DOI: 10.28919/ejma.2025.5.2

Eur. J. Math. Appl. (2025)5:2
URL: http://ejma.euap.org

SECOND ORDER ASYMPTOTICS FOR THE VASICEK MODEL DRIVEN BY LEVY PROCESSES

JAYA P. N. BISHWAL

Abstract. In the Vasicek model driven by Levy processes, we obtain the bound on the Kolmogorov distance, i.e., the rate of weak convergence to normality for the maximum likelihood estimator of the speed of mean reversion parameter.

1. Introduction and Preliminaries

Levy processes are stochastic processes with stationary independent increments. Levy Ornstein-Uhlenbeck (LOU) process generalizes the Ornstein-Uhlenbeck process to include jumps, see Jacod and Shiryayev [26]. The Levy Ornstein-Uhlenbeck (LOU) process, is an extension of Ornstein-Uhlenbeck process with Levy process driving term. Levy driven processes of Ornstein-Uhlenbeck type have been extensively studied over the last decade and widely used in finance, see Barndorff-Neilsen and Shephard [1]. In finance, it is useful as a generalization of Vasicek model, as one-factor short-term interest rate model which could take into account the jump of the interest rate. It also generalizes stochastic volatility model where the volatility has jumps.

Jump processes are of two types: Finite activity processes and infinite activity processes. Finite activity processes have finite number of jumps in a finite time interval, e.g., a Poisson process and infinite activity processes have infinite number of jumps in a finite time interval, e.g., gamma process, inverse Gaussian process and tempered stable process.

It is well known that the suitably parametrized autoregresive (AR) process with Gaussian error has the continuous limit the Vasicek model. Wolfe (1982) studied continuous analogue of the stochastic difference equation of AR type with Levy type innovations whose limit is a Levy driven OU Process. Gourieroux and Jasiak [24] studied autoregressive gamma (ARG) process and showed that its continuous time limit is the Cox-Ingersoll-Ross (CIR) model. Thus the stationary ARG process is a discretized version of the CIR process. Gourieroux and Jasiak(2006) studied pseudo-maximum likelihood estimation in autoregressive gamma (ARG) process. This process can also be used for application in series of squared returns and intertrade durations for high-frequency data, i.e., it is a stochastic duration model. ARG model also fits a series of volumes per trade, which is an alternative proxy for liquidity. This is different from gamma autoregressive process (GAR) process studied in Sim [41] and Gaver and Lewis [23] where just the noise of the linear autoregressive process is Gamma distributed. For intertrade durations,

Department of Mathematics and Statistics, University of North Carolina at Charlotte, 376 Fretwell Bldg., 9201 University City Blvd., Charlotte, NC 28223, USA

E-mail address: J.Bishwal@uncc.edu.

Key words and phrases. Itô stochastic differential equation, Vasicek model, Levy process, Mean-reversion speed, Kolmogorov distance, Berry-Esseen bound.

Received 24/06/2024.

the most popular model is autoregressive conditional duration (ACD) model introduced by Engle and Russell (1998).

For the exponential AR(1) model, the ratio estimator was introduced in Davis and McCormick (1989). This estimator is motivated by the extreme value theory for the correlation parameter of an AR(1) process whose innovation distribution is positive. In the case of exponential AR(1) process, it coincides with the maximum likelihood estimator. See Neilsen and Shephard [38].

The weak consistency of the ratio estimator in the LOU process was studied in Jongbloed *et al.* [27]. The strong consistency and asymptotic Weibullness was studied in Brockwell, Davis and Yang [15] in the case of Gamma innovations.

Parameter estimation for directly observed stochastic differential equations is studied in Bishwal [6]. Parameter estimation in partially observed stochastic volatility models is studied in Bishwal [13]. Hypothesis testing for stochastic differential equations is studied in Linkov [29, 30]. Parameter estimation and hypotheses testing in ergodic diffusion processes is studied in Kutoyants [28]. Luschgy [32–37] and Linkov [29] studied inference for semimartingales.

Berry-Esseen inequalities for discretely observed diffusion was studied in Bishwal [7]. For the standard Ornstein-Uhlenbeck process, sufficiency and Rao-Blackwellization was studied in Bishwal [10] where also a time transformation to reduce the general problem to a fixed time case and the asymptotics were studied in large parameter case. Rate of weak convergence of the approximate minimum contrast estimators for discretely observed Ornstein-Uhlenbeck process was studied in Bishwal [4]. Maximum quasi-likelihood estimation in fractional Levy stochastic volatility model was studied in Bishwal [9]. Berry-Esseen inequalities for the fractional Black-Karasinski model of term structure of interest rates were studied in Bishwal [12].

The standard $O(T^{-1})$ rate holds in the singular case. In the model driven by standard Brownian motion innovation case, the limit distribution is Dicky-Fuller distribution or White distribution, see White [42], which does not have a closed form, see Feigin [20]. Recall that in the singular case for zero innovation mean locally asymptotically Brownian functional (LABF) condition holds while for nonzero innovation mean, local asymptotic normality (LAN) condition holds, see Bishwal [11].

Consider the Vasicek model

$$dX_t = (\theta_0 - \theta_1 X_t)dt + dV_t(\theta_2),\ X_0 = 0$$

In financial literature, this model is heavily used in term structure of interest rates and in bond pricing, see Brigo and Mercurio [14]. With a positive innovation process, this model is used for modeling stochastic volatility, see Barndorff-Nielsen and Shephard [1] and Bishwal [13]. This is a mean reverting model. The parameter θ_0 is called the mean reversion level and the parameter θ_1 is called the speed of mean reversion. For simplicity first assume that $\theta_0 = 0$. Let X_t be described by the Ornstein-Uhlenbeck type linear SDE

$$dX_t = -\theta_1 X_t dt + dV_t(\theta_2),\ X_0 = 0$$

where $\theta_1 \in \Theta_1 = \mathbb{R}$ and $V(\theta_2)$ is a process with stationary independent increments with $V_0(\theta_2) = 0$ and Levy characteristics $(b(\theta_2)t,\ ct,\ L(\theta_2)t)$ depending on a parameter θ_2 from an arbitrary set Θ_2. We assume that the trajectories of $V(\theta_2)$ are right continuous with left limits. This model provides the natural analogue of the discrete time of AR(1) models with i.i.d. innovations.

For $\theta = (\theta_1, \theta_2) \in \Theta = \Theta_1 \times \Theta_2$, let $P_{\theta,t}$ be the distribution of X when observed up to time t. For $\theta \in \Theta$, let P_Θ denote the distribution of the unique solution of the SDE on $\Omega = D(\mathbb{R}_+, \mathbb{R})$ equipped with the coordinate process $X = (X_t)_{t\geq 0}$ and the σ-algebra $\mathcal{F} = \sigma(X_t : t \geq 0)$ where $D(\mathbb{R}_+, \mathbb{R})$ is the space of real valued functions on $\mathbb{R}_+$ which are right continuous with left limits. Let $(\mathcal{F}_t)_{t\geq 0}$ denote the right continuous filtration generated by X.

Under P_θ, the process $V(\theta_1)$ defined by $V_t(\theta_1) = X_t + \theta_1 \int_0^t X_s ds$ has independent stationary increments and Levy characteristics $(b(\theta_2)t, ct, L(\theta_2)t)$ relative to some fixed continuous bounded truncation function $h : \mathbb{R} \to \mathbb{R}$ with compact support satisfying $h(x) = x$ in a neighborhood of zero, that is,

$$E_\theta \exp(izV_t(\theta_1)) = \exp\left(t\left[izb(\theta_2) - \frac{1}{2}cz^2 + \int (\exp(izx) - 1 - izh(x))L(\theta_2)(dx)\right]\right), \quad z \in \mathbb{R},$$

where $b(\theta_2) \in \mathbb{R}, c > 0$, and $L(\theta_2)$ is a Levy measure on $\mathbb{R}$ satisfying $L(\theta_2)(\{0\}) = 0$ and $\int (x^2 \wedge 1)L(\theta_2)(dx) < \infty$ (See Jacod and Shiryayev [26] II.4.19, III.2.26). Thus X is a solution of the above SDE with respect to $V(\theta_1)$. Assume for simplicity that $c = 1$.

Now fix $\tau, \theta \in \Theta$ such that $P_\tau \neq P_\theta$. Let $m(\tau_2) = E_\tau(V_1(\tau_1))$ and $\sigma^2(\tau_2) = \mathrm{Var}_\tau V_1(\tau_1)$.

Then

$$m(\tau_2) = b(\tau_2) + \int (x - h(x))L(\tau_2)(dx), \quad \sigma^2(\tau_2) = 1 + \int x^2 L(\tau_2)dx.$$

We call $(m(\tau_2)t,\ t,\ L(\tau_2)t)$ Levy characteristics of $V(\tau_1)$ without truncation.

The basic regularity conditions are the following: We assume the conditions (A1)–(A4).

(A1) $L(\tau_2)$ and $L(\theta_2)$ are mutually absolutely continuous and $\int (f(\tau_2, \theta_2))^{1/2} - 1)^2 dL(\theta_2) < \infty$ where $f(\tau_2, \theta_2) = dL(\tau_2)/dL(\theta_2)$.

Let

$$a(\tau_2, \theta_2) := b(\tau_2) - b(\theta_2) - \int (f(\tau_2, \theta_2) - 1)h dL(\theta_2).$$

(A1) implies that $\int |(f(\tau_2, \theta_2) - 1)h| dL(\theta_2) < \infty$. Hence $a(\tau_2, \theta_2) \in \mathbb{R}$. Note also that $a(\tau_2, \theta_2) = -a(\tau_2, \theta_2)$.

Define the Kullback- Leibler information of Levy measures by

$$K(L(\tau_2, \theta_2)) = \int (f(\theta_2, \tau_2) - 1 - \log f(\theta_2, \tau_2))dL(\tau_2).$$

(A2) $K((L(\tau_2), L(\theta_2)) < \infty$.

(A3) $\int (\log f(\tau_2, \theta_2))^2 dL(\tau_2) < \infty$.

(A4) $\int x^2 L(\tau_2)(dx) < \infty$.

(A4) is equivalent to $E_\tau V_t^2(\tau_1) < \infty$ for every $t \geq 0$. Let $m(\tau_2) = E_\tau V_t(\tau_1)$. Then $m(\tau_2) = b(\tau_2) + \int (x - h(x))L(\tau_2)(dx)$ and $\sigma^2(\tau_2) = 1 + \int x^2 L(\tau_2)(dx)$.

Under (A1), $P_{\tau,t}$ and $P_{\tau,t}$ are mutually absolutely continuous and he log-likelihood ratio $\Lambda_T(\tau, \theta) = \log(dP_{\tau,t}/dP_{\theta,t})$ admits the representation

$$\Lambda_T(\tau, \theta) = -\Lambda_T(\theta, \tau) = \int_0^T [(\theta_1 - \tau_1)X_{s-} + a(\tau_2, \theta_2)]dX_s^c(\tau) + \frac{1}{2}\int_0^T [(\theta_1 - \tau_1)X_s + a(\tau_2, \theta_2)]^2 ds$$

$$+ \int (1 - f(\theta_2, \tau_2))d(\mu - \nu(\tau_2)) + \int_0^T \int (f(\theta_2, \tau_2) - 1 - \log f(\tau_2, \theta_2))d\mu$$

where μ is the Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ associated with the jumps of X by $\mu = \sum_{t\geq 0} \varepsilon_{(t,\Delta X_t)} I_{\{\Delta X_t \neq 0\}}$ with with $\Delta X_t = X_t - X_{t-}, \quad \Delta X_0 = 0$ and $d\nu = \nu(\omega, dt, dx)$ is the compensator.

$$\sum_{0<s\leq t} I_{(|\Delta X_s|>1)} = \int_0^t \int_{|x|>1} x d\mu.$$

This gives

$$\Lambda_T(\tau, \theta) = Y_T + \frac{1}{2}\int_0^T [(\theta_1 - \tau_1)X_s + a(\tau_2, \theta_2)]^2 ds + K(L(\tau_2), L(\theta_2))T$$

where

$$Y_T = \int_0^T [(\theta_1 - \tau_1)X_{s-} + a(\tau_2, \theta_2)]dX_s^c(\tau) + \int_0^T \int \log f(\tau_2, \theta_2) d(\mu - \nu(\tau_2)).$$

We assume

$$\int |x| dL(\theta_3, dx) < \infty \quad \text{and} \quad E_\theta V(\theta) = \int x dL(\theta_3)$$

for every θ.

The maximum likelihood estimator (MLE) based on the observations in $[0, T]$ is given by

$$\hat{\theta}_{1,T} := \frac{T - X_T^2 + \sum_{s\leq T} \Delta X_s^2}{2\int_0^T X_s^2 ds}.$$

Bishwal [10] studied Berry-Esseen inequalities for the maximum likelihood estimator in Ornstein-Uhlenbeck driven by Gamma process in the ergodic case by the truncation method.

The MLE based on the observations in $[0, T]$ for the Vasicek model

$$dX_t = (\theta_0 - \theta_1 X_t)dt + dV_t(\theta_2), \; X_0 = 0$$

are given by

$$\widehat{\theta}_{1,T} := \frac{T J_{1,T} + \int_0^T X_s ds J_{2,T}}{T\int_0^T X_s^2 ds - (\int_0^T X_s ds)^2},$$

$$\widehat{\theta}_{0,T} := \frac{\int_0^T X_s ds J_{1,T} + \int_0^T X_s^2 ds J_{2,T}}{T\int_0^T X_s^2 ds - (\int_0^T X_s ds)^2},$$

where

$$J_{1,T} := (\sum_{s\leq T} \Delta X_s^2 + T - X_T^2)/2, \quad J_{2,T} := X_T - \sum_{s\leq T} \Delta X_s.$$

Suppose $V(\theta) - X^c(\theta)$ is a gamma process with parameter $1/\theta_3$ under P_θ, with Lebesgue density $\theta_2^{-t} x^{t-1} e^{-x/\theta_2} I_{(0,\infty)}(x)/\Gamma(t)$, Levy measure $L(\theta)$ has the Lebesgue density $x^{-1}e^{-x/\theta_2} I_{(0,\infty)}(x)$, $E(V(\theta)) = \int x dL(\theta_2) = \theta_2$, $\int x^2 L(\theta_2, dx) = \theta_2^2$ or a Poisson process with intensity θ_2, $E(V(\theta)) = 1/\theta_2$. Then the MLE of θ_3 is

$$\widehat{\theta}_{2,T} := T^{-1} \sum_{s\leq T} \Delta X_s.$$

This estimator is regular and efficient.

The Kullback-Leibler information $K(P_{\tau,t}, P_{\theta,t}) = E_\tau \Lambda_t(\tau, \theta)$ of $P_{\tau,t}$ with respect to $P_{\theta,t}$ is finite under the above conditions.

$$K(\tau, \theta) = \frac{1}{2}(\theta_1 - \tau_1)^2 \left(\frac{\sigma^2(\tau_2)}{2\tau_1} + \frac{m(\tau_2)^2}{\tau_1^2}\right) + (\theta_1 - \tau_1)a(\tau_2, \theta_2)\frac{m(\tau_2)}{\tau_1} + K(L(\tau_2), L(\theta_2)) + \frac{1}{2}a(\tau_2, \theta_2)^2.$$

Let

$$\begin{aligned} H(\tau,\theta) \;&:=\; \left[m(\tau_2)\frac{\theta_1^2}{\tau_1}+(a(\tau_2,\theta_2)-m(\tau_2))\frac{\theta_1}{\tau_1}\right]^2 \\ &+\sigma^2(\tau_2)\frac{(\theta_1^2-\tau_1^2)^2}{8\tau_1^3}+\int\Bigg[\log f(\tau_2,\theta_2)(x)+\frac{(\theta_1-\tau_1)^2}{4\tau_1}x^2 \\ &+\left\{m(\tau_2)\frac{\theta_1^2}{\tau_1}+(a(\tau_2,\theta_2)-2m(\tau_2))\frac{\theta_1}{\tau_1}+m(\tau_2)-a(\tau_2,\theta_2)\right\}x\Bigg]^2 L(\tau_2)(dx) \\ &+\sigma^2(\tau_2)\int x^2L(\tau_2)(dx)\frac{(\theta_1-\tau_1)^4}{8\tau_1^3}. \end{aligned}$$

$$K(P_{\tau,T},P_{\theta,T})=\frac{1}{6}\theta_1^2m(\tau_2)^2T^3+O(T^2)=\bar{K}(\tau,\theta)T^3+O(T^2)$$

where

$$\bar{K}(\tau,\theta):=\frac{1}{6}\theta_1^2m(\tau_2)^2,\quad \bar{H}(\tau,\theta):=\frac{2}{15}\theta_1^4m(\tau_2)^2\sigma^2(\tau_2).$$

Let

$$\bar{H}^{-5}/2=:\beta.$$

Consider the critical case: $\theta_1=0$ and $EV(\theta)\neq 0$.

For simplicity of notations, let $m=m(\tau_2),\sigma^2=\sigma^2(\tau_2),a=a(\tau_2,\theta_2),f=f(\tau_2,\theta_2),L=L(\tau_2),\nu=\nu(\tau_2),X^c=X^c(\tau),V=V(\tau_1)$. Hence $\bar{H}=\frac{2}{15}\theta_1^4m^2\sigma^2$.

Define a P_τ martingale M by $M_t:=V_t-mt$. Then $E_\tau M_t^2=\langle M\rangle_t=\sigma^2$ and integration by parts under P_τ yields

$$X_t=\frac{m}{\tau_1}(1-e^{-\tau_1 t})+e^{-\tau_1 t}\int_0^t e^{\tau_1 s}dM_s$$

if $\tau_1\neq 0$.

The decomposition of M into a continuous martingale and purely discontinuous martingale is equal to

$$M_t=X_t^c+\int_0^t\int x(\mu-\nu)(ds,dx).$$

$$\begin{aligned}\Lambda_T(\tau,\theta)=-\Lambda_T(\theta,\tau)=\int_0^T[(\theta_1-\tau_1)X_{s-}+a(\tau_2,\theta_2)]dX_s^c(\tau)+\frac{1}{2}\int_0^T[(\theta_1-\tau_1)X_s+a(\tau_2,\theta_2)]^2ds \\ +\int(1-f(\theta_2,\tau_2))d(\mu-\nu(\tau_2))+\int_0^T\int(f(\theta_2,\tau_2)-1-\log f(\tau_2,\theta_2))d\mu\end{aligned}$$

Under the assumption $\theta_1=0$ and $EV(\theta)\neq 0$ we have $X=V$.

$$\Lambda_T(\tau,\theta)=Y_T+\frac{1}{2}\int_0^T[(\theta_1-\tau_1)X_s+a(\tau_2,\theta_2)]^2ds+K(L(\tau_2),L(\theta_2))T$$

where

$$Y_T=\int_0^T[(\theta_1-\tau_1)X_{s-}+a(\tau_2,\theta_2)]dX_s^c(\tau)+\int_0^T\int\log f(\tau_2,\theta_2)d(\mu-\nu(\tau_2))$$

is a P_τ martingale with the bracket process

$$\langle Y\rangle_T=\int_0^T[(\theta_1-\tau_1)X_{s-}+a(\tau_2,\theta_2)]^2ds+\int(\log f(\tau_2,\theta_2))^2dLT.$$

$$\Lambda_T(\tau,\theta)=Y_T+\frac{1}{2}\theta_1^2\int_0^T M_s^2ds+\theta_1^2m\int_0^T M_s s ds+\theta_1 a\int_0^T M_s ds$$

$$+[K((L(\tau_2),L(\theta_2))+\frac{1}{2}a^2]T+\frac{1}{2}\theta_1 amT^2+\bar{K}T^3.$$

$$T^{-5/2}[\Lambda_T(\tau,\theta)-\bar{K}T^3]=T^{-5/2}Y_T+T^{-5/2}\frac{1}{2}\int_0^T M_s^2ds+T^{-5/2}\theta_1^2 m\int_0^T M_s sds$$
$$+T^{-5/2}\theta_1 a\int_0^T M_s ds+T^{-3/2}\left[K((L(\tau_2),L(\theta_2))+\frac{1}{2}a^2\right]+T^{-1/2}\frac{1}{2}\theta_1 am.$$

For $T>0$, define M^T by $M_s^T=T^{-1/2}M_{sT}$. By the functional limit theorem for martingales, under P_τ, $M^T\to^{\mathcal{D}}\sigma B$ as $T\to\infty$ and hence by the continuous mapping theorem

$$T^{-5/2}\int_0^T M_s sds=\int_0^1 M_s^T sds\to^{\mathcal{D}}\sigma\int_0^1 B_s sds,$$
$$T^{-2}\int_0^T M_s^2 ds=\int_0^1 (M_s^T)^2 ds\to^{\mathcal{D}}\sigma^2\int_0^1 B_s^2 ds.$$

Since

$$E_\tau\int_0^\infty (1+s^2)^{-2}d\langle Y\rangle_s<\infty$$

it follows from the SLLN for martingales

$$T^{-2}Y_T\to 0\quad P_\tau-\text{a.s.}$$

Also

$$T^{-2}\int_0^T M_s ds\to 0\quad P_\tau-a.s.$$

Since $\int_0^1 B_s sds$ is $\mathcal{N}(0,2/15)$-distributed, by the functional CLT for martingales,

$$\lim_{T\to\infty}T^{-5/2}[\Lambda_T(\tau,\theta)-\bar{K}T^3]=\mathcal{N}(0,\bar{H}).$$

Next the derivative of the likelihood is given by

$$\Lambda_T'(\tau,\theta)=\int_0^T X_{s-}dX_s^c+\theta_1\int_0^T M_s^2ds+2\theta_1 m\int_0^T M_s sds+a\int_0^T M_s ds+\frac{1}{2}amT^2+\frac{1}{3}\theta_1 m^2T^3.$$

Solution of the likelihood equation $\Lambda_T'(\tau,\theta)=0$ provides the MLE

$$\widehat{\theta}_{1,T}=\frac{-\int_0^T X_{s-}dX_s^c-a\int_0^T M_s ds-\frac{1}{2}amT^2}{\int_0^T M_s^2ds+2\int_0^T M_s sds+\frac{1}{3}m^2T^3}.$$

Thus

$$\widehat{\theta}_{1,T}-\theta_1=\frac{-\int_0^T X_{s-}dX_s^c-a\int_0^T M_s ds-\frac{1}{2}amT^2-\theta_1\int_0^T M_s^2ds-2\theta_1\int_0^T M_s sds-\frac{1}{3}\theta_1 m^2T^3}{\int_0^T M_s^2ds+2\int_0^T M_s sds+\frac{1}{3}m^2T^3}$$

which gives

$$\left(\frac{T}{2\beta}\right)^{1/2}(\hat{\theta}_{1,T}-\theta)$$
$$=\frac{\left(\frac{2\beta}{T}\right)^{5/2}\left(-\int_0^T X_{s-}dX_s^c-a\int_0^T M_s ds-\frac{1}{2}amT^2-\theta_1\int_0^T M_s^2ds-2\theta_1\int_0^T M_s sds-\frac{1}{3}\theta_1 m^2T^3\right)}{\left(\frac{2\beta}{T}\right)^5\left(\int_0^T M_s^2ds+2\int_0^T M_s sds+\frac{1}{3}m^2T^3\right)}.$$

Hence the MLE does not converge to normal distribution in the singular case. The asymptotic normality of the MLE only holds in the ergodic case $\theta_1>0$. In the following section we only focus on the ergodic case.

We need the following lemma in the sequel.

Lemma 1.2 (Esseen's Smoothing Lemma:)

Let F be a non-decreasing function and G be a differentiable function of bounded variation on the real line with $F(\pm\infty) = G(\pm\infty)$. Denote the corresponding Fourier-Stieltjes transforms by $\widehat{F}$ and $\widehat{G}$, respectively. Then for all $U > 0$,

$$\sup_{x\in\mathbb{R}} |F(x) - G(x)| \leq \frac{1}{\pi}\int_{-U}^{U} \frac{\left|\widehat{F}(u) - \widehat{G}(u)\right|}{|u|} du + \frac{24}{\pi U} \sup_{x\in\mathbb{R}} |G'(x)|.$$

Proof: See Petrov (1975) or Feller (1971). □

2. Main Results

Let $\Phi(\cdot)$ denote the standard normal distribution function. Throughout the paper C denotes a generic constant (perhaps depending on θ, but not on anything else). In this section, we assume $\theta_1 > 0$ and $\theta_0 = 0$.
Denote

$$\left(\frac{T}{2\beta}\right)^{1/2} (\widehat{\theta}_{1,T} - \theta) = -\frac{\left(\frac{2\beta}{T}\right)^{5/2} Z_T}{\left(\frac{2\beta}{T}\right)^{5} I_T}$$

where

$$Z_T := \int_0^T X_t dV_t \quad \text{and} \quad I_T := \int_0^T X_t^2 dt.$$

Next we have some lemmas on large deviations and Fourier distance of the terms in the MLE to obtain bounds on the Kolmogorov distance between the MLE distribution and normal distribution.

The first lemma is on large deviations for the quadratic variation integral whose proof is similar to Florens-Landais and Pham [22] and Bercu *et al* [2].

Lemma 2.1 For every $\delta > 0$,

$$P\left\{\left|\frac{I_T}{T}\right| \geq \delta\right\} \leq C \exp\left(-CT\delta^2\right).$$

The following lemma gives the bounds on the distance of the characteristic functions of the terms in the MLE and the normal characteristic function.

Lemma 2.2 (a) Let $\phi_T(z_1, z_2) := E\exp\{(z_1 I_T + z_2(X_T^2 + J_T)\}, z_1, z_2 \in \mathbb{C}$ where $J_T := \sum_{s\leq T} \Delta X_s^2$. Then $\phi_T(z_1, z_2)$ exists for $|z_i| \leq \delta$, 1 = 1,2 for some $\delta > 0$ and is given by

$$\phi_T(z_1, z_2) = \exp\left(\frac{\beta T}{2}\right) \left[\frac{2\gamma}{(\gamma - \beta + 2z_2)e^{-\gamma T} + (\gamma + \beta - 2z_2)e^{\gamma T}}\right]^{1/2}$$

where $\gamma = (\beta^2 - 2z_1)^{1/2}$ and we choose the principal branch of the square root.
(b) Let

$$G_{T,x} := -\left(\frac{2\beta}{T}\right)^{1/2} Z_T - \left((\frac{2\beta}{T})^5 I_T - 1\right) x$$

Then for $|x| \le 2(\log T)^{1/2}$ and for $|t| \le \epsilon T^{1/2}$, where ϵ is sufficiently small,

$$\left| E\exp(itG_{T,x}) - \exp(\frac{-t^2}{2})\right| \le C\exp(\frac{-t^2}{4})(|t| + |t|^3)T^{-1/2}.$$

(c) For $|t| \le \epsilon_1 T^{\frac{1}{2}}$, where ϵ_1 is sufficiently small, we have as $T \to \infty$,

$$\left| E\exp\left\{it\left(\frac{2\beta}{T}\right)^{1/2}\left(\beta I_T - \frac{T}{2}\right)\right\} - \exp(-\frac{t^2}{2})\right| \le C\exp(-\frac{t^2}{4})(|t| + |t|^3)T^{-1/2}.$$

(d) For $|t| \le \epsilon_1 T^{\frac{1}{2}}$, where ϵ_1 is sufficiently small, we have as $T \to \infty$,

$$\left| E\exp\left\{it\left(\frac{2\beta}{T}\right)^{1/2} Z_T\right\} - \exp(-\frac{t^2}{2})\right| \le C\exp(-\frac{t^2}{4})(|t| + |t|^3)T^{-1/2}.$$

Proof : Part (a) is given in Liptser and Shiryayev (1978) for $z_1 \in \mathbb{R}, z_2 = 0$ for the Brownian case. We shall prove part (b) in details. Proof of part (c) and (d) are very similar to part (b) and will be omitted.

By Itô formula,

$$Z_T = \theta I_T + \frac{X_T^2}{2} - \frac{T}{2} + \frac{1}{2}\sum_{s\le T}\Delta X_s^2.$$

Note that

$$\begin{aligned}
&E\exp(itG_{T,x})\\
=\;& E\exp\left[-it\left(\frac{2\beta}{T}\right)^{1/2} Z_T - it\left((\frac{2\beta}{T})I_T - 1\right)x\right]\\
=\;& E\exp\left[-it\left(\frac{2\beta}{T}\right)^{1/2}\left\{\theta I_T + \frac{X_T^2}{2} - \frac{T}{2} + \frac{1}{2}\sum_{s\le T}\Delta X_s^2\right\} - it\left((\frac{2\beta}{T})I_T - 1\right)x\right]\\
=\;& E\exp[z_1 I_T + z_2(X_T^2 + \sum_{s\le T}\Delta X_s^2) + z_3])\\
=\;& \exp(z_3)\phi_T(z_1, z_2)
\end{aligned}$$

where

$$z_1 = -it\theta\delta_{T,x}, \quad z_2 = -\frac{it}{2}\left(\frac{2\beta}{T}\right)^{1/2}, \quad z_3 = \frac{itT}{2}\delta_{T,x}, \;\; \delta_{T,x} = \left(\frac{2\beta}{T}\right)^{1/2} + \frac{2x}{T}.$$

Note that (z_1, z_2) satisfies the conditions of (a) by choosing ϵ sufficiently small. Let $\alpha_{1,T}(t), \alpha_{2,T}(t)$, $\alpha_{3,T}(t)$ and $\alpha_{4,T}(t)$ be functions which are $O(|t|T^{-1/2}), O(|t|^2T^{-1/2}), O(|t|^3T^{-3/2})$ and $O(|t|^3T^{-1/2})$ respectively. Note that for the given range of values of x and t, the conditions on z_i for part (a) of Lemma are satisfied. Note also that $z_2 = \alpha_{1,T}(t)$. Further, with

$$\varpi_T(t) = 1 + it\frac{\delta_{T,x}}{\beta} + \frac{t^2\delta_{T,x}^2}{2\beta^2},$$

$$\begin{aligned}\gamma &= (\beta^2 - 2z_1)^{1/2}\\ &= \theta\left[1 - \frac{z_1}{\beta^2} - \frac{z_1^2}{2\beta^4} + \frac{z_1^3}{2\beta^8} + \cdots\right]\\ &= \beta\left[1 + it\frac{\delta_{T,x}}{\beta} + \frac{t^2\delta_{T,x}^2}{2\beta^2} + \frac{it^3\delta_{T,x}^3}{2\beta^3} + \cdots\right]\\ &= \beta[1 + \alpha_{1,T}(t) + \alpha_{2,T}(t) + \alpha_{3,T}(t)]\\ &= \beta\varpi_T(t) + \alpha_{3,T}(t)\\ &= \beta[1 + \alpha_{1,T}(t)].\end{aligned}$$

Thus

$$\gamma - \beta = \alpha_{1,T},\ \ \gamma + \beta = 2\beta + \alpha_{1,T}.$$

Hence the above expectation equals

$$\begin{aligned}&\exp\left(z_3 + \frac{\beta T}{2}\right)\left[\frac{2\beta\varpi_T(t) + \alpha_{3,T}(t)}{\alpha_{1,T}\exp\{-\theta T\varpi_T(t) + \alpha_{4,T}(t)\} + (2\beta + \alpha_{1,T}(t))\exp\{\beta T\varpi_T(t) + \alpha_{4,T}(t)\}}\right]^{1/2}\\ = &\left[\frac{1 + \alpha_{1,T}(t)}{\alpha_{1,T}\exp(\chi_T(t)) + (1 + \alpha_{1,T}(t))\exp(\psi_T(t))}\right]^{1/2}\end{aligned}$$

where

$$\begin{aligned}\chi_T(t) &= -\beta T\varpi_T(t) + \alpha_{4,T}(t) - 2z_3 - \beta T\\ &= -2\beta T + \alpha_{1,T}(t) + t^2\alpha_{1,T}(t).\end{aligned}$$

and

$$\begin{aligned}\psi_T(t) &= \beta T\varpi_T(t) + \alpha_{4,T}(t) - 2z_3 - \beta T\\ &= \beta T\left[1 + it\frac{\delta_{T,x}}{\beta} + \frac{t^2\delta_{T,x}^2}{2\beta^2}\right] + \alpha_{4,T}(t) - itT\delta_{T,x} - \beta T\\ &= \frac{t^2 T}{2\beta}\left[\left(\frac{2\beta}{T}\right)^{5/2} + \frac{2x}{T}\right]^2\\ &= t^2 + t^2\alpha_{1,T}(t).\end{aligned}$$

Hence, for the given range of values of t, $\chi_T(t) - \psi_T(t) \leq -\beta T$.

Hence the above expectation equals

$$\begin{aligned}&\exp(-\frac{t^2}{2})(1 + \alpha_{1,T})^{1/2}\left[\alpha_{1,T}\exp\{-2\beta T + \alpha_{1,T} + t^2\alpha_{1,T}\} + (1 + \alpha_{1,T}(t))\exp\{t^2\alpha_{1,T}(t)\}\right]^{-1/2}\\ = &\exp(-\frac{t^2}{2})\left[1 + \alpha_{1,T})(1 + \alpha_{1,T}(1 + \alpha_{1,T})\exp\{-\beta T + \alpha_{1,T} + t^2\alpha_{1,T}\}\right]\exp(t^2\alpha_{1,T}(t)).\quad\square\end{aligned}$$

Parts (c) and (d) of Lemma 2.2 give the Berry-Esseen rate for Z_T and I_T immediately by using the Esseen's lemma.

Corollary 2.3

$$\text{(a)}\quad \sup_{x\epsilon\mathbb{R}}\left|P\left\{\left(\frac{2\beta}{T}\right)^{1/2} Z_T \leq x\right\} - \Phi(x)\right| \leq CT^{-1/2}.$$

$$\text{(b)}\quad \sup_{x\in\mathbb{R}}\left|P\left\{(\frac{2\beta}{T})^{1/2}(\theta I_T - \frac{T}{2}) \leq x\right\} - \Phi(x)\right| \leq CT^{-1/2}.$$

Before we prove the results on the Berry-Esseen bound for the MLE with nonrandom norming, we need the following estimate on the tail behaviour of the MLE.

Lemma 2.4

$$P\left\{(\frac{T}{2\beta})^{1/2}|\theta_T-\theta|\geq 2(\log T)^{1/2}\right\}\leq CT^{-1/2}$$

Proof :

$$\begin{aligned}
& P\left\{(\frac{T}{2\beta})^{1/2}|\theta_T-\theta|\geq 2(\log T)^{1/2}\right\} \\
= \ & P\left\{\left|\frac{(\frac{2\beta}{T})^{1/2}Z_T}{(\frac{2\beta}{T})I_T}\right|\geq 2(\log T)^{1/2}\right\} \\
\leq \ & P\left\{\left|(\frac{2\beta}{T})^{1/2}Z_T\right|\geq (\log T)^{1/2}\right\}+P\left\{\left|(\frac{2\beta}{T})I_T\right|\leq\frac{1}{2}\right\} \\
\leq \ & \left|P\left\{(\frac{2\beta}{T})^{1/2}|Z_T|\geq(\log T)^{1/2}\right\}-2\Phi(-(\log T)^{1/2})\right| \\
& +2\Phi(-(\log T)^{1/2})+P\left\{\left|(\frac{2\beta}{T})I_T-1\right|\geq\frac{1}{2}\right\} \\
\leq \ & \sup_{x\in\mathbb{R}}\left|P\left\{(\frac{2\beta}{T})^{1/2}|Z_T|\geq x\right\}-2\Phi(-x)\right| \\
\leq \ & \sup_{x\epsilon\mathbb{R}}\left|P\left\{(\frac{2\beta}{T})^{1/2}|Z_T|\geq x\right\}-2\Phi(-x)\right| \\
& +2\Phi(-(\log T)^{1/2})+P\left\{\left|(\frac{2\beta}{T})I_T-1\right|\geq\frac{1}{2}\right\} \\
\leq \ & CT^{-1/2}+C(T\log T)^{-1/2}+Ce^{-CT} \\
\leq \ & CT^{-1/2}.
\end{aligned}$$

The bounds for the first and the third terms come from Corollary 2.3 (a) and Lemma 2.1 respectively and that for the middle term comes from Feller ([21], p. 166). □

We are now in a position to obtain the Berry-Esseen bound of the order $O(T^{-5/2})$ for the MLE.

Theorem 2.5

$$(a)\sup_{x\in\mathbb{R}}\left|P\left\{(\frac{T}{2\beta})^{1/2}(\widehat{\theta}_{1,T}-\theta)\leq x\right\}-\Phi(x)\right|=O(T^{-1/2}).$$

Proof : We shall consider two possibilities: (i) $|x|>2(\log T)^{1/2}$ and (ii) $|x|\leq 2(\log T)^{1/2}$.
(i) We shall give a proof for the case $x>2(\log T)^{1/2}$. The proof for the case $x<-2(\log T)^{1/2}$ runs similarly. Note that

$$\left|P\left\{(\frac{T}{2\beta})^{1/2}(\widehat{\theta}_{1,T}-\theta)\leq x\right\}-\Phi(x)\right|\leq P\left\{(\frac{T}{2\beta})^{1/2}(\widehat{\theta}_{1,T}-\theta)\geq x\right\}+\Phi(-x)$$

But $\Phi(-x)\leq\Phi(-2(\log T)^{1/2})\leq CT^{-2}$. See Feller ([21], p. 166).
Moreover by Lemma 2.4, we have

$$P\left\{(\frac{T}{2\beta})^{1/2}(\widehat{\theta}_{1,T}-\theta)\geq 2(\log T)^{1/2}\right\}\leq CT^{-1/2}.$$

Hence

$$\left|P\left\{(\frac{T}{2\beta})^{1/2}(\widehat{\theta}_{1,T}-\theta)\le x\right\}-\Phi(x)\right|\le CT^{-1/2}.$$

(ii)

$$\text{Let } A_T:=\left\{(\frac{T}{2\beta})^{1/2}|\widehat{\theta}_{1,T}-\theta|\le 2(\log T)^{1/2}\right\}\text{ and } B_T:=\left\{\frac{I_T}{T}>c_0\right\}$$

where $0<c_0<\frac{1}{2\beta}$. By Lemma 2.4, we have

$$P(A_T^c)\le CT^{-1/2}. \tag{2.1}$$

By Lemma 2.1, we have

$$P(B_T^c)=P\left\{(\frac{2\beta}{T})I_T-1<2\theta c_0-1\right\}<P\left\{|(\frac{2\beta}{T})I_T-1|>1-2\theta c_0\right\}\le Ce^{-CT}. \tag{2.2}$$

Let b_0 be some positive number. For $\omega\in A_T\cap B_T$ and for all $T>T_0$ with $4b_0(\log T_0)^{5/2}(\frac{2\beta}{T_0})^{1/2}\le c_0$, we have

$$\begin{aligned}
&(\frac{T}{2\beta})^{1/2}(\widehat{\theta}_{1,T}-\theta)\le x\\
\Rightarrow\;& I_T+b_0T(\widehat{\theta}_{1,T}-\theta)<I_T+(\frac{T}{2\beta})^{1/2}2b_0\theta x\\
\Rightarrow\;& (\frac{T}{2\beta})^{1/2}(\widehat{\theta}_{1,T}-\theta)[I_T+b_0T(\widehat{\theta}_{1,T}-\theta)]<x[I_T+(\frac{T}{2\beta})^{1/2}2b_0\theta x]\\
\Rightarrow\;& (\widehat{\theta}_{1,T}-\theta)I_T+b_0T(\widehat{\theta}_{1,T}-\theta)^2<(\frac{2\beta}{T})^{1/2}I_Tx+2b_0\theta x^2\\
\Rightarrow\;& Z_T+(\widehat{\theta}_{1,T}-\theta)I_T+b_0T(\widehat{\theta}_{1,T}-\theta)^2<Z_T+(\frac{2\beta}{T})^{1/2}I_Tx+2b_0\theta x^2\\
\Rightarrow\;& 0<Z_T+(\frac{2\beta}{T})^{1/2}I_Tx+2b_0\theta x^2
\end{aligned}$$

since

$$\begin{aligned}
&I_T+b_0T(\widehat{\theta}_{1,T}-\theta)\\
>\;& Tc_0+b_0T(\widehat{\theta}_{1,T}-\theta)\\
>\;& 4b_0(\log T)^{1/2}(\frac{2\beta}{T})^{1/2}-2b_0(\log T)^{1/2}(\frac{2\beta}{T})^{1/2}\\
=\;& 2b_0(\log T)^{1/2}(\frac{2\beta}{T})^{1/2}>0.
\end{aligned}$$

On the other hand, for $\omega\in A_T\cap B_T$ and for all $T>T_0$ with $4b_0(\log T_0)^{1/2}(\frac{2\beta}{T_0})^{1/2}\le c_0$, we have

$$\begin{aligned}
&(\frac{T}{2\beta})^{1/2}(\widehat{\theta}_{1,T}-\theta)>x\\
\Rightarrow\;& I_T-b_0T(\widehat{\theta}_{1,T}-\theta)<I_T-(\frac{T}{2\beta})^{1/2}2b_0\theta x\\
\Rightarrow\;& (\frac{T}{2\beta})^{1/2}(\widehat{\theta}_{1,T}-\theta)[I_T-b_0T(\widehat{\theta}_{1,T}-\theta)]>x[I_T-(\frac{T}{2\beta})^{1/2}2b_0\theta x]\\
\Rightarrow\;& (\widehat{\theta}_{1,T}-\theta)I_T-b_0T(\widehat{\theta}_{1,T}-\theta)^2>(\frac{2\beta}{T})^{1/2}I_Tx-2b_0\theta x^2\\
\Rightarrow\;& Z_T+(\widehat{\theta}_{1,T}-\theta)I_T-b_0T(\widehat{\theta}_{1,T}-\theta)^2>Z_T+(\frac{2\beta}{T})^{1/2}I_Tx-2b_0\theta x^2
\end{aligned}$$

$$\Rightarrow \quad 0 > Z_T + (\frac{2\beta}{T})^{1/2} I_T x - 2b_0\theta x^2$$

since

$$\begin{aligned} & I_T - b_0 T(\widehat{\theta}_{1,T} - \theta) \\ > \ & Tc_0 - b_0 T(\widehat{\theta}_{1,T} - \theta) \\ > \ & 4b_0(\log T)^{1/2}(\frac{2\beta}{T})^{1/2} - 2b_0(\log T)^{1/2}(\frac{2\beta}{T})^{1/2} \\ = \ & 2b_0(\log T)^{1/2}(\frac{2\beta}{T})^{1/2} > 0. \end{aligned}$$

Hence

$$0 < Z_T + (\frac{2\beta}{T})^{1/2} I_T x - 2b_0\theta x^2 \Rightarrow (\frac{T}{2\beta})^{1/2}(\widehat{\theta}_{1,T} - \theta) \leq x.$$

We use Pfanzagl (1971)'s squeezing method developed for the minimum contrast estimator in the i.i.d. case.

Let us introduce the piecewise quadratic random functions involving the martingale and quadratic variation part of $\widehat{\theta}_{1,T} - \theta$:

$$g^{\pm}(x) := Z_T + (\frac{2\beta}{T})^{1/2} I_T x \pm 2b_0\theta x^2.$$

Let us introduce the events

$$D^{\pm}_{T,x} := \left\{ Z_T + (\frac{2\beta}{T})^{1/2} I_T x \pm 2b_0\theta x^2 > 0 \right\}.$$

we obtain

$$D^-_{T,x} \cap A_T \cap B_T \subseteq A_T \cap B_T \cap \left\{ (\frac{T}{2\beta})^{1/2}(\widehat{\theta}_{1,T} - \theta) \leq x \right\} \subseteq D^+_{T,x} \cap A_T \cap B_T. \tag{2.3}$$

This gives

$$P(D^-_{T,x} \cap A_T \cap B_T) \leq P\left(A_T \cap B_T \cap \left\{ \left((\frac{2\beta}{T})^{1/2} \right) (\widehat{\theta}_{1,T} - \theta) \leq x \right\} \right) \leq P(D^+_{T,x} \cap A_T \cap B_T)$$

so that

$$\begin{aligned} & \left| P\left(A_T \cap B_T \cap \left\{ \left((\frac{2\beta}{T})^{1/2} \right) (\widehat{\theta}_{1,T} - \theta) \leq x \right\} \right) - \Phi(x) \right| \\ & \leq \max\left\{ |P(D^-_{T,x} \cap A_T \cap B_T) - \Phi(x)|, \ |P(D^+_{T,x} \cap A_T \cap B_T) - \Phi(x)| \right\} \\ & \leq \max\left\{ |P(D^-_{T,x}) - \Phi(x)|, \ |P(D^+_{T,x}) - \Phi(x)| \right\} + P(A_T \cap B_T)^c. \end{aligned}$$

From (2.1) and (2.2), we have

$$P(A_T \cap B_T)^c \leq CT^{-1/2}$$

for all $T > T_0$ and $|x| \leq 2(\beta T)^{1/2}$.

If it is shown that

$$\left| P\left\{ D^{\pm}_{T,x} \right\} - \Phi(x) \right| \leq CT^{-1/2} \tag{2.4}$$

for all $T > T_0$ and $|x| \leq 2(\log T)^{1/2}$, then the theorem would follow from (2.1) – (2.3).

We shall prove (2.4) for $D^+_{T,x}$. The proof for $D^-_{T,x}$ is analogous.

Note that

$$\begin{aligned} & \left|P\left\{D^+_{T,x}\right\} - \Phi(x)\right| \\ = & \left|P\left\{-(\frac{2\beta}{T})^{1/2} Z_T - \left((\frac{2\beta}{T}) I_T - 1\right) x < x + 2(\frac{2\beta}{T})^{1/2} b_0 \theta x^2\right\} - \Phi(x)\right| \\ \leq & \sup_{y\in\mathbb{R}} \left|P\left\{-(\frac{2\beta}{T})^{1/2} Z_T - \left((\frac{2\beta}{T}) I_T - 1\right) x \leq y\right\} - \Phi(y)\right| + \left|\Phi\left(x + (\frac{2\beta}{T})^{1/2} b_0 \theta x^2\right) - \Phi(x)\right| \\ =: & \Delta_1 + \Delta_2. \end{aligned} \tag{2.5}$$

Lemma 2.2 (b) and Esseen's lemma 1.2 immediately yield

$$\Delta_1 \leq CT^{-1/2}. \tag{2.6}$$

On the other hand, for all $T > T_0$,

$$\Delta_2 \leq 2(\frac{2\beta}{T})^{1/2} b_0 \theta x^2 (2\pi)^{-1/2} \exp(-\overline{x}^2/2)$$

where

$$|\overline{x} - x| \leq 2(\frac{2\beta}{T})^{1/2} b_0 \theta x^2.$$

Since $|x| \leq 2(\log T)^{1/2}$, it follows that $|\bar{x}| > |x|/2$ for all $T > T_0$ and consequently

$$\begin{aligned} \Delta_2 & \leq 2(\frac{2\beta}{T})^{1/2} b_0 \theta x^2 (2\pi)^{-1/2} x^2 \exp(-x^2/8) \\ & \leq CT^{-1/2}. \end{aligned} \tag{2.7}$$

From (2.5) – (2.7), we obtain

$$\left|P\left\{D^+_{T,x}\right\} - \Phi(x)\right| \leq CT^{-1/2}.$$

This completes the proof of the theorem. □

Remarks 1) The deterministic norming we used in this paper, though useful for testing hypotheses about the unknown parameter, may not necessarily give a confidence interval. One may use random norming and obtain the Berry-Esseen bounds, see Bishwal [3] in the Brownian motion innovation case.

2) Sequential estimation unifies the ergodic, nonergodic and singular case and gives asymptotic normality in all cases. Sequential maximum likelihood estimation in semimartingales was studied in Bishwal [5].

3) For fixed T case, Berry-Esseen rate of the order $O(N^{-3/2})$ is obtained in Es-Sebaiy *et al.* [19] for the MLE in a linear SPDE model where N is the number of Fourier coefficients in the expansion of the solution.

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