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EXISTENCE AND REGULARITY OF SOLUTIONS IN α-NORM FOR SOME SECOND ORDER PARTIAL NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH FINITE DELAY IN BANACH SPACES

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Abstract. The purpose of this work is to investigate the existence and regularity of solutions in the α-norm for some second order partial neutral functional differential equations in Banach spaces with finite delay using fractional α-power and the theory of the cosine family. As result, we obtain a generalization of work of Herman R. Henriquez *et al.* (*Journal of Mathematics, Vol. 41, No. 6 (2011)*) without alpha norm and regularity. Our results extend and complement many other important results in the literature. Finally, a concrete example is given to illustrate the application of the main results.

1. Introduction

In this work, we study the existence and regularity in α-norm of solutions for the following second order neutral partial functional differential equation

$$\begin{cases} \dfrac{d^2}{dt^2}[u(t) - g(t, u_t)] = Au(t) + f(t, u_t, u'_t) \text{ for } t \geq 0 \\ u_0 = \varphi \in \mathcal{C}_\alpha \\ u'_0 = \varphi' \in \mathcal{C}_\alpha \end{cases} \tag{1.1}$$

where A is the (possibly unbounded) infinitesimal generator of strongly continous cosine family of linear operators in X. $\mathcal{C}_\alpha = C^1([-r,0], D((-A)^\alpha))$, $0 < \alpha < 1$, denotes the space of continuous differentiable functions from $[-r,0]$ into $D((-A)^\alpha)$, $(-A)^\alpha$ is the fractional α-power of A. This operator $((-A)^\alpha, D((-A)^\alpha))$ will be describe later. C_α is endowed with the following norm $\|h\|_{\mathcal{C}_\alpha} = \|h\|_\alpha + \|h'\|_\alpha$ for all $h \in \mathcal{C} = C^1([-r,0], X)$,where $\|\varphi\|_\alpha = \sup\limits_{-r\leq\theta\leq 0} |\varphi(\theta)|_\alpha$. The

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norm $|.|_\alpha$ will be specified later. For every $t \geq 0$, u_t denotes the history function of $\mathcal{C}_\alpha$ defined by

$$u_t = u(t+\theta) \text{ for } \theta \in [-r, 0],$$

$f : \mathbb{R}^+ \times \mathcal{C}_\alpha \times \mathcal{C}_\alpha \to X$ and $g : \mathbb{R}^+ \times \mathcal{C}_\alpha \to X_\alpha$ are given functions.

In [9] the authors study firstly the abstract semi-linear second order initial value problem and secondly they unify and simplify some ideas from strongly continuous cosine families of linear operators in Banach spaces.

In [1], the authors reveal three properties of cosine families, distinguishing them from semigroups of operators.

Recently, in [12], Zabsonre Issa *et al.* considered the following nonlinear second order differential equation

$$\begin{cases} u'(t) = Au(t) + f(t, u_t, u'_t) \text{ for } t \geq 0, \\ u_0 = \varphi \in \mathcal{C} = C^1([-r, 0], X), \\ u'_0 = \varphi' \in \mathcal{C}. \end{cases} \tag{1.2}$$

Using the cosine family theory and the Banach fixed point Theorem, the authors established the existence and regularity of solutions.

More recently, in [7], D. Mbainadji *et al.* considered the following second order partial neutral functional differential equation:

$$\begin{cases} \dfrac{d}{dt}[u'(t) - g(t, u_t)] = Au(t) + f(t, u_t, u'_t) \text{ for } t \geq 0, \\ u_0 = \varphi \in \mathcal{C}_\alpha, \\ u'_0 = \varphi' \in \mathcal{C}_\alpha. \end{cases} \tag{1.3}$$

The authors investigated the existence and regularity of solutions in the α-norm using cosine family theory and Schauder's fixed point theorem.

The present work is motived by the papers of Issa Zabsonre *et al.* [13] and Travis and Webb [9]. This is paper is a generalization of [10] and a continuation of [5].

Using the theory of strongly continuous cosine families of linear operators in Banach space, in this paper we will prove the existence of mild and strict solutions. The organisation of this paper is as follows, in section 2 we recall some preliminary results on cosine families and fractional α-power, in section 3 we prove the existence and uniqueness of the mild solution in the α-norm for (1.1). In section 4 we study the regularity of the solutions, we give sufficient conditions to obtain the existence of a strict solution. Finally, in Section 5 we illustrate our results by examining an example.

2. Preliminary Results

Let $(X, \|.\|)$ be a Banach space and α be a constant such that $0 < \alpha < 1$ and $-A$ be the infinitesimal generator of strongly continuous $(C(t))_{t\geq 0}$ on X. We assume without loss of generality that $0 \in \rho(-A)$. Note that if the assumption $0 \in \rho(-A)$ is not satisfied, one can substitute the operator $-A$ by the operator $(-A-\sigma I)$ with σ large enough such that $0 \in \rho(-A-\sigma I)$. This allows us to define the fractional power $(-A)^{\alpha}$ for $0<\alpha<1$, as a closed linear invertible operator with domain $D((-A)^{\alpha})$ dense in X. The closeness of A^{α} implies that $D((-A)^{\alpha})$, endowed with the graph norm of $(-A)^{\alpha}$, $|x| = \|x\| + \|(-A)^{\alpha}x\|$, is a Banach space. Since $(-A)^{\alpha}$ is invertible, its graph norm $|.|$ is equivalent to the norm $|x|_{\alpha} = \|(-A)^{\alpha}x\|$. Thus, $D((-A)^{\alpha})$ equipped with the norm $|.|_{\alpha}$, is a Banach space, which we denote by X_{α}.

Definition 1. *[9] A one parameter family $\{C(t), t \in \mathbb{R}\}$ of bounded linear operators mapping the Banach space X into itself is called a strongly continuous cosine family if and only if*

i) $C(s+t) + C(s-t) = 2C(s)C(t)$ *for all* $s, t \in \mathbb{R}$

ii) $C(0) = I$

iii) $C(t)x$ *is continuous on* $\mathbb{R}$ *for each fixed* $x \in X$.

The strongly continuous sine family $\{S(t), t \in \mathbb{R}\}$ associated to the given strongly continuous cosine family $\{C(t), t \in \mathbb{R}\}$ by

$$S(t)x = \int_0^t C(s)x ds, \text{ for } x \in X, t \in \mathbb{R} \tag{2.1}$$

Definition 2. *The infinitesimal generator of strongly continuous cosine family $\{C(t), t \in \mathbb{R}\}$ is the operator $A : X \longrightarrow X$ define by*

$$Ax = \frac{d^2C(t)x}{dt^2}\Big|_{t=0}.$$

$D(A) = \{x \in X : \ C(t)x$ *is a twice continuously differentiable function of* $t\}$.

We shall also make use of the set

$$E = \{x : C(t)x \text{ is a once continuously differentiable function of t}\}$$

Lemma 1. *Let $C(t), \in \mathbb{R}$ be a strongly continuous cosine family in X with infinitesimal generator A. The following are true.*

i) $D(A)$ *is dense in* X *and* A *is closed operator in* X;

ii) *if* $x \in X$ *and* $s, r \in \mathbb{R}$ *then* $z = \int_s^r = S(u)xdu \in D(A)$ *and* $Az = C(s)x - C(r)x$;

iii) *if* $x \in X$, $s, r \in \mathbb{R}$ *then* $z = \int_0^s \int_0^r C(u)C(v)xdudv \in D(A)$ *and*

$Az = \frac{1}{2}(C(s+r)x - C(s-r)x)$;

iv) *if* $x \in X$, $S(t)x \in E$;

v) *if* $\in X$, *the* $S(t)x \in D(A)$ *and* $\frac{dC(t)}{dt} = AS(t)x$:

vi) *if* $x \in D(A)$, *then* $C(t)x \in D(A)$ *and* $\frac{d^2C(t)}{dt^2} = AC(t)x = C(t)Ax$;

vii) *if* $x \in E$, *then* $\lim_{t\to 0} AS(t) = 0$;

viii) *if* $x \in E$, *then* $S(t)x \in D(A)$ *and* $\frac{d^2S(t)}{dt^2} = AS(t)x$;

ix) if $x \in D(A)$*, then* $S(t)x \in D(A)$ *and* $AS(t)x = S(t)Ax$*;*
x) $C(t+s) + C(t-s) = 2AS(t)S(s)$ *for all* $s, t \in \mathbb{R}$*.*

In [9], for $0 < \alpha < 1$ the fractional powers $(-A)^{\alpha}$ exist as closed linear operators in X,

$D((-A)^{\alpha}) \subset D((-A)^{\beta})$ for $0 \leq \beta \leq \alpha \leq 1$ and $(-A)^{\alpha}(-A)^{\beta} = (-A)^{\alpha+\beta}$ for $0 \leq \alpha + \beta \leq 1$.

For our objevtive we assume that
($\mathbf{H}_0$) $-A$ is the infinitesimal generator of a strongly continuous cosine family of linear operators on a Banach space X.

By Lemma 1, ($\mathbf{H}_{0)}$) implies that the operator A is densely defined in X, i.e $\overline{D(A)} = X$. We have the following result

Lemma 2. *[9] Assume that* (H_0) *hols. Then there are constants* $M \geq 1$ *and* $\omega \geq 0$ *such that* $\|C(t)\| \leq Me^{\omega|t|}$ *and* $\|S(t_1) - S(t_2)\| \leq M\left|\int_{t_1}^{t_2} e^{\omega|s|}ds\right|$*, for all* $t_1, t_2 \in \mathbb{R}$*.*

From previous inequality, since $S(0) = 0$ we can deduce that

$$\|S(t)\| \leq \frac{M}{\omega}e^{\omega t} \text{ for } t \in \mathbb{R}^+$$

In the sequel, let us pose $M_1 = \max\left(M, \dfrac{M}{\omega}\right)$.

Theorem 1. *[9] If* $k : \mathbb{R}^+ \to X$ *is continous,* $h : \mathbb{R}^+ \to X$ *is continuous and* u *is a solution of equation* (1.1)*, the* u *is a solution of integral equation*

$$u(t) = C(t)x + S(t)y + \int_0^t AS(t-s)k(s)ds + \int_0^t S(t-s)h(s)ds.$$

($\mathbf{A}_1$): For $0 < \alpha < 1$, $(-A)^{\alpha}$ maps onto X and $1-1$, so that $D((-A)^{\alpha})$ endowed with the norm $|x|_{\alpha} = \|(-A)^{\alpha}x\|$ is a Banach space. We denote by X_{α} this space. In addition we assume that A^{-1} is compact. To establish our results, we need the following Lemmas.

Lemma 3. *[10] Assume that* ($\boldsymbol{H}_0$) *holds. The following are true*
(i) For $0 < \alpha < 1$*,* $(-A)^{\alpha}$ *is compact if and only if* A^{-1} *is compact.*
(ii) For $0 < \alpha < 1$*, and* $t \in \mathbb{R}$ $(-A)^{\alpha}C(t) = C(t)(-A)^{\alpha}$ *and* $(-A)^{\alpha}S(t) = S(t)(-A)^{\alpha}$

Recall from [4], $(-A)^{\alpha}$ is given by the following formula

$$(-A)^{\alpha} = \frac{\sin \pi\alpha}{\pi}\int_0^{+\infty} t^{-\alpha}(tI - A)^{-1}dt.$$

Lemma 4. *[10] Assume that* ($\boldsymbol{H}_0$) *holds. Let* $v : \mathbb{R} \longrightarrow x$ *such that* v *is continuously differentiable and let* $q(t) = \int_0^t S(t-s)v(s)ds$*. Then*
(i) q *is twice continuously differentiable and for* $t \in \mathbb{R}$*,* $q(t) \in D(A)$*,*

$$q'(t) = \int_0^t C(t-s)v(s)ds$$

and

$$q''(t)=\int_0^t C(t-s)v'(s)ds+C(t)v(0)=Aq(t)+v(t)$$

(ii) For $0<\alpha<1$ and $t\in\mathbb{R}$, $(-A)^{\alpha-1}q'(t)\in E$.

Theorem 2. *(Heine's theorem)*
Let f be a continous function on a compact set K, then f is uniformly continuous on K.

Theorem 3. *(Arzela-Ascoli theorem)*
Let (X,d_X) and (Y,d_Y) be compact metric spaces, $C(X,Y)$ be the set of continuous functions from X to Y and Let $\mathcal{F}$ be q subset of $C(X,Y)$. If $\mathcal{F}$ is closed and equicontinuous then, it is compact.

Let $\mathbb{E}$ be a Banach space. We define

$$\chi(\Omega)=\inf\{\varepsilon>0:\Omega \text{ has finite cover diameter } <\varepsilon\},$$

where $\chi(\Omega)$ is a Kuratowski measure of noncompactness of a set $\Omega\subset\mathbb{E}$.

Definition 3. *A mapping $\mathcal{K}$ from a set $\mathcal{C}$ in Banach space $\mathbb{E}$ is called a condensing operator if it is continuous and for every bounded noncompact set $\Omega\subseteq\mathcal{C}$ the inequality holds*

$$\chi[\mathcal{K}(\Omega)]<\chi(\Omega).$$

Theorem 4. *(Sadovskii's fixed point theorem)*
If a condensing operator $\mathcal{K}$ maps a bounded convex set $\mathcal{C}$ of Banach space $\mathbb{E}$ into itself (i.e $\mathcal{K}(\mathcal{C})\subseteq\mathcal{C}$), then $\mathcal{K}$ has least one fixed point in $\mathcal{C}$.

3. Existence of mild solutions

Definition 4. *A continuous function $u:]-r,+\infty[\to X_\alpha$, for $b>0$ is said to a mild solution of equation (1.1) if*

$$i)\ u(t) = C(t)(\varphi(0)-g(0,\varphi))+S(t)(\varphi'(0)-g'(0,\varphi))+g(t,u_t)+\int_0^t AS(t-s)g(s,x_s)ds$$
$$+\int_0^t S(t-s)f(s,u_s,u'(s))ds \text{ for } t\in[0,b]$$

ii) $u_0=\varphi$, $u'_0=\varphi'$.

Proposition 1. *Assume that $(\boldsymbol{H}_{0)})$ holds. If u is a solution of equation (1.1), then*
(3.1)
$$u(t)=C(t)(\phi(0)-g(0,\phi)+S(t)(\phi'(0)-\eta)+\int_0^t AS(t-s)g(s,u_s)ds+\int_0^t S(t-s)f(s,u_s,u'_s)ds$$

Proof. It is just the consequence of Theorem 1. In fact, let us pose $k(t)=g(t,u_t)$ and $h(t)=f(t,u_t,u'_t)$ for $t\geq 0$. The we get the desired results.■

Remark 1. *The converse is not true. In fact if u satisfies equation (3.1), u may be not twice continuously differentiable, that is why we distinguish between mild and strict solutions*

Definition 5. *A continuous function* $u :]-r, +\infty[\to X_\alpha$*, for* $b > 0$ *is said to a mild solution of equation* (1.1) *if*

i) $u(.) \in C^1([0,b], X_\alpha)$
ii) $\dfrac{d^2}{dt^2}[u(t) - g(t,u_t)] = Au(t) + f(t, u_t, u'_t),\ t \in [0,b]$
iii) $u_0 = \varphi,\ u'_0 = \varphi'$.

In the following, we give a local existence of mild solutions of equation(1.1). We will use the Sadovskii's fixed point theorem which generalize the Schauder's fixed point and the contraction principle.
For this purpose, we make this following assumptions.
$(\mathbf{H}_1)$The function $f : [0,b] \times \mathcal{C}_\alpha \to X$ satisfies the following conditions

i) $f : [0,b] \times \mathcal{C}_\alpha \times \mathcal{C}_\alpha \to X$ is continuously differentiable.
ii) There exists a continuous nondecreasing function $\beta : [0,b] \to \mathbb{R}^+$ such that

$$\|f(t,\varphi,\varphi')\| \leq \beta(t)\|\varphi\|_\alpha \text{ for } (t,\varphi) \in [0,b] \times \mathcal{C}_\alpha.$$

$(\mathbf{H}_2)$ $g : [0,b] \times \mathcal{C}_\alpha \to X_\alpha$ is continuously differentiable and for each $b > 0$ there exist $0 < L_g < 1$ such that

(i) $|g(t,\varphi) - g(t,\psi)|_\alpha \leq L_g|\varphi - \psi\|_\alpha$ for every $t \in [0,b]$ and $\varphi, \psi \in \mathcal{C}_\alpha$.

(ii) $\dfrac{d}{dt} g(t,u_t)\Big|_{t=0} = \eta$

$(\mathbf{H}_3)$ A^{-1} is compact.

Theorem 5. *Assume that* $(\boldsymbol{H}_0)$*,* $(\boldsymbol{H}_1)$*,* $(\boldsymbol{H}_2)$*,* $(\boldsymbol{H}_3)$ *and hold. Let* $\varphi \in \mathcal{C}_\alpha$ *such that* $\varphi(0) - g(,\varphi)$*,* $\varphi'(0) - \eta \in E$ *and assume that*

$$L_g(1 + M_1 e^{\omega b}) + \|(-A)^{\alpha-1}\| \sup_{t\in[0,b]} \Big[\big(\beta(t)(1 + 2M_1 e^{\omega b}) + M e^{\omega b}\Big] < 1.$$

Then equation (1.1) *has at least one mild solution on* $[0,b]$.

Proof. Let $k > \|\varphi\|_{\mathcal{C}_\alpha}$, we define the following set

$$B_k = \{u \in C([0,b], X_\alpha) : \ u(0) = \varphi(0) \text{ and } |u|_\infty \leq k\},$$

B_k is a closed subet of $C([0,b], X_\alpha)$, where $C([0,b], X_\alpha)$ is the space of continuous functions from $[0,b]$ to X_α equipped with the norm topology

$$|u|_\infty = \sup_{t\in[0,b]} |u(t)|_\alpha.$$

For $u \in B_k$, define the $\widetilde{u}(t) : [0,b] \to X_\alpha$ by

$$\widetilde{u}(t) = \begin{cases} u(t) \text{ for } t \in [0,b] \\ \\ \varphi(t) \text{ for } t \in [-r,0]. \end{cases}$$

The function $t \to \widetilde{u}_t$ is continuous from $[0,b]$ to $\mathcal{C}_\alpha$. Now, define the operator $\mathcal{K}$ on B_k by

$$\begin{aligned}\mathcal{K}(u)(t) &= C(t)(\varphi(0)-g(0,\varphi)) + S(t)(\varphi'(0)-\eta) + g(t,\widetilde{u}_t)\\ &\quad + \int_0^t AS(t-s)g(s,\widetilde{u}_s)ds + \int_0^t S(t-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)ds \text{ for } t\in[0,b].\end{aligned}$$

It is sufficient to show that $\mathcal{K}$ has a fixed point in B_k. We give the proof in several steps.

Step 1: There is a positive $k > \|\varphi\|_\alpha$ such that $\mathcal{K}(B_k) \subset B_k$.

If not, then for each $k > \|\varphi\|_\alpha$, there exist $u_k \in B_k$ and $t_k \in [0,b]$ such that $|(\mathcal{K}u_k)(t_k)|_\alpha > k$.

$$\begin{aligned}
k < |(\mathcal{K}u_k)(t_k)|_\alpha &= \Big|C(t_k)(\varphi(0)-g(0,\varphi)) + S(t_k)(\varphi'(0)-\eta) + g(t_k,\widetilde{u}_k) + \int_0^{t_k} AS(t_k-s)g(s,\widetilde{u}_s)ds\\
&\quad + \int_0^{t_k} S(t_k-s)f(s,\widetilde{u}_s)ds\Big|_\alpha\\
&< |C(t_k)(\varphi(0)-g(0,\varphi))|_\alpha + |S(t_k)(\varphi'(0)-\eta)|_\alpha + |g(t_k,\widetilde{u}_k)|_\alpha\\
&\quad + \Big|\int_0^{t_k} AS(t_k-s)g(s,\widetilde{u}_s)ds\Big|_\alpha + \Big|\int_0^{t_k} S(t_k-s)f(s,\widetilde{u}_s)ds\Big|_\alpha\\
&< |C(t_k)(\varphi(0)-g(0,\varphi))|_\alpha + |S(t_k)(\varphi'(0)-\eta)|_\alpha + |g(t_k,\widetilde{u}_k)-g(t_k,0)|_\alpha + |g(t_k,0)|_\alpha\\
&\quad + \Big|\int_0^{t_k} \frac{d}{ds}\big(C(t_k-s)g(s,\widetilde{u}_s)\big)ds - \int_0^{t_k} C(t_k-s)\frac{d}{ds}\big(g(s,\widetilde{u}_s)\big)ds\Big|_\alpha\\
&\quad + \Big\| -(-A)^{\alpha-1}\int_0^{t_k} AS(t_k-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)ds\Big\|\\
&< |C(t_k)(\varphi(0)-g(0,\varphi))|_\alpha + |S(t_k)(\varphi'(0)-\eta)|_\alpha + |g(t_k,\widetilde{u}_k)-g(t_k,0)|_\alpha + |g(t_k,0)|_\alpha\\
&\quad + \Big|\int_0^{t_k} \frac{d}{ds}\big(C(t_k-s)g(s,\widetilde{u}_s)\big)ds - \int_0^{t_k} C(t_k-s)\frac{d}{ds}\big(g(s,\widetilde{u}_s)\big)ds\Big|_\alpha\\
&\quad + \Big\|(-A)^{\alpha-1}\Big[\int_0^{t_k} \frac{d}{ds}\big(C(t_k-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)\big)ds - \int_0^{t_k} C(t_k-s)\frac{d}{ds}\big(f(s,\widetilde{u}_s,\widetilde{u'}_s)\big)\Big]\Big\|\\
&< |C(t_k)(\varphi(0)-g(0,\varphi))|_\alpha + |S(t_k)(\varphi'(0)-\eta)|_\alpha + L_g|\widetilde{u}_{t_k}|_\alpha + \sup_{s\in[0,b]}|g(s,0)|_\alpha\\
&\quad + |g(t_k,\widetilde{u}_{t_k}) - C(t_k)g(0,\widetilde{u}_0)|_\alpha + M_1e^{\omega b}|g(t_k,\widetilde{u}_{t_k}) - g(0,\widetilde{u}_0)|_\alpha\\
&\quad + \|(-A)^{\alpha-1}\|\Big(\|f(t_k,\widetilde{u}_{t_k},\widetilde{u'}_{t_k})\| + \|C(t_k)f(0,\widetilde{u}_0,\widetilde{u'}_0)\|\\
&\quad + M_1e^{\omega b}\|f(t_k,\widetilde{u}_{t_k},\widetilde{u'}_{t_k}) - f(0,\widetilde{u}_0,\widetilde{u'}_0)\|\Big)\\
&< M_1e^{\omega b}\Big(|(\varphi(0)-g(0,\varphi))|_\alpha + |(\varphi'(0)-\eta)|_\alpha\Big) + L_g\|\widetilde{u}_{t_k}\|_\alpha + 2\sup_{s\in[0,b]}|g(s,0)|_\alpha
\end{aligned}$$

$$+M_1e^{\omega b}L_g\|\widetilde{u}_{t_k}\|_\alpha+\|(-A)^{\alpha-1}\|\Big[\big(\beta(t_k)+M_1e^{\omega b})|\widetilde{u}_{t_k}|_\alpha+2M_1e^{\omega b}\beta(0)\|\widetilde{u}_0\|_\alpha\Big]$$

Since $|\widetilde{u}_t|_\infty\leq k$ for all $t\in[0,b]$ and $u\in B_k$. The we have

$$k \;<\; M_1e^{\omega b}\Big(|(\varphi(0)-g(0,\varphi))|_\alpha+|(\varphi'(0)-\eta)|_\alpha\Big)+L_g\big(1+M_1e^{\omega b}\big)k+2\sup_{s\in[0,b]}|g(s,0)|_\alpha$$

$$++M_1e^{\omega b}|g(0,\widetilde{u}_0)|_\alpha\|(-A)^{\alpha-1}\|\sup_{t\in[0,b]}\Big[\big(\beta(t)(1+2M_1e^{\omega b})+M_1e^{\omega b}\Big]k$$

Divinding above sides of obove inequality by k, it follows that

$$1 \;<\; \frac{M_1e^{\omega b}\Big(|(\varphi(0)-g(0,\varphi))|_\alpha+|(\varphi'(0)-\eta)|_\alpha\Big)}{k}+L_g\big(1+M_1e^{\omega b}\big)$$

$$+\frac{2\sup_{s\in[0,b]}|g(s,0)|_\alpha}{k}+\frac{M_1e^{\omega b}|g(0,\widetilde{u}_0)|_\alpha}{k}+\|(-A)^{\alpha-1}\|\sup_{t\in[0,b]}\Big[\big(\beta(t)(1+2M_1e^{\omega b})+M_1e^{\omega b}\Big].$$

When $k\to 0$, we have

$$1<L_g\big(1+M_1e^{\omega b}\big)+\|(-A)^{\alpha-1}\|\sup_{t\in[0,b]}\Big[\big(\beta(t)(1+2M_1e^{\omega b})+M_1e^{\omega b}\Big],$$

which gives contradiction.
Now we decompose $\mathcal{K}$ as follows $\mathcal{K}=\mathcal{K}_1+\mathcal{K}_2$, where

$$\mathcal{K}_1(u)(t)=g(t,\widetilde{u}_t)\text{ for }t\in[0,b]$$

and

$$\mathcal{K}_2(u)(t)=C(t)(\varphi(0)-g(0,\varphi))+S(t)(\varphi'(0)-\eta)+\int_0^t AS(t-s)g(s,\widetilde{u}_s)+\int_0^t S(t-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)\text{ for }t\in[0,b]$$

Then, we shall show $\mathcal{K}_1$ is a strict contraction and $\mathcal{K}_2$ is continous.
Step 2: $\mathcal{K}_1$ is strict contraction and $\mathcal{K}_2$ is continuous

For $t\in[0,b]$ and $u,v\in B_k$ and by $(\mathbf{H}_2)$ we have

$$\begin{aligned}|\mathcal{K}_1(u)(t)-\mathcal{K}_1(v)(t)|_\alpha &= |g(t,\widetilde{u}_t)-g(t,\widetilde{v}_t)|_\alpha\\ &\leq L_g\|\widetilde{u}_t-\widetilde{v}_t\|_\alpha\\ &\leq L_g\sup_{0\leq\tau\leq b}|u(\tau)-v(\tau)|_\alpha.\end{aligned}$$

Then

$$|\mathcal{K}_1(u)(t)-\mathcal{K}_1(v)(t)|_\infty\leq L_g|u-v|_\infty$$

This means $\mathcal{K}_1$ is a strict contraction.

Let $(u^n)_n\in B_k$ with $u^n\to u$ in B_k. Then, the set

$$\Delta=\{(s,\widetilde{u}^n_s,\widetilde{u'}^n_s)),\;(s,\widetilde{u}_s,\widetilde{u}_s)):\;s\in[0,b],\;n\geq 1\}$$

and

$$\wedge = \{(s,\widetilde{u}^n_s),\ (s,\widetilde{u}_s):\ s\in[0,b],\ n\geq 1\}$$

are compact respectively in $[0,b]\times\mathcal{C}_\alpha\times\mathcal{C}_\alpha$ and $[0,b]\times\mathcal{C}_\alpha$. Heine's theorem implies that f and g are uniformly continuous respectively in Δ and $\wedge$. Then, we have

$$\begin{aligned}
&|\mathcal{K}_2(u^n)(t)-\mathcal{K}_2(u)(t)|_\infty\\
\leq\ & \sup_{t\in[0,b]}\Big|\int_0^t AS(t-s)\Big(g(s,\widetilde{u}^n_s)-g(s,\widetilde{u}_s)\Big)ds\Big|_\alpha\\
&+\sup_{t\in[0,b]}\Big\|-(-A)^{\alpha-1}\int_0^t AS(t-s)\Big(f(s,\widetilde{u}^n_s,\widetilde{u'}^n_s)-f(s,\widetilde{u}_s,\widetilde{u'}_s)ds\Big)\Big\|\\
\leq\ & \sup_{t\in[0,b]}\Big|\int_0^t \frac{d}{ds}\Big(C(t-s)g(s,\widetilde{u}^n_s)-g(s,\widetilde{u}_s)\Big)ds\\
&-\int_0^t C(t-s)\frac{d}{ds}\Big(g(s,\widetilde{u}^n_s)-g(s,\widetilde{u}_s)\Big)ds\Big|_\alpha\\
&+\sup_{t\in[0,b]}\Big\|(-A)^{\alpha-1}\Big[\int_0^t\frac{d}{ds}\big(C(t-s)f(s,\widetilde{u}^n_s,\widetilde{u'}^n_s)-f(s,\widetilde{u}_s,\widetilde{u'}_s)\big)ds\\
&-\int_0^t C(t-s)\frac{d}{ds}\big(f(s,\widetilde{u}^n_s,\widetilde{u'}^n_s)-f(s,\widetilde{x}_s,\widetilde{u'}_s)ds\big)\Big]\Big\|\\
\leq\ & \sup_{t\in[0,b]}\Big[|g(t,\widetilde{u}^n_t)-g(t,\widetilde{u}_t)|_\alpha+|C(t)\big(g(0,\widetilde{u}^n_0)-g(0,\widetilde{u}_0)\big)|_\alpha\\
&+Me^{\omega b}\Big(|g(t,\widetilde{u}^n_t)-g(t,\widetilde{u}_t)|_\alpha-|g(0,\widetilde{u}^n_0)-g(0,\widetilde{u}_0)|_\alpha\Big]\\
&+\sup_{t\in[0,b]}\|(-A)^{\alpha-1}\|\Big[\big(f(t,\widetilde{u}^n_t,\widetilde{u'}^n_t)-f(t,\widetilde{u}_t,\widetilde{u'}_t)\big)-C(t)\big(f(0,\widetilde{u}^n_0,\widetilde{u'}^n_0)\\
&-f(0,\widetilde{u}_0,\widetilde{u'}_0)\big)\big)\|\\
&+M_1e^{\omega b}\|f(t,\widetilde{u}^n_t,\widetilde{u'}^n_t)-f(t,\widetilde{u}_t,\widetilde{u'}_t))-\big(f(0,\widetilde{u}^n_0,\widetilde{u'}^n_0)-f(0,\widetilde{u}_0,\widetilde{u'}_0)\big)\big)\Big]\\
\leq\ & \sup_{t\in[0,b]}\Big[(1+M_1e^{\omega b})|g(t,\widetilde{u}^n_t)-g(t,\widetilde{u}_t)|_\alpha+2Me^{\omega b}|g(0,\widetilde{u}^n_0)-g(0,\widetilde{u}_0)|_\alpha\Big]\\
&+\sup_{t\in[0,b]}\|(-A)^{\alpha-1}\|\Big[(1+M_1e^{\omega b})\|f(t,\widetilde{u}^n_t,\widetilde{u'}^n_t)-f(t,\widetilde{u}_t,\widetilde{u'}_t)\|\\
&+2M_1e^{\omega b}\|f(0,\widetilde{u}^n_0,\widetilde{u'}^n_0)-f(0,\widetilde{u}_0,\widetilde{u'}_0)\|\Big]\to 0 \text{ as } n\to\infty,
\end{aligned}$$

and this yield the continuity of $\mathcal{K}_2$ on B_k.

Step 3: The set $\{\mathcal{K}_2(u)(t):\ u\in B_k\}$ is relatively compact for each $t\in]0,b]$.

Let $t \in]0, b]$ be fixed and $\gamma > 0$ be such that $\alpha < \gamma < 1$. Using the same reasoning like previously, it follows that

Step 1

$$\begin{aligned}\|(-A)^{\gamma}\mathcal{K}_2(u)\| \;\leq\; & \|(-A)^{\gamma-1}\|\Big[M_1e^{\omega b}\Big(\|A(\varphi(0)-g(0,\varphi))\|+\|A(\varphi'(0)-\eta)\| \\ & +\sup_{t\in[0,b]}\Big[\Big(\beta(t)(1+2M_1e^{\omega b})+M_1e^{\omega b}\Big]k+L_gM_1e^{\omega b}k+\sup_{s\in[0,b]}|g(s,0)|_{\gamma} \\ & +M_1e^{\omega b}|g(0,\widetilde{u}_0)|_{\gamma}<\infty\end{aligned}$$

Consequently for $t \in]0, b]$ fixed, the set $\{(-A)^{\gamma}\mathcal{K}_2(u)(t) : \; u \in B_k\}$ is bounded in X. By $(\mathbf{H}_3)$, we deduce that $(-A)^{-\gamma} : X \to X_{\alpha}$ is compact. It follows that the set $\{\mathcal{K}_2(u)(t) : \; u \in B_k\}$ is relatively compact for each $t \in]0, b]$ in X_{α}.

Step 4: The set $\{\mathcal{K}_2(u) : \; u \in B_k\}$ is an equicontinuous family of functions.

Let $u \in B_k$ and $0 \leq \tau_1 < \tau_2 \leq b$ then we have

$$\begin{aligned} & |\mathcal{K}_2(u)(\tau_2)-\mathcal{K}_2(u)(\tau_1)|_{\alpha} \\ \leq\; & |[C(\tau_2)-C(\tau_1)](\varphi(0)-g(0,\varphi))|_{\alpha}+|[S(\tau_2)-S(\tau_1)](\varphi'(0)-\eta)|_{\alpha} \\ & +\Big|\int_0^{\tau_2}AS(\tau_2-s)g(s,\widetilde{u}_s)ds-\int_0^{\tau_1}AS(\tau_1-s)g(s,\widetilde{u}_s)ds\Big|_{\alpha} \\ & +\Big|\int_0^{\tau_2}S(\tau_2-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)ds-\int_0^{\tau_2}S(\tau_2-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)ds\Big| \\ \leq\; & |[C(\tau_2)-C(\tau_1)](\varphi(0)-g(0,\varphi))|_{\alpha}+|[S(\tau_2)-S(\tau_1)](\varphi'(0)-\eta)|_{\alpha} \\ & +\Big|\int_0^{\tau_1}A[S(\tau_2-s)-S(\tau_1-s)]g(s,\widetilde{u}_s)ds-\int_{\tau_1}^{\tau_2}A[S(\tau_2-s)g(s,\widetilde{u}_s)ds\Big|_{\alpha} \\ & +\Big|\int_0^{\tau_1}[S(\tau_2-s)-S(\tau_1-s)]f(s,\widetilde{x}_s,\widetilde{u'}_s)ds\Big| \\ & +\Big|\int_{\tau_2}^{\tau_2}S(\tau_2-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)ds\Big|\end{aligned}$$

It follows that

$$\begin{aligned} & |\mathcal{K}_2(u)(\tau_2)-\mathcal{K}_2(u)(\tau_1)|_{\alpha} \\ \leq\; & |[C(\tau_2)-C(\tau_1)](\varphi(0)-g(0,\varphi))|_{\alpha}+|[S(\tau_2)-S(\tau_1)](\varphi'(0)-\eta)|_{\alpha} \\ & +\Big|\int_0^{\tau_1}\frac{d}{ds}\Big([C(\tau_2-s)-C(\tau_1-s)]g(s,\widetilde{u}_s)\Big)ds\end{aligned}$$

$$
\begin{aligned}
&-\int_0^{\tau_1}[C(\tau_2-s)-C(\tau_1-s)]\frac{d}{ds}g(s,\widetilde{u}_s)ds\Big|_\alpha\\
&+\Big\|(-A)^{\alpha-1}\Big[\int_0^{\tau_1}\frac{d}{ds}\Big([C(\tau_2-s)-C(\tau_1-s)]f(s,\widetilde{u}_s,\widetilde{u'}_s)ds\\
&-\int_0^{\tau_1}[C(\tau_2-s)-C(\tau_1-s)]\frac{d}{ds}\big(]f(s,\widetilde{u}_s,\widetilde{u'}_s)\big)ds\Big\|\\
&+\Big\|(-A)^{\alpha}\int_{\tau_1}^{\tau_2}\frac{d}{ds}\big(C(\tau_2-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)\big)ds\\
&-\int_{\tau_1}^{\tau_2}C(\tau_2-s)\frac{d}{ds}\big(f(s,\widetilde{u}_s,\widetilde{u'}_s)\big)ds\Big\|.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
&|\mathcal{K}_2(u)(\tau_2)-\mathcal{K}_2(u)(\tau_1)|_\alpha\\
\leq\ &|[C(\tau_2)-C(\tau_1)](\varphi(0)-g(0,\varphi))|_\alpha+|[S(\tau_2)-S(\tau_1)](\varphi'(0)-\eta)|_\alpha\\
&+|(C(\tau_1)-I)g(\tau_1,\widetilde{u}_{\tau_1})|_\alpha\\
&+|[C(\tau_2)-C(\tau_1)]g(0,\widetilde{u}_0)|_\alpha+|[C(\tau_2)-C(\tau_1)](g(\tau_1,\widetilde{u}_{\tau_1}))|_\alpha\\
&+|g(\tau_2,\widetilde{u}_{\tau_2})-C(\tau_2-\tau_1)g(\tau_1,\widetilde{u}_{\tau_1})|_\alpha+Me^{\omega b}|g(\tau_2,\widetilde{u}_{\tau_2})-g(\tau_1,\widetilde{u}_{\tau_1})|_\alpha\\
&+\|(-A)^{\alpha-1}\|\Big[\|(C(\tau_2-\tau_1)-I)f(\tau_1,\widetilde{x}_{\tau_1},\widetilde{u'}_{\tau_1})\|\\
&+\|[C(\tau_2)-C(\tau_1)]f(0,\widetilde{u}_0,\widetilde{u'}_0)\|\\
&+\|f(\tau_2,\widetilde{u}_{\tau_2},\widetilde{u'}_{\tau_2})-C(\tau_2-\tau_1)f(\tau_1,\widetilde{u}_{\tau_1},\widetilde{u'}_{\tau_1})\|\\
&+M_1e^{\omega b}\|f(\tau_2,\widetilde{u}_{\tau_2},\widetilde{u'}_{\tau_2)})-f(\tau_1,\widetilde{u}_{\tau_1},\widetilde{u'}_{\tau_1})\Big]\to 0\ \text{ as }\tau_1\to\tau_2.
\end{aligned}
$$

Since $(-A)^{\alpha-1}$ is compact from X to X and $(C(t)_{t\in\mathbb{R}})$ is uniformly continuous on compact subset of X. Thus $\mathcal{H}$ maps B_k into an equicontinuous family of functions.
So from **Step 1** to **Step 4** and by Ascoli-Arzela theorem we can conclude that $\mathcal{K}_2 : B_k \to B_k$ is compact and $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$ is an condensing operator. Hence by Sadvoskii's fixed point theorem 4, we conclude that $\mathcal{K}$ has least one fixed point in B_k which is a mild solution of equation (1.1) on $[0, b]$. □

Our next objective is to prove the uniqueness of mild solution. To do this, we assume that

$(\mathbf{H}_4)$: $f : [0,b] \times \mathcal{C}_\alpha \times \mathcal{C}_\alpha \to X$ is contunuously differentiable and lipschitzian with the respect on second variable. Then there exists $c_0(r) > 0$ such that for $\varphi,\ \psi \in \mathcal{C}_\alpha$ with $\|\varphi\|_\alpha,\ \|\psi\|_\alpha \le r$, we have

$$\|f(t,\varphi_1,\varphi_1',) - f(t,\varphi_2,\varphi_2',) \le c_0(r)\|\varphi_1 - \varphi_2\|_\alpha \text{ for } \in [0,b],\ \varphi_1,\varphi_2 \in \mathcal{C}_\alpha$$

$(\mathbf{H}_5)$ The function $g : [0,b] \times \mathcal{C}_\alpha \to X_\alpha$ is continuously differentialble and $D_1 g$ and $D_2 g$ are locally lipschitz. Then there exists $c_0(r) > 0$ such that for $\varphi,\ \psi \in \mathcal{C}_\alpha$ with $\|\varphi\|_{\mathcal{C}_\alpha},\ \|\psi\|_{\mathcal{C}_\alpha} \le r$, we have

$$|D_t g(t,\varphi) - D_t g(t,\psi)|_\alpha \le c_0(r)\|\varphi - \psi\|_\alpha \text{ for } t \in [0,b].$$
$$|D_\varphi g(t,\varphi) - D_\varphi g(t,\psi)|_\alpha \le c_0(r)\|\varphi - \psi\|_\alpha \text{ for } t \in [0,b].$$

$(\mathbf{H}_6)$ The maps $t \mapsto AC(t)$ is locally bounded.

Theorem 6. *Assume that $(\boldsymbol{H}_0)$, $(\boldsymbol{H}_2)$, $(\boldsymbol{H}_3)$, $(\boldsymbol{H}_4)$, $(\boldsymbol{H}_5)$ and $(\boldsymbol{H}_6)$ hold. Let $\varphi \in \mathcal{C}_\alpha$ sucth that $\varphi(0) - g(0,\varphi) \in D(A)$ and $\varphi'(0) - \eta \in E$ Then Eq.(1.1) has unique mild solution.*

Proof. Let us consider the following set

$$\mathbb{F}(\varphi) = \{u \in C^1([0,b]), X_\alpha) : u(0) = \varphi(0)\}$$

endowed with the norm $\|u\|_{\mathbb{F}(\varphi)} = \sup_{0\le s\le b} |u(s)|_\alpha + \sup_{0\le s\le b} |u'(s)|_\alpha$.
For $u \in \mathbb{F}(\varphi)$ we define $\widetilde{u} : [-r,b] \to X_\alpha$ by

$$\widetilde{u}(t) = \begin{cases} u(t) \text{ for } t \in [0,b] \\ \\ \varphi(t) \text{ for } t \in [-r,0] \end{cases}$$

Now, we define the operator $\Phi : \mathcal{F}(\varphi) \to \mathcal{F}(\varphi)$ by

$$\begin{aligned} \Phi(u)(t) &= C(t)(\varphi(0) - g(0,\varphi)) + S(t)(\varphi'(0) - \eta) + g(t,\widetilde{u}_t) + \int_0^t AS(t-s)g(s,\widetilde{u}_s)ds \\ &\quad + \int_0^t S(t-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)ds \text{ for } t \in [0,b]. \end{aligned}$$

We will show that Φ is a strict contarction. Let $u, v \in \mathbb{F}(\varphi)$ and μ be a positive real number such that $\|AC(t)\| \le \mu$ for $t \in [0,b]$. Then we have

$$\begin{aligned} \Phi(u)(t) - \Phi(v)(t) &= g(t,\widetilde{u}_t) + g(t,\widetilde{v}_t) + \int_0^t AS(t-s)[g(s,\widetilde{u}_s) - g(s,\widetilde{u}_s)]ds \\ &\quad + \int_0^t S(t-s)[f(s,\widetilde{u}_s,\widetilde{u'}_s) - f(s,\widetilde{v}_s,\widetilde{v'}_s)]ds. \end{aligned}$$

Then

$$\begin{aligned} |\Phi(u)(t) - \Phi(v)(t)|_\alpha &\le |g(t,\widetilde{u}_t) + g(t,\widetilde{v}_t)|_\alpha + \Big|\int_0^t AS(t-s)[g(s,\widetilde{u}_s) - g(s,\widetilde{u}_s)]ds\Big|_\alpha \\ &\quad + \Big|\int_0^t S(t-s)[f(s,\widetilde{u}_s,\widetilde{u'}_s) - f(s,\widetilde{v}_s,\widetilde{v'}_s)]ds\Big|_\alpha \\ &\le L_g\|\widetilde{u}_t - \widetilde{v}_t\|_\alpha + \Big|\int_0^t \Big(\int_0^{t-s} AC(\sigma)[g(s,\widetilde{u}_s) - g(s,\widetilde{u}_s)]d\sigma\Big)ds\Big|_\alpha \end{aligned}$$

$$+\Big|\int_0^t\Big(\int_0^{t-s}C(\sigma)[f(s,\widetilde{u}_s,\widetilde{u'}_s)-f(s,\widetilde{v}_s,\widetilde{v'}_s)]d\sigma\Big)ds\Big|_\alpha$$

$$\leq \quad L_g\|\widetilde{u}_t-\widetilde{v}_t\|_\alpha+\mu L_g b\int_0^t\|\widetilde{u}_s-\widetilde{v}_s\|_\alpha ds$$

$$+\|(-A)^{\alpha-1}\|bM_1e^{wb}L_F\int_0^t\|\widetilde{u}_s-\widetilde{v}_s\|_\alpha ds.$$

It follows that

$$|\Phi(u)(t)-\Phi(v)(t)|_\alpha \quad\leq\quad \gamma_1(b)\|u-v\|_{\mathbb{F}(\varphi)}, \tag{3.2}$$

where

$$\gamma_1=\Big[L_g(1+\mu b^2)+\|(-A)^{\alpha-1}\|M_1e^{wb}b^2L_F\Big].$$

On the other hand, by use of Eq. (2.1) and Proposition 1, we have

$$(\phi(u))'(t) \quad=\quad C'(t)(\varphi(0)-g(0,\varphi))+S'(t)(\varphi'(0)-g'(0,\varphi))+\frac{d}{dt}g(t,\widetilde{u}_t)+\int_0^t AC(t-s)g(s,\widetilde{u}_s)ds$$

$$+\int_0^t C(t-s)f(s,\widetilde{u}_s,\widetilde{u'}_s)ds.$$

Now let us pose

$$P(u)(t)=\frac{d}{dt}g(t,u_t)=D_tg(t,u_t)+D_\varphi g(t,u_t)u'_t.$$

Then we have

$$\begin{aligned}
|P(u)(t)-P(v)(t)|_\alpha \quad&\leq\quad |D_tg(t,u_t)-D_tg(t,v_t)|_\alpha+|D_\varphi g(t,u_t)-D_\varphi g(t,v_t)|_\alpha\\
&\leq\quad |D_tg(t,u_t)-D_tg(t,v_t)|_\alpha+|D_\varphi g(t,u_t)u'_t-D_\varphi g(t,v_t)u'_t+D_2g(t,v_t)u'_t-D_2g(t,v_t)|_\alpha\\
&\leq\quad |D_tg(t,u_t)-D_tg(t,v_t)|_\alpha+|D_\varphi g(t,u_t)-D_\varphi g(t,v_t)||u'_t|_\alpha+|D_\varphi g(t,v_t)|_\alpha|u'_t-v'_t|_\alpha\\
&\leq\quad c_0(r)\|u_t-v_t\|_\alpha+c_0(r)\|u_t-v_t\|_{\mathcal{C}_\alpha}|u'_t|_\alpha\\
&\quad\quad +|D_\varphi g(t,v_t)-D_\varphi g(t,\varphi)+D_\varphi g(t,\varphi)|_\alpha|u'_t-v'_t|_\alpha\\
&\leq\quad c_0(r)\|u_t-v_t\|_\alpha+c_0(r)\|u_t-v_t\|_\alpha|u'_t|_\alpha\\
&\quad\quad +\Big(|D_\varphi g(t,v_t)-D_2g(t,\varphi)|_\alpha+|D_\varphi g(t,\varphi)|_\alpha\Big)|u'_t-v'_t|_\alpha\\
&\leq\quad c_0(r)\|u_t-v_t\|_\alpha+c_0(r)\|u_t-v_t\|_\alpha|u'_t|_\alpha+\Big(c_0(r)\|u_t-\varphi\||_\alpha+|D_\varphi g(t,\varphi)|_\alpha\Big)|u'_t-v'_t|_\alpha.
\end{aligned}$$

By the local lipschitz of $D_\varphi g$, there exists for each $r>0$ a positive constant $c_0(r)$ such that for $\varphi\in\mathcal{C}_\alpha$ with $\|\varphi\|_{\mathcal{C}_\alpha}\leq r$

$$|D_\varphi g(t,\varphi)|_\alpha\leq c_0(r)\|\varphi\|_\alpha+|D_\varphi g(s,0)|_\alpha\leq c_0(r)r+\sup_{s\in[0,b]}|D_\varphi g(s,0)|_\alpha=c_1(r)$$

Moreover

$$\|u_t-\varphi\|_\alpha\leq\sup_{0\leq\tau\leq b}|u(\tau)-\varphi(0)|_\alpha\leq\|u-\varphi\|_{\mathbb{F}(\varphi)}$$

because of $u(\tau) = \varphi(0)$ for $\tau \in [-r, 0]$.

Likewise, we have

$$\|u_t - v_t\|_\alpha \leq \sup_{0\leq\tau\leq b} |u(\tau) - v(\tau)|_\alpha \leq \|u - v\|_{\mathbb{F}(\varphi)}$$

because of $u(\tau) = v(\tau)$ for $\tau \in [-r, 0]$.

Since by the definition of $\|.\|_{\mathbb{F}(\varphi)}$, we have

$$|u'_t|_\alpha \leq \|u\|_{\mathbb{F}(\varphi)}.$$

Thus it follows that

$$|P(u)(t) - P(v)(t)|_\alpha \;\leq\; \Big[c_0(r)\big(1 + \|u\|_{\mathbb{F}(\varphi)} + \|u - \varphi\|_{\mathbb{F}(\varphi)}\big) + c_1(r)\Big]\|u - v\|_{\mathbb{F}(\varphi)}$$

$$(\Phi(u))'(t) - (\Phi(v))'(t) \;=\; P(u)(t) - P(v)(t) + \int_0^t AC(t-s)[g(s, \widetilde{u}_s) - g(s, \widetilde{v}_s)]ds$$
$$+ \int_0^t C(t-s)[f(s, \widetilde{u}_s, \widetilde{u'}_s) - [f(s, \widetilde{v}_s, \widetilde{v'}_s)]ds$$

Using the same reasoning like previously, then we have

$$|(\Phi(u))'(t) - (\Phi(v))'(t)|_\alpha \;\leq\; \gamma_2(b)\|u - v\|_{\mathbb{F}(\varphi)}, \tag{3.3}$$

where

$$\gamma_2(b) = \Big[c_0(r)\Big(1 + r + \|u - \varphi\|_{\mathbb{F}(\varphi)}\Big) + c_1(r)\Big] + b\big(\mu L_g + L_F M_1 e^{wb}\|(-A)^{\alpha-1}\|\big)\Big]$$

Adding equation (3.2) and equation (3.3), then we have

$$\|\Phi(u)(t) - \Phi(u)(t)\|_{\mathbb{F}(\varphi)} \leq \gamma(b)\big)\|u - v\|_{\mathbb{F}(\varphi)},$$

where $\gamma(b) = \gamma_1(b) + \gamma_2(b)$.

We choose b sufficiently small such that $\gamma(b) < 1$.

This means Φ is a strict contraction. By principle contraction, we can deduce that Φ has a unique fixed point in $\mathbb{F}(\varphi)$. Then Eq.(1.1) has a unique mild solution on $[0, b]$. □

4. Existence of strict solutions

Theorem 7. *Assume that* $(\boldsymbol{H}_0)$, $(\boldsymbol{H}_2)$, $(\boldsymbol{H}_3)$, $(\boldsymbol{H}_4)$ *and* $(\boldsymbol{H}_6)$ *hold and f is continuously differentiable. Moreover assume that the partial derivatives f_1, f_2 and f_3 are locally lipschitz in classical sens. i.e, there exists positive constant L_1 such for $\varphi_1, \varphi_2 \in \mathcal{C}_\alpha$,*

$$\|f_i(t, \varphi, \varphi') - f_i(t, \psi, \psi')\| \leq L_1\|\varphi - \psi\|_\alpha \; for \, t \geq 0, \; i = 1, \; 2, \; 3.$$

Let $\varphi \in C^2([-r, 0], D((-A)^\alpha))$ such that $\varphi(0), \varphi''(0) \in D(A)$, $\varphi'(0) - \eta \in E$ and

$$\varphi''(0) - \partial_{tt}g(0, \varphi) - \partial_{\varphi t}g(0, \varphi)\varphi' - \partial_{t\varphi}g(0, \varphi)\varphi' - \partial_{\varphi\varphi}g(0, \varphi)\varphi'' = A\varphi(0) + f(0, \varphi, \varphi')$$

and Then the corresponding of mild solution u becomes a strict solution of equation (1.1) on $[0, b]$.

Proof. Let $\varphi \in C^2([-r,0], D((-A)^\alpha))$ such that $\varphi(0), \varphi''(0) \in D(A)$, $\varphi'(0) - \eta \in E$ and

$$\varphi''(0) - \partial_{tt} g(0,\varphi) - \partial_{\varphi t} g(0,\varphi)\varphi' - \partial_{t\varphi} g(0,\varphi)\varphi' - \partial_{\varphi\varphi} g(0,\varphi)\varphi'' = A\varphi(0) + f(0,\varphi,\varphi').$$

Let v be the corresponding mild solution of equation (1.1) which is defined on $[0,b]$ by

$$\begin{cases} v(t) = C(t)\Big[A(\varphi(0) + f(0,\varphi,\varphi')\Big] + S(t)A(\varphi'(0) - \eta) + \dfrac{d}{dt}P(u)(t) + \displaystyle\int_0^t AC(t-s)[D_t g(s,u_s) \\ \qquad -D\varphi g(s,u_s)v_s]ds + \displaystyle\int_0^t C(t-s)[f_1(s,u_s,u_s') + f_2(s,u_s,u_s')u_s' + f_3(s,u_s,u_s')v_s]ds \\ v_0 = \varphi'' \end{cases}$$

Now, we define w by

$$w(t) = \begin{cases} \psi'(0) + \displaystyle\int_0^t v(s)ds \text{ if } t \in [0,b] \\ w(t) = \psi'(t) \text{ if } -r \le t \le 0 \\ w'(t) = \psi''(t) \text{ if } -r \le t \le 0. \end{cases} \tag{4.1}$$

Where $\psi'(0) = (\varphi'(0) - \eta)$

Then we can see that $w_t = \psi' + \displaystyle\int_0^t v_s ds$ for $t \in [0,b]$.

Consequently the map $t \mapsto w_t$ and $t \mapsto \displaystyle\int_0^t C(t-s)f(s,u_s,u_s')ds$ are continuously differentiable.

Then we have

$$\begin{aligned} & \frac{d}{dt}\int_0^t C(t-s)f(s,u_s,u_s')dss \\ = \; & \frac{d}{dt}\int_0^t C(s)f(t-s,u_{t-s},u_{t-s}')ds \\ = \; & C(t)f(0,u_0,w_0) + \int_0^t C(t-s)\Big[f_1(s,u_s,w_s) + f_s(s,u_s,u_s')u_s' \\ & + f_3(s,u_s,w_s)v_s\Big]ds \\ = \; & C(t)f(0,\varphi,\varphi') + \int_0^t C(t-s)\Big[f_1(s,u_s,w_s) + f_s(s,u_s,u_s')u_s' \\ & + f_3(s,u_s,w_s)v_s\Big]ds, \end{aligned}$$

it follows that

$$\begin{aligned}
&\int_0^t C(s)f(0,\varphi,\varphi')ds \\
=\ &\int_0^t C(s)f(s,u_s,u_s')ds - \int_0^t\int_0^s C(s-\tau)\Big[f_1(\tau,u_\tau,w_\tau)+f_2(\tau,u_\tau,w_\tau)u_\tau' \\
&+f_3(\tau,u_\tau,w_\tau)v_\tau\Big]d\tau ds.
\end{aligned}$$

On other hand one has

$$\begin{aligned}
&\frac{d}{dt}\int_0^t AC(t-s)g(s,u_s)ds \\
=\ &\frac{d}{dt}\int_0^t AC(s)g(t-s,u_{t-s})dss = C(t)Ag(0,\varphi)+\int_0^t AC(t-s)[D_tg(s,u_s) \\
&+D_\varphi g(s,u_s)v_s]ds
\end{aligned}$$

which implies that

$$\int_0^t\int_0^s AC(t-s)[D_1g(\tau,u_\tau)+D_2g(\tau,u_\tau)v_\tau]d\tau ds = \int_0^t AC(t-s)g(s,u_s)ds - AS(t)g(0,\varphi).$$

Consequently we have

$$\begin{aligned}
w(t) =\ & \psi'(0)+\int_0^t C(s)(A(\varphi(0)-g(0,\varphi)ds+\int_0^t S(s)A(\varphi'(0)-\eta)ds+P(u)(t) \\
&+\int_0^t AC(t-s)g(s,u_s)ds - AS(t)g(0,\varphi)+\int_0^t C(t-s)f(s,u_s,u_s')ds-\int_0^t C(t-s)f(s,u_s,w_s)ds \\
&+\int_0^t\int_0^s C(s-\tau)\Big[f_1(\tau,u_\tau,u_\tau')+f_2(\tau,u_\tau,u_\tau')u_s'+f_3(\tau,u_\tau,u_\tau')v_\tau\Big]d\tau ds \\
&-\int_0^t\int_0^s C(s-\tau)\Big[f_1(\tau,u_\tau,w_\tau)+f_2(\tau,u_\tau,w_\tau)u_\tau' \\
&+f_3(\tau,u_\tau,w_\tau)v_\tau\Big]d\tau ds.
\end{aligned}$$

Moreover by Lemma 1, we have

$$\int_0^t C(s)A\varphi(0)ds = S(t)A\varphi(0)$$

$$\int_0^t S(t)A(\varphi'(0)-\eta)ds = C(t)(\varphi'(0)-\eta)-(\varphi'(0)-\eta) = C(t)(\varphi'(0)-\eta)-\psi'(0)$$

It follows that

$$\begin{aligned}
w(t) =\ & S(t)A(\varphi(0)-g(0,\varphi))+C(t)(\varphi'(0)-\eta)+\int_0 C(t-s)f(s,u_s,w_s)ds+P(u)(t) \\
&+\int_0^t AC(t-s)g(s,u_s)ds+\int_0^t\int_0^s C(s-\tau)\Big[(f_1(\tau,u_\tau,u_\tau')-f_1(\tau,u_\tau,w_\tau)) \\
&+(f_2(\tau,u_\tau,u_\tau')-f_2(\tau,u_\tau,u_\tau'))u_\tau'+(f_3(\tau,u_\tau,u_\tau')-f_3(\tau,u_\tau,w_\tau))v_\tau d\tau\Big]ds.
\end{aligned}$$

Furthermore for $t \geq 0$, we know that

$$u'(t) = AS(t)(\varphi(0) - g(0,\varphi)) + C(t)(\varphi'(0) - \eta) + P(u)(t) + \int_0^t AC(t-s)g(s,u_s)ds + \int_0^t C(t-s)f(s,u_s,u_s')ds,$$

then for $t \in [0,b]$, we have

$$\begin{aligned} & u'(t) - w(t) \\ = & \int_0 C(t-s)[f(s,u_s,u_s') - f(s,u_s,w_s)]ds + \int_0^t \int_0^s C(s-\tau)\Big[(f_1(\tau,u_\tau,u_\tau') - f_1(\tau,u_\tau,w_\tau)) \\ & +(f_2(\tau,u_\tau,u_\tau') - f_2(\tau,u_\tau,u_\tau'))u_\tau' \\ & +(f_3(\tau,u_\tau,u_\tau') - f_3(\tau,u_\tau,w_\tau))v_\tau d\tau\Big]ds. \end{aligned}$$

$$\begin{aligned} & |u'(t) - w(t)|_\alpha \\ \leq & \int_0^t |C(t-s)[f(s,u_s,u_s') - f(s,u_s,w_s)]|_\alpha ds \\ & + \int_0^t \int_0^s |C(s-\tau)(f_1(\tau,u_\tau,u_\tau') - f_1(\tau,u_\tau,w_\tau))|_\alpha d\tau ds \\ & + \int_0^t \int_0^s |(f_2(\tau,u_\tau,u_\tau') - f_2(\tau,u_\tau,u_\tau'))u_\tau'|_\alpha d\tau ds \\ & + \int_0^t \int_0^s |(f_3(\tau,u_\tau,u_\tau') - f_3(\tau,u_\tau,w_\tau))v_\tau|_\alpha d\tau ds. \end{aligned}$$

(4.2)

Let us choose $F = \Big\{u_s', w_s : s \in [0,b]\Big\}$. Then F is compact set. It follows that f_1, f_2 and f_3 are globally lipschitz on F. Let $L_1 > 0$ be such that for $t \in [0,b]$ and $x, y \in H$ Then we have

$$\begin{aligned} \|f(t,x,x') - f(t,y,y')\| &\leq L_1\|x-y\|_\alpha \\ \|f_1(t,x,x') - f_1(t,y,y')\| &\leq L_1\|x-y\|_\alpha \\ \|f_2(t,x,x') - f_2(t,y,y')\| &\leq L_1\|x-y\|_\alpha \\ \|f_3(t,x,x') - f_3(t,y,y')\| &\leq L_1\|x-y\|_\alpha. \end{aligned}$$

Consequently, using equation (4.2), we one can find a positive constance $k(b)$ such that by Gronwall's lemma,

$$|u_\tau' - w_\tau|_\alpha \leq k(b)\int_0^t \|u_s' - w_s\|_\alpha ds,$$

then we deduce that $u' = w$. Consequently, it follows that the mild solution is twice continuous differentiable from $[-r,b]$ to X_α and the function $t \longrightarrow f(t,u_t,u_t')$ and $t \longrightarrow g(t,u_t)$ are continuous differentiable on $[0,b]$, thus according to Theorem (1), we conclude that u is a strict solution of equation (1.1) on $[0,b]$. □

5. Application

For our illustration, we propose to study the existence of solutions for the following model

(5.1)
$$\begin{cases} \dfrac{\partial^2}{\partial t^2}[z(t,x) - \displaystyle\int_{-r}^{0} k(t, z(t+\theta, x))d\theta] = \dfrac{\partial^2}{\partial x^2} z(t,x)] + \displaystyle\int_{-r}^{0} h(t, \dfrac{\partial}{\partial x} z(t+\theta, x), \dfrac{\partial}{\partial x} z'(t+\theta, x))d\theta \\ \qquad \text{for } t \geq 0 \text{ and } x \in [0,\pi] \\ z(t,0) - \displaystyle\int_{-r}^{0} k(t, z(t+\theta, x))d\theta = 0 \text{ for } t \geq 0 \\ z(t,\pi) - \displaystyle\int_{-r}^{0} k(t, z(t+\theta, x))d\theta = 0 \\ z(\theta, x) = \varphi_0(\theta)(x) \text{ for } \theta \in [-r, 0] \text{ and } x \in [0,\pi] \end{cases}$$

where $h : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exists a positive constant L such that for $x, y, x_1, y_1 \in \mathbb{R}$,

$$|h(t,x,y) - h(t,x_1,y_2)| \leq L\Big(|x - x_1| + |y - y_1|\Big)$$

We can choose for exemple

$$h(t,x,y) = e^{-t^2}[\sin(\frac{x}{2}) + \sin(\frac{y}{2})] \text{ for } (\theta, x, y) \in \mathbb{R}^- \times \mathbb{R} \times \mathbb{R}$$

we can observe that

$$|h(t,x_1,y_1) - h(t,x_2,y_2)| \leq \frac{1}{2}\Big(|x_1 - x_2| + |y_1 - y_2|\Big)$$

$k : \mathbb{R}^- \times \mathbb{R} \longrightarrow \mathbb{R}$ is lipschizian with respect to the second argument.

In the oder to rewrite equation (5.1) in the abstract form, we introduce the space $X = L^2([0,\pi]; \mathbb{R})$ vanishing at 0 and π, equipped with the L^2 norm that is to say for all $x \in X$,

$$\|x\|_{L^2} = \Big(\int_0^{\pi} |x(s)|^2 ds\Big)^{\frac{1}{2}}.$$

Let $e_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$, $x \in [0,\pi]$, $n \geq 1$, then $(e_n)_{n\geq 1}$ is an orthogonal base for X.

Let $A : X \to X$ be defined by

$$\begin{cases} Ay = y'' \\ D(A) = \big\{y \in X : y, y' \text{ are absolutuely continuous},\ y'' \in X,\ y(0) = y(\pi) = 0\big\} \end{cases}$$

Then the operator is computed by

$$Ay = \sum_{n=1}^{+\infty} -n^2(y, e_n)e_n,\ y \in D(A),$$

where

$$(u,v) = \int_0^{\pi} u(s)v(s)ds \ \text{ for } u, v \in X$$

It is well known that A is the infinitesimal generator of strongly continuous cosine family $C(t), \in \mathbb{R}$ in X which is given by

$$C(t)y = \sum_{n=1}^{+\infty} \cos nt(y, e_n)e_n,\ y \in X.$$

and that the associated sine family is given by

$$S(t)y = \sum_{n=1}^{+\infty} \frac{1}{n} \sin nt(y, e_n)e_n,\ y \in X.$$

If we choose $\alpha = \dfrac{1}{2}$. then $(\mathbf{H}_0)$ and $(\mathbf{A}_1)$ are satisfied since

$$(-A)^{\frac{1}{2}}y = \sum_{n=1}^{+\infty}(y, e_n)e_n,\ y \in D((-A)).$$

and

$$(-A)^{\frac{1}{2}}y = \sum_{n=1}^{+\infty} \frac{1}{n}(y, e_n)e_n,\ y \in X.$$

Frome [10], the compactness of A^{-1} follows from Lemma 3 and the fact that the eigenvalues of $(-A)^{\frac{1}{2}}$ are $\lambda_n = \dfrac{1}{n}$, $\ n = 1, 2 \ldots$, the $(\mathbf{H}_3)$ is satisfied.

We define the space

$$\mathcal{C} = C^1([-r, 0], X)$$

where $C^1([-r, 0], X)$ is the space of bounded uniformly continuous differentiable from $[-r, 0]$ into X whith the norm

$$|\varphi| = \sup_{-r\leq\theta\leq 0} |\varphi(\theta)|.$$

Let $f : \mathbb{R} \times \mathcal{C}_{\frac{1}{2}} \times \mathcal{C}_{\frac{1}{2}} \longrightarrow X$ and $g : \mathbb{R} \times \mathcal{C}_{\frac{1}{2}}$ define by

$$f(t, \varphi, \varphi')(x) = \int_{-r}^{0} h(t, \frac{\partial}{\partial x}\varphi(\theta)(x), \frac{\partial}{\partial x}\varphi'(\theta)(x))d\theta \text{ for } x \in [0, \pi], t \geq 0,\ \varphi, \in \mathcal{C}_{\frac{1}{2}}$$

and

$$g(t, \varphi,)(x) = \int_{-r}^{0} k(t, \varphi(\theta)(x))d\theta \text{ for } x \in [0, \pi], t \geq 0,\ \varphi, \in \mathcal{C}_{\frac{1}{2}}$$

where $\varphi, \in \mathcal{C}_{\frac{1}{2}}$ define by

$$\varphi(\theta)(x) = \varphi_0(\theta, x)$$

and the norm in $\mathcal{C}_{\frac{1}{2}}$ is given by

$$\|\varphi\|_{\mathcal{C}_{\frac{1}{2}}} = \sup_{\theta\in[-r,0]} \Big(\int_0^{\pi} \big|\frac{\partial}{\partial x}[\varphi(\theta)(x)]\big|^2 dx\Big)^{\frac{1}{2}} + \sup_{\theta\in[-r,0]} \Big(\int_0^{\pi} \big|\frac{\partial}{\partial x}[\varphi'(\theta)(x)]\big|^2 dx\Big)^{\frac{1}{2}}$$

Let us pose $v(t) = z(t, x)$. Then equation (5.1) takes the following abstract form

$$\text{(5.2)} \qquad \begin{cases} \dfrac{d^2}{dt}[v(t) - g(t, v_t)] = Av(t) + f(t, v_t, v_t') \text{ for } t \geq 0 \\ \\ v_0 = \varphi \in \mathcal{C}_{\frac{1}{2}} \\ \\ v_0' = \varphi' \in \mathcal{C}_{\frac{1}{2}} \end{cases}$$

From [10], for all $y \in X_{\frac{1}{2}}$, y is absolutely continuous and $|y|_{\frac{1}{2}} = |y|_{L^2}$ Let $\varphi, \psi \in C^1([-r,0], X_{\frac{1}{2}})$, since
$|h(t,x_1,y_1) - h(t,x_2,y_2)| \leq \frac{1}{2}\Big(|x_1 - x_2| + \|y_1 - y_2|\Big)$

$$\begin{aligned} |f(t,\varphi,\varphi') - f(t,\psi,\psi')|_{L^2} &\leq \Big(\int_0^{\pi}\Big(\int_{-r}^{0} h(t, \frac{\partial}{\partial x}[\varphi(\theta)(x)], \frac{\partial}{\partial x}[\varphi'(\theta)(x)]d\theta\Big) \\ &\quad + \Big(\int_0^{\pi}\Big(\int_{-r}^{0} h(t, \frac{\partial}{\partial x}[\psi(\theta)(x)], \frac{\partial}{\partial x}[\psi'(\theta)(x)])d\theta\Big)^2 dx\Big)^{\frac{1}{2}} \\ &\leq \frac{1}{2}r\Big[\Big(\int_0^{\pi}\Big|\frac{\partial}{\partial x}[\varphi(\theta)(x)] - \frac{\partial}{\partial x}[\psi(\theta)(x)]\Big|^2 dx\Big)^{\frac{1}{2}} \\ &\quad + \Big(\int_0^{\pi}\Big|\frac{\partial}{\partial x}[\varphi'(\theta)(x)] - \frac{\partial}{\partial x}[\psi'(\theta)(x)]\Big|^2 dx\Big)^{\frac{1}{2}}\Big] \end{aligned}$$

By Minkawski Lemma, we have

$$\begin{aligned} |f(t,\varphi,\varphi') - f(t,\psi,\psi')|_{L^2} &\leq \frac{1}{2}r\Big[\Big(\int_0^{\pi}\Big|\frac{\partial}{\partial x}[\varphi(\theta)(x)] - \frac{\partial}{\partial x}[\psi(\theta)(x)]\Big|^2 dx\Big)^{\frac{1}{2}} \\ &\quad + \Big(\int_0^{\pi}\Big|\frac{\partial}{\partial x}[\varphi'(\theta)(x)] - \frac{\partial}{\partial x}[\psi'(\theta)(x)]\Big|^2 dx\Big)^{\frac{1}{2}}\Big] \\ &\leq \frac{1}{2}r\Big[\sup_{\theta\in[-r,0]}\Big(\int_0^{\pi}\Big|\frac{\partial}{\partial x}[\varphi(\theta)(x)] - \frac{\partial}{\partial x}[\psi(\theta)(x)]\Big|^2 dx\Big)^{\frac{1}{2}} \\ &\quad + \sup_{\theta\in[-r,0]}\Big(\int_0^{\pi}\Big|\frac{\partial}{\partial x}[\varphi'(\theta)(x)] - \frac{\partial}{\partial x}[\psi'(\theta)(x)]\Big|^2 dx\Big)^{\frac{1}{2}}\Big], \end{aligned}$$

which implies that

$$|f(t,\varphi,\varphi') - f(t,\psi,\psi')|_{L^2} \leq \frac{1}{2}r\|\varphi - \psi\|_{\mathcal{C}_{\frac{1}{2}}}.$$

$(\mathbf{H}_7)$ $0 < rL_h < 1$

We claim that g is a contraction function with respect to the second argument with value in $X_{\frac{1}{2}}$. Indeed let $\varphi_1, \varphi_2 \in \mathcal{C}_{\frac{1}{2}}$ and L_k the constant lipschitz of k. Then we have

$$|g(t,\varphi) - g(t,\psi)|_{L^2} \leq L_k r\|\varphi - \psi\|_{\mathcal{C}_{\frac{1}{2}}}.$$

Consequently, assumption $(\mathbf{H}_7)$ implies that g is a strict contraction. Moreover

$$D_t(g(t,\varphi)(x)) = \int_{-r}^{0}\frac{\partial}{\partial t}k(t,\varphi(\theta)(x))d\theta,$$

and

$$D_\varphi(g(t,\varphi)(\varphi)(x)) = \int_{-r}^{0}\frac{\partial}{\partial v}k(t,\varphi(\theta)(x))(\psi(\theta))(x)d\theta,$$

which implies that the partial derivates of g are locally lipschitzian with the respect of second argument. Then the equation (5.2) has a unique mild solution.

Proposition 2. *Under the above assumptions, equation* (5.2) *has a unique mild solution which is defined for all* $t \geq 0$

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