehat{F}_-(s)} = [1 - \widehat{F}(s)]\widehat{U}_-(s),$$

whence

$$F_+ = F * U_- - U_- - \delta_0. \tag{42}$$

Obviously, $U_- \in Z^0_\varphi(\tau)$ since the measure U_- is concentrated on $(-\infty, 0]$. Hence (42) implies $F_+ \in Z^0_\varphi(\tau)$. The proof of the theorem is complete. □

Theorem 12. *Suppose that $F \in Z_\varphi(\tau)$, where φ and τ satisfy the hypotheses of Theorem 1. Then $F_+ \in Z_\varphi(\tau)$. In particular,*

$$f_+(x) = O\Big[\frac{\tau(x)}{\varphi(x)}\Big] \qquad \text{as } x \to \infty.$$

Proof. Replace $Z^0_\varphi(\tau)$ with $Z_\varphi(\tau)$ in the proof of the preceding theorem. □

Theorem 13. *Suppose that $F \in Z_\varphi(\tau, L)$, where φ and τ satisfy the hypotheses of Theorem 2. Then $F_+ \in Z_\varphi(\tau, L)$. Moreover,*

$$L(F_+) = \frac{L(F)}{1 - \widehat{F}_-[r_+(\varphi)]}. \tag{43}$$

Proof. Acting as in the proof of Theorem 11, we obtain $U_- \in Z_\varphi(\tau, L)$. Obviously, $L(U_-) = 0$. Equality (43) now follows from (42) and (11). □

Remark 2. Provided that $L(F) > 0$, relation (43) may be rewritten in other terms as follows:

$$f_+(x) \sim \frac{f(x)}{1 - \widehat{F}_-[r_+(\varphi)]} \qquad \text{as } x \to \infty,$$

where $f_+(x)$ is the density of F_+ and $f(x)$ is the density of the restriction $F|_{(0,\infty)}$.

Remark 3. Notice that in the Theorems 11–13, we do not require that the underlying distribution F be absolutely continuous of the *whole* line.

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