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ON STATE-DEPENDENT DELAY, FRACTIONAL ORDER NEUTRAL INTEGRODIFFERENTIAL SYSTEMS WITH NON-INSTANTANEOUS IMPULSES AND NONLOCAL CONDITIONS: EXISTENCE AND CONTROLLABILITY RESULTS

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Abstract. The current paper is concerned by the existence and controllability of mild solution for a new class of state-dependent delay, fractional order, neutral integrodifferential systems with non-instantaneaous impulses and nonlocal conditions. More precisely, using the $\alpha-$resolvent operator theory, combinated with the theory of measures of noncompactness, we derive some conditions which guarantee the existence and the controllability of the mild solutions. One of the major innovations of this system is that the nonlinear term is a function of both the state function and the control function. We provide an example which shows the applicability of the results.

1. Introduction

The theory of differential equations has been widely developed and now applies in all fields. Whether it is physics, climate sciences, medecine, including economics, finance, energy, etc..., it is now obvious that we need to model the phenomena that we meet in order to make rational decisions to achieve the targeted objectives. However, there are complex phenomena for which the classical differential and partial differential equations fail for their modeling. For the latter, the use of fractional order differential equations is essential.

Fractional calculus is the theory of integrals and derivatives of arbitrary real (and even complex) order and was first suggested in works by mathematicians such as Leibniz, L'Hospital, Abel, Liouville, Riemann, etc. The importance of fractional derivatives for modeling phenomena in different branches of science and engineering is due to their nonlocality nature, an intrinsic property of many complex systems. Unlike the derivative of integer order, fractional derivatives do not take into account only local characteristics of the dynamics but considers the global evolution of the system; for that reason, when dealing with certain phenomena, they provide more accurate models of real-world behavior than standard derivatives. They arise in many scientific and engineering areas such as physics, chemistry, biology, biophysics, economics, signal and image processing, etc. Particularly, nonlinear systems describing different phenomena can be modeled with fractional derivatives. Chaotic behavior has also been reported in some fractional models. There exist theoretical results related to existence and uniqueness of solutions

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to initial and boundary value problems with fractional differential equations. For instance, in [12], V. Lakshmikantham and A.S.Vatsala, discussed about the basic theory of fractional differential equations. In [6], K. Diethelm and J. Ford studied the existence and structural stability of solutions of nonlinear differential equations of fractional order. Using the Grünwald-Letnikov definition of fractional derivatives, R. Sherer et al. discussed in [17] on finite difference schemes for the approximation of solutions for fractional order differential systems. For more informations on the theory of frcational differential systems, we refer the readers to the book of N'Guerekata et al. [1].

On the other hand, following the exemple of the resolvent operator developed by R. Grimmer [8] in the 1980s, to study the solutions of classical integrodifferential systems, R. P. Agarwal et al. [2] and J. P. Dos Santos [7] introduced the $\alpha-$resolvent operator for the fractional case. This last tool indeed generalizes Grimmer's theory to the case where the order of the derivative is fractional, first for $\alpha \in (1, 2)$ then for $\alpha \in (0, 1)$.

Moreover, in real life, it is common to find phenomena that in addition to the impulsive effects, have effects of delays. Modeling such phenomena involves impulsive (integro)differential systems with delay. These equations have also been widely studied, notably by Hernandez et al. [10], Ren et al. [13] and references therein.

However, it is important to point out that from a theoretical point of view, the study of impulsive differential equations has long focused on the case where impulses appear at fixed moments. It was not until very recently in 2013 that O'Regan and Hernandez developed in [16] a larger study, taking into account the case where impulses are non-instantaneous. This new study, which has the merit of covering a broad category of real phenomena, has been further developed by numerous publications including those due to Agarwal et al. [2], Nieto et al. [4] and refernces therein. Unlike the classical local conditions, nonlocal conditions are involved in phenomena with functional dependance in the boundary conditions or in the equation.

The controllability is one of the fundamental concepts in mathematical control theory. It is a qualitative behaviour of dynamical systems and is of particular importance in control theory of dynamical systems. A control system is said to be controllable if it is possible to steer the solution of the system from an arbitrary initial state to an arbitrary final state using the set of admissible controls, where the initial and final states may vary over the entire space. Different types of controllability have been defined, such as approximate, null, local null and local approximate null controllability. Thus, the study of controllability for such systems is important for many applications. Using various approaches, reseachers have studied different types of controllability for several classes of integrodifferential systems. See for instance the papers [9, 21] and references therein. But for the differential systems for which the nonlinear term is a function of both the state function and the control function, we had to wait until 2011, and the paper of Sukavanam [19]. Later, in 2022 Wen et al. [20] will extend the previous work to the case of a fractional order differential system of neutral type. However, to the best of our knowledge, there is no work combining both the notions of state dependent delay, neutral integrodifferential systems with non-instantaneous impulses, nonlocal conditions and in which the nonlinear term depend on the control function.

Motivated by the considerations mentioned above, in this paper, using an approach introduced in the 1980s by Banas and Goebel (see [5]) and which mainly uses the theory of measures of noncompactness, we consider a controllability problem for the following class of state-dependent delay, fractional order, neutral integrodifferential systems with non-instantaneous impulses and nonlocal conditions and for which the nonlinear term is a function of both the state function and the control function.

$$(1)\quad \begin{cases} {}^cD_t^{\alpha}\left[v(t)-p\left(t,v_{\rho(t,v_t)}\right)\right]=A\left[v(t)-p\left(t,v_{\rho(t,v_t)}\right)\right]+\displaystyle\int_0^t\gamma(t-s)\left[v(t)-p\left(t,v_{\rho(t,v_t)}\right)\right] \\ \qquad +\zeta\left(t,v_{\rho(t,v_t)},u(t)\right)+Bu(t),\ t\in \mathsf{I}=\displaystyle\bigcup_{i=0}^{n}\mathsf{I}_i:=\bigcup_{i=0}^{n}(s_i,t_{i+1}] \\ v(t)=h_i\left(t,v_{\rho(t,v_t)}\right),\ t\in\displaystyle\bigcup_{i=1}^{n}(t_i,s_i] \\ v_0=g\left(\beta(v),v\right)\in\mathsf{C}=\mathscr{C}([-\tau,0],\mathbb{H}). \end{cases}$$

In this system, ${}^cD_t^{\alpha}$ is the Caputo fractional derivative of order $\alpha\in(0,1)$; the state variable $v(\cdot)$ belongs to a Hilbert space $(\mathbb{H},\|\cdot\|)$; $A:\mathsf{D}(A)\subset\mathbb{H}\to\mathbb{H}$ is an infinitesimal generator of an analytic semigroup $\{T(t),\ t\geq 0\}$ in $\mathbb{H}$; for $t\geq 0$, $\gamma(t)$ is a closed linear operator with domain at least $\mathsf{D}(A)$. $B:\ \mathscr{U}\to\mathbb{H}$ is a bounded linear operator, $\mathscr{U}$ is a Hilbert space and the control function u belongs to $\mathsf{L}^2(J,\mathscr{U})$. Let $0=t_0=s_0<t_1\leq s_1\leq t_2<\cdots<t_n\leq s_n\leq t_{n+1}=\mathsf{T}$, the mappings $\beta:\mathscr{C}([-\tau,\mathsf{T}],\mathbb{H})\to[0,\mathsf{T}]$; $\rho:[0,\mathsf{T}]\times\mathsf{C}\to[0,\mathsf{T}]$; $g:[0,\mathsf{T}]\times\mathscr{C}([-\tau,\mathsf{T}],\mathbb{H})\to\mathsf{C}$; $p:\ \bigcup_{i=0}^{n}(s_i,t_{i+1}]\times\mathsf{C}\to\mathbb{H}$; $\zeta:\ \bigcup_{i=0}^{n}(s_i,t_{i+1}]\times\mathsf{C}\times\mathscr{U}\to\mathbb{H}$ and for $i\in[\![1,n]\!]$, $h_i:\ (t_i,s_i]\times\mathsf{C}\to\mathbb{H}$ are given. The History $v_t:[-\tau,0]\to\mathbb{H}$ is an element of C given by $v_t(\theta)=v(t+\theta)$.

In the following, we introduce in section 1, the notations and preliminary notions that will be used in the rest of the work, mainly the theory of measures of noncompactness and the theory of $\alpha-$resolvent operators. In section 2, we present and prove our main results. Section 3 focuses on the applicability of the results presented in the previous section and we conclude in the last section.

2. Preliminaries

First, we give some notations which will be used in the sequel of this work.

Let introduce the space
$\varXi=\{v:[-\tau,0]\to\mathbb{H}:\ v$ is continuous except at s where $v(s^-)$ and $v(s^+)$ exists and $v(s^-)=v(s)\}$, furnished with the norm $\|v\|_{\varXi}=\sup_{t\in[-\tau,0]}\{\|v(t)\|\}$.

We denote by $\varXi_{PC}$, the space
$\varXi_{PC}=\{v:[-\tau,\mathsf{T}]\to\mathbb{H}$ such that $v|_{(s_i,t_{i+1}]}$ is continuous and $v(t_i)=v(t_i^-)$, $v(t_i^+)$ exist; $v_0\in\mathsf{C}$ and $\sup_{t\in[-\tau,\mathsf{T}]}\|v(t)\|<\infty\}$, given with the norm $\|v\|_{\varXi_{PC}}=\sup_{t\in[-\tau,\mathsf{T}]}\{\|v(t)\|\}$. In the following, we will denote this norm by $\|\cdot\|$ is there is no possible confusion.

2.1. The Kuratowski measure of noncompactness.

This section concerns the Kuratowski measure of noncompactness. We give some basic definitions and lemmas.

Definition 2.1. (Kuratowski [11]) *Let $(\mathbb{H}, d)$ be a complete metric space. The Kuratowski measure of noncompactness of a nonempty and bounded subsets D of $\mathbb{H}$, denoted by $\mu(\mathsf{D})$, is the infimum of all numbers $\varepsilon > 0$ such that D can be covered by a finite number of sets with diameters less than ε, i.e.,*

$$\mu(\mathsf{D}) = \inf\left\{\varepsilon > 0 : \ \mathsf{D} \subset \bigcup_{i=1}^{n} \mathsf{B}_i : \ \mathsf{B}_i \subset \mathbb{H}, \ diam(\mathsf{B}_i) < \varepsilon, \ i \in [\![1, n]\!], \ n \in \mathbb{N}\right\}.$$

The function μ defined on the set of all nonempty and bounded subsets of $(\mathbb{H}, d)$, is called Kuratowski's measure of noncompactness.

Throughout this paper, we denote by $\mu(\cdot)$ and $\mu_{[a,b]}(\cdot)$ the Kuratowski mesure of noncompactness on the bounded subsets of $\mathbb{H}$ and $\mathscr{C}([a, b], \mathbb{H})$, respectively.

Let $t \in [a, b]$. If $\mathsf{D} \subset \mathscr{C}([a, b], \mathbb{H})$ is bounded, then $\mathsf{D}(t) := \{x(t) : x \in \mathsf{D}\}$ is bounded in $\mathbb{H}$ and $\mu(\mathsf{D}(t)) \leq \mu_{[a,b]}(\mathsf{D})$.

Next, we give the following results on the Kuratowski measure of noncompactness which will be very useful in the sequel.

Lemma 2.1. [5] *On the Banach space $\mathbb{H}$, let's consider the bounded and equicontinuous set $\mathsf{D} \subset \mathscr{C}([a, b], \mathbb{H})$. Then $\mu(\mathsf{D}(t))$ is continuous on $[a, b]$, and*

$$\mu_{[a,b]}(\mathsf{D}) = \sup_{t\in[a,b]} \mu(\mathsf{D}(t)).$$

Lemma 2.2. [18] *On the Banach space $\mathbb{H}$, if $\mathsf{D} = \{x_n\} \subset \mathscr{C}([a, b], \mathbb{H})$ is a bounded and countable set, then $\mu(\mathsf{D}(t))$ is Lebesgue integrable on $[a, b]$, and we have:*

$$\mu\left(\left\{\int_a^b x_n(t)dt : n \in \mathbb{N}\right\}\right) \leq 2\int_a^b \mu(\mathsf{D}(t))dt.$$

Lemma 2.3. [3] *If D is a bounded subset of a Banach space $\mathbb{H}$, then there is a countable subset D^* of D, such that $\mu(\mathsf{D}) \leq 2\mu(\mathsf{D}^*)$.*

Lemma 2.4. [14] *Considering two bounded subsets B_1 and B_2 of $\mathbb{H}$, with norms respectively $\|\cdot\|_{B_1}$ and $\|\cdot\|_{B_2}$, if there is surjective map $\varXi : B_1 \to B_2$ such that for any $x, y \in B_1$, $\|\varXi(x) - \varXi(y)\|_{B_2} \leq \|x - y\|_{B_1}$, then $\mu(B_2) \leq \mu(B_1)$.*

For more details on the measures of noncompactness and their properties, we refer the reader to [11] and references therein.

To end this part, we give the Mönch fixed point theorem, which is usefull to prove our existence result.

Theorem 2.5. [15] *Let B be a bounded closed and convex subset of $\mathbb{H}$, $0 \in \mathsf{B}$. $\varLambda : \mathsf{B} \to \mathsf{B}$ is continuous, such that for any countable set $\mathsf{D} \subseteq \mathsf{B}$, $\mathsf{D} \subseteq \overline{conv}\left(\{0\} \cup \varLambda(\mathsf{D})\right)$, D is relatively compact. Then $\varLambda$ has a fixed point in B.*

2.2. **Fractional order integrodifferential systems in Banach spaces.** Now, for the question of existence of mild solutions of the integrodifferential Eq.(1), we recall some needed fundamental results. Regarding the theory of $\alpha-$resolvent operator, we refer the reader to [7].

First, we consider the following homogeneous integrodifferential Cauchy problem:

$$\begin{cases} D_t^{\alpha} v(t) = Av(t) + \int_0^t \gamma(t-s)v(s)ds \text{ for } t \in [0, \mathsf{T}], \\ v(0) = v_0 \in \mathbb{H}, \end{cases} \tag{2}$$

where $\alpha \in (0,1)$ and $D_t^{\alpha} v(t)$ represents the Caputo derivative of v.

The Caputo derivative of v is defined for $\alpha > 0$ by:

$$D_t^{\alpha} v(t) := \int_0^t l_{n-\alpha}(t-s) \frac{d^n}{ds^n} v(s) ds,$$

where n is the smallest integer greater than or equal to α and $l_{n-\alpha}$ is the Gelfand-Shilov function given by $l_{\beta}(t) = \frac{t^{\beta-1}}{\Gamma(\beta)},\ t > 0,\ \beta \geq 0$. These functions satisfy the semigroup property

$$l_{\alpha} \star l_{\beta} = l_{\alpha+\beta}.$$

Denoted by

$$J_t^{\alpha} f(t) = (l_{\alpha} \star f)(t) = \int_0^t l_{\alpha}(t-s) f(s) ds,$$

we get

$$D_t^{\alpha} J_t^{\alpha} f(t) = f(t) \quad \text{and} \quad J_t^{\alpha} D_t^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0) \frac{t^k}{k!}.$$

The properties of the Laplace transform and the fact that $\widehat{l_{\alpha}}(\Lambda) = \Lambda^{-\alpha}$ lead to:

$$\widehat{D_t^{\alpha}} f(\Lambda) = \Lambda^{\alpha} f(\Lambda) - \sum_{k=0}^{n-1} f^{(k)}(0) \Lambda^{\alpha-1-k}.$$

Throughout this paper, let $(\mathsf{Z}, \|\cdot\|_{\mathsf{Z}})$ and $(\mathsf{W}, \|\cdot\|_{\mathsf{W}})$ be Banach spaces. We denote by $\mathcal{L}(\mathsf{Z}, \mathsf{W})$ the space of bounded linear operators from Z into W endowed with norm of operators, and we write simply $\mathcal{L}(\mathsf{Z})$ when $\mathsf{Z} = \mathsf{W}$. By $R(Q)$ we denote the range of a map Q and for a closed linear operator $P : \mathsf{D}(P) \subseteq \mathsf{Z} \to \mathsf{W}$, the notation $[\mathsf{D}(P)]$ represents the domain of P endowed with the graph norm, $\|z\|_1 = \|z\|_{\mathsf{Z}} + \|Pz\|_{\mathsf{W}},\ z \in \mathsf{D}(P)$. The notation, $\mathsf{B}(v, R)$ and $\mathsf{B}[v, R]$ represent the open ball and the closed ball, respectively, with center at v and radius $R > 0$ in $\mathbb{H}$. Let $\mathsf{J} \subset \mathbb{R}$, by $\mathscr{C}(\mathsf{J}, \mathbb{H})$ we denote the space of continuous functions defined on J into $\mathbb{H}$, and $\mathscr{C}^1(\mathsf{J}, \mathbb{H})$ stands for the space of continuous functions from J to $\mathbb{H}$ having continuous derivative. We define the space $\mathscr{C}^{\alpha}(\mathsf{J}, \mathbb{H})$, by $\mathscr{C}^{\alpha}(\mathsf{J}, \mathbb{H}) := \{v \in \mathscr{C}(\mathsf{J}, \mathbb{H}) :\ D_t^{\alpha} v \in \mathscr{C}(\mathsf{J}, \mathbb{H})\}$.

We denote by $\mathsf{L}^p(\mathsf{J}, \mathbb{H})$ the set of all measurable functions $v(\cdot)$ on J into $\mathbb{H}$ such that $\|v(t)\|^p$ is integrable, and its norm is given by $\|v\|_{\mathsf{L}^p(\mathsf{J},\mathbb{H})} = \left(\int_{\mathsf{J}} \|v(t)\|^p\right)^{\frac{1}{p}}$, similarly, by $\mathsf{L}^p_{loc}(\mathbb{R}^+, \mathbb{H})$ we denote the space of the functions belonging $\mathsf{L}^p(\mathsf{J}, \mathbb{H})$, for any compact set $\mathsf{J} \subset \mathbb{R}^+$. When $\mathbb{H} = \mathbb{R}^n$, for some n, we denote for simplicity by $\mathscr{C}(\mathsf{J})$, $\mathscr{C}^1(\mathsf{J})$, $\mathscr{C}^{\alpha}(\mathsf{J})$, $\mathsf{L}^p(\mathsf{J})$ and $\mathsf{L}^p_{loc}(\mathbb{R}^+)$, respectively.

The notation $\rho(P)$ stands for the resolvent set of P and $R(\Lambda, P) = (\Lambda \mathsf{I}_{\mathcal{L}(\mathbb{H})} - P)^{-1}$ is the resolvent operator of P. Furthermore, for appropriate functions $K : [0, \infty) \to \mathsf{Z}$ and $S : [0, \infty) \to \mathcal{L}(\mathsf{Z}, \mathsf{W})$, the notation $\widehat{K}$ denotes the Laplace transform of K, and $S \star K$ the convolution between S and K, which is defined by $S \star K(t) = \int_0^t S(t-s)K(s)ds$.

We introduce the following concept of resolvent operator for the abstract fractional integro-differential problem (2).

Definition 2.2. [7] *A $\alpha-$resolvent operator of* (2) *is a family of bounded linear operators* $(\mathsf{R}_\alpha(t))_{t\geq 0}$ *on* $\mathbb{H}$*, satisfying the following properties:*

(1) *The function* $\mathsf{R}_\alpha : \mathbb{R}^+ \to \mathcal{L}(\mathbb{H})$ *is strongly continuous and* $\mathsf{R}_\alpha(0)v = v$ *for all* $v \in \mathbb{H}$ *and* $\alpha \in (0,1)$.

(2) *for each* $v \in \mathsf{D}(A)$, $\mathsf{R}_\alpha(\cdot)v \in \mathscr{C}([0,\infty), [\mathsf{D}(A)]) \cap \mathscr{C}^\alpha((0,\infty), \mathbb{H})$, *and*

$$D_t^\alpha \mathsf{R}_\alpha(t)v = A\mathsf{R}_\alpha(t)v + \int_0^t \gamma(t-r)\mathsf{R}_\alpha(s)vds$$

$$D_t^\alpha \mathsf{R}_\alpha(t)v = \mathsf{R}_\alpha(t)Av + \int_0^t \mathsf{R}_\alpha(t-r)\gamma(s)vds, \quad t \geq 0.$$

To study the existence of the relsolvent operator of the system (1), we suppose the following hypothesis:

A$_1$ The operator $A : \mathsf{D}(A) \subseteq \mathbb{H} \to \mathbb{H}$ is a closed linear operator with $[\mathsf{D}(A)]$ dense in $\mathbb{H}$, for some $\phi \in \left(\frac{\pi}{2}, \pi\right)$ there is positive constants $C_0 = C_0(\phi)$ such that $\Lambda \in \rho(A)$ for each

$$\Sigma_{0,\phi} = \{\Lambda \in \mathbb{C} : \ |\arg(\Lambda)| < \phi\} \subset \rho(A),$$

and $\|R(\Lambda, A)\| \leq \frac{C_0}{|\Lambda|}$ for all $\Lambda \in \Sigma_{0,\phi}$.

A$_2$ For all $t \geq 0$, $\gamma(t) : \mathsf{D}(\gamma(t)) \subset \mathbb{H} \to \mathbb{H}$ is closed linear operator, $\mathsf{D}(A) \subseteq \mathsf{D}(\gamma(t))$ and $\gamma(\cdot)v$ is strongly measurable on $(0,\infty)$ for each $v \in \mathsf{D}(A)$. There exists a function $b(\cdot) \in \mathsf{L}^1_{loc}(\mathbb{R}^+)$ such that $\widehat{b}(\Lambda)$ exists for $Re(\Lambda) > 0$ and $\|\gamma(t)v\| \leq b(t)\|v\|_1$ for all $t > 0$ and $v \in \mathsf{D}(A)$. Moreover, the operator valued function $\widehat{\gamma} : \ \Sigma_{0,\pi/2} \to \mathcal{L}([\mathsf{D}(A)], \mathbb{H})$ has an analytical extension (still denoted by $\widehat{\gamma}$) to $\Sigma_{0,\phi}$, such that $\|\widehat{\gamma}(\Lambda)v\| \leq \|\widehat{\gamma}(\Lambda)\|\|v\|_1$ for all $v \in \mathsf{D}(A)$, and $\|\widehat{\gamma}(\Lambda)\| = O\left(\frac{1}{|\Lambda|}\right)$, as $|\Lambda| \to \infty$.

A$_3$ There exists a subspace $\mathsf{D} \subseteq \mathsf{D}(A)$ dense in $[\mathsf{D}(A)]$ and positive constant C_1, such that $A(\mathsf{D}) \subseteq \mathsf{D}(A)$, $\widehat{\gamma}(\Lambda)(\mathsf{D}) \subseteq \mathsf{D}(A)$, $\|A\widehat{\gamma}(\Lambda)v\| \leq C_1\|v\|$ for every $v \in \mathsf{D}$ and all $\Lambda \in \Sigma_{0,\phi}$.

In the sequel, for $r > 0$ and $\theta \in (\frac{\pi}{2}, \phi)$,

$$\Sigma_{r,\theta} = \{\Lambda \in \mathbb{C} : \ |\Lambda| \geq r, \ \text{and} \ |\arg(\Lambda)| < \theta\}.$$

In addition, $\rho(F_\alpha)$ and $\rho(G_\alpha)$ are the sets

$$\rho(F_\alpha) = \{\Lambda \in \mathbb{C} : \ F_\alpha(\Lambda) := (\Lambda^\alpha \mathsf{I}_{\mathcal{L}(\mathbb{H})} - A - \widehat{\gamma}(\Lambda))^{-1} \in \mathcal{L}(\mathbb{H})\}$$

and

$$\rho(G_\alpha) = \{\Lambda \in \mathbb{C} : \ G_\alpha(\Lambda) := \Lambda^{\alpha-1}(\Lambda^\alpha - A - \widehat{\gamma}(\Lambda))^{-1} \in \mathcal{L}(\mathbb{H})\}.$$

Assuming that the conditions $(\mathbf{A}_i)$, $i = 1,2,3$, holds, r, θ are numbers such that $r > r_1$ and $\theta \in (\pi/2, \phi)$. Moreover, we denote by $\Gamma_{r,\theta}$, $\Gamma^i_{r,\theta}$, $i = 1,2,3$, we define the paths

$$\Gamma^1_{r,\theta} = \{te^{i\theta} : \ t \geq r\}, \quad \Gamma^2_{r,\theta} = \{re^{i\xi} : \ -\theta \leq \xi \leq \theta\} \text{ and } \Gamma^3_{r,\theta} = \{te^{-i\theta} : \ t \geq r\},$$

and $\Gamma_{r,\theta} = \bigcup_{i=1}^{3} \Gamma^i_{r,\theta}$ oriented counterclockwise.

In the following, we give the generalization of the analytic resolvent operator associated a integrodifferential equations [8] for the fractional integro-differential problem (2) with $\alpha \in (0,1)$.

Definition 2.3. [7] *We define the operator family* $(\mathsf{R}_\alpha(t))_{t\geq 0}$ *by*

$$\mathsf{R}_\alpha(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\Lambda t} G_\alpha(\Lambda) d\Lambda, \ t \geq 0, \tag{3}$$

and the auxiliary resolvent operator family $(\mathsf{S}_\alpha(t))_{t\geq 0}$ *by*

$$\mathsf{S}_\alpha(t) = \frac{t^{1-\alpha}}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\Lambda t}F_\alpha(\Lambda)d\Lambda,\ t \geq 0,$$

These operators satisfy the following properties.

Theorem 2.6. [7] *The operator function* $\mathsf{R}_\alpha(\cdot)$ *is:*

(1) *exponentially bounded in* $\mathcal{L}(\mathbb{H})$;
(2) *exponentially bounded in* $\mathcal{L}([\mathsf{D}(A)])$;
(3) *strongly continuous on* $[0,\infty)$ *and uniformly continuous on* $(0,\infty)$;
(4) *strongly continuous on* $[0,\infty)$ *in* $\mathcal{L}([\mathsf{D}(A)])$.

Theorem 2.7. [7] *The operator funtion* $t \mapsto t^{\alpha-1}\mathsf{S}_\alpha(t)$ *is exponentially bounded in* $\mathcal{L}(\mathbb{H})$ *and uniformly (strong) continuous on* $(0,\infty)$.

Theorem 2.8. [7] *The function* $\mathsf{R}_\alpha(\cdot)$ *is a* $\alpha-$*resolvent operator for the system* (2).

Remark 2.1. *By the theorems 2.6(i) and 2.7, we conclude that there is some constants* M *and* N_α *such that:*

$$\forall\ t \in [0,\mathsf{T}],\quad \|\mathsf{R}_\alpha(t)\|_{\mathcal{L}(\mathbb{H})} \leq \mathsf{M}\quad \text{and}\quad \|t^{\alpha-1}\mathsf{S}_\alpha(t)\|_{\mathcal{L}(\mathbb{H})} \leq \mathsf{N}_\alpha.$$

The existence of resolvent operator implies in the existence of solutions for problem (2).

Theorem 2.9. [7] *Let* $v_0 \in [\mathsf{D}(A)]$ *and define* $v(t) = \mathsf{R}_\alpha(t)v_0$. *Then* $v \in \mathscr{C}([0,\infty),[\mathsf{D}(A)]) \cap \mathscr{C}^\alpha((0,\infty),\mathbb{H})$, *and is a solutions of* (2).

Let $f : [0,\mathsf{T}] \times \mathbb{H} \to \mathbb{H}$ be a appropriate function and we consider the following non-homogeneous fractional order integrodifferential system:

$$\begin{cases} D_t^\alpha v(t) = Av(t) + \int_0^t \gamma(t-s)v(s)ds + f(t,v(t)) \text{ for } t \in [0,\mathsf{T}], \\ v(0) = v_0 \in \mathbb{H}. \end{cases} \tag{4}$$

We give the following results.

Theorem 2.10. [7] *The functions* $\mathsf{R}_\alpha(\cdot)$ *and* $\mathsf{S}_\alpha(\cdot)$ *are respectively* $\alpha-$*resolvent and auxiliary* $\alpha-$*resolvent operators for the system* (4).

Now we derive the appropriate definition of mild solutions of (4).

Definition 2.4. [7] *Let* $\tau > 0$, *a function* $v : (0,\tau) \to \mathbb{H}$ *is called mild solution of* (4) *in* $(0,\tau)$ *if* $v \in \mathscr{C}((0,\tau),\mathbb{H})$ *and* v *satisfies the following variation of constants formula:*

$$v(t) = \mathsf{R}_\alpha(t)v_0 + \int_0^t (t-s)^{\alpha-1}\mathsf{S}_\alpha(t-s)f(s,v(s))ds,$$

holds for all $t \in (0,\tau)$.

3. Main results

In this section, we establish and prove the existence of mild solution for the system (1). First we give the definition of the mild solution for (1).

Definition 3.1. *A mild solution of the fractional order integrodifferential system (1) is a function $v \in \Xi_{PC}$ such that $v_0(t) = g(\beta(v), v)(t)$ for $t \in [-\tau, 0]$ and satisfying the following integral system*

(5)

$$v(t) = \begin{cases} \mathsf{R}_\alpha(t)\left[g(\beta(v),v)(0) - p\left(0, v_{\rho(0,v_0)}\right)\right] + p\left(t, v_{\rho(t,v_t)}\right) + \displaystyle\int_0^t (t-s)^{\alpha-1}\mathsf{S}_\alpha(t-s)\times \\ \qquad\qquad \times\left[\zeta\left(s, v_{\rho(s,v_s)}, u(s)\right) + Bu(s)\right]ds, \quad t \in [0, t_1] \\ h_i\left(t, v_{\rho(t,v_t)}\right), \quad t \in (t_i, s_i] \\ \mathsf{R}_\alpha(t-s_i)\left[h_i\left(t, v_{\rho(t,v_t)}\right) - p\left(s_i, v_{\rho(s_i,v_{s_i})}\right)\right] + p\left(t, v_{\rho(t,v_t)}\right) + \displaystyle\int_{s_i}^t (t-s)^{\alpha-1}\mathsf{S}_\alpha(t-s)\times \\ \qquad\qquad \times\left[\zeta\left(s, v_{\rho(s,v_s)}, u(s)\right) + Bu(s)\right]ds, \quad t \in \displaystyle\bigcup_{i=1}^n (s_i, t_{i+1}]. \end{cases}$$

To prove our results, we need the following conditions

$\mathbf{H}_1$ The function $\zeta : \mathsf{I} \times \mathsf{C} \times \mathscr{U} \to \mathbb{H}$ satisfies the following conditions:

(a) ζ is of caratheodory, this is: for any $(v, u) \in \mathsf{C} \times \mathscr{U}$, the function $t \mapsto \zeta(t, v, u)$ is strongly measurable; and for any $t \in \mathsf{I}$, the function $(v, u) \mapsto \zeta(t, v, u)$ is continuous.

(b) There is a bounded function $\varsigma : \mathsf{I} \to (0, +\infty)$, and a continuous nondecreasing function $\varpi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $(t, s, v, y, u, x) \in (\mathsf{I}^2 \times \mathsf{C}^2 \times \mathscr{U}^2)$,

$$\|\zeta(t, v, u) - \zeta(s, y, x)\| \le \varsigma(t-s)\left(\varpi(\|v - y\|) + \|u - x\|_{\mathscr{U}}\right), \quad \text{and} \quad \liminf_{n\to\infty} \frac{\varpi(n)}{n} = 0.$$

Where $\zeta(0, 0, u) = 0, \ \forall\ u \in \mathscr{U}$.

(c) There is $\delta_\zeta : \mathsf{I} \to \mathbb{R}^+$, such that for any bounded set $\mathsf{D} \subset \Xi_{PC}$, and $(t, u) \in (\mathsf{I} \times \mathscr{U})$,

$$\mu(\zeta(t, \mathsf{D}_t, u)) \le \delta_\zeta(t)\mu(\mathsf{D}_t),$$

where $\mathsf{D}_t := \{v_t : \ v \in \mathsf{D}\} \subset \Xi$.

$\mathbf{H}_2$ The function $p : \mathsf{I} \times \mathsf{C} \to \mathbb{H}$ satisfies:

(a) For any $v \in \mathsf{C}$, the function $t \mapsto p(t, v)$ is piecewise continuous and for any $t \in \mathsf{I}$, the function $v \mapsto p(t, v)$ is continuous.

(b) There is a bounded function $\eta : \mathsf{I} \to \mathbb{R}^+$, and a continuous nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $t \in \mathsf{I}$ and $v \in \mathsf{C}$,

$$\|p(t, v)\| \le \eta(t)\phi(\|v\|), \quad \liminf_{n\to\infty} \frac{\phi(n)}{n} = 0.$$

(c) There is $\delta_p : \mathsf{I} \to \mathbb{R}^+$, such that for any bounded set $\mathsf{D} \subset \Xi_{PC}$ and $t \in \mathsf{I}$,

$$\mu(p(t, \mathsf{D}_t)) \le \delta_p(t)\mu(\mathsf{D}_t),$$

where $\mathsf{D}_t := \{v_t : \ v \in \mathsf{D}\} \subset \Xi$.

$\mathbf{H}_3$ For any $i \in [\![1, n]\!]$, the functions $h_i : (t_i, s_i] \times \mathsf{C} \to \mathbb{H}$ satisfy:

(a) There is a bounded function $\kappa_i : (t_i, s_i] \to \mathbb{R}^+$, and a continuous nondecreasing function $\psi_i : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $t \in (t_i, s_i]$ and $v \in \mathsf{C}$,

$$\|h_i(t,v)\| \le \kappa_i(t)\psi_i(\|v\|), \quad \liminf_{n\to\infty} \frac{\psi_i(n)}{n} = 0.$$

(b) There is $\delta_{h_i} : (t_i, s_i] \to \mathbb{R}^+$, non decreasing, such that for any bounded set $\mathsf{D} \subset \Xi_{PC}$ and $t \in (t_i, s_i]$,

$$\mu(h_i(t, \mathsf{D}_t)) \le \delta_{h_i}(t)\mu(\mathsf{D}_t),$$

where $\mathsf{D}_t := \{v_t : \ v \in \mathsf{D}\} \subset \Xi$.

$\mathbf{H}_4$ The function $g : [0, \mathsf{T}] \times \mathscr{C}([-\tau, \mathsf{T}], \mathbb{H}) \to \mathsf{C}$ satisfy:

(a) There is a bounded function $\lambda > 0$, such that for any $v \in \mathscr{C}([-\tau, \mathsf{T}], \mathbb{H})$,

$$\|g(\beta(v), v)\| \le \lambda(1 + (\|v\|);$$

(b) There is $\delta_g > 0$, such that for any bounded set $\mathsf{D} \subset \mathscr{C}([-\tau, \mathsf{T}], \mathbb{H})$,

$$\mu(g(\beta(v), \mathsf{D})) \le \delta_g \mu(\mathsf{D}).$$

$\mathbf{H}_5$ $B : \mathscr{U} \to \mathbb{H}$ is a bounded linear operator, $\Theta_i : \mathsf{L}^2(\mathsf{I}, \mathscr{U}) \to \mathbb{H}$ is a linear operator defined by:

$$\Theta_i u = \int_{s_i}^{t_{i+1}} (t_{i+1} - s)^{\alpha-1} \mathsf{S}_\alpha(t_{i+1} - s) B u(s) ds, \quad i \in [\![0, n]\!],$$

and we have the following asumptions:

(a) The operator Θ_i has an inverse Θ_i^{-1} which takes values in $\mathsf{L}^2(\mathsf{I}, \mathscr{U}) \setminus \ker \Theta_i$ and there is some positive constants L_B and L_Θ, such that:

$$\|B\| \le \mathsf{L}_B, \quad \|\Theta_i^{-1}\| \le \mathsf{L}_\Theta;$$

(b) There is $\delta_\Theta \in \mathsf{L}^1(\mathsf{I}, \mathbb{R}^+)$ and $\delta_B \ge 0$ such that for any bounded sets $\mathsf{D}_1 \subset \mathbb{H}$ and $\mathsf{D}_2 \subset \mathscr{U}$,

$$\mu\left(\left(\Theta_i^{-1}\mathsf{D}_1\right)(t)\right) \le \delta_\Theta(t)\mu(\mathsf{D}_1(t)), \quad \mu(B(\mathsf{D}_2)) \le \delta_B \mu_{\mathscr{U}}(\mathsf{D}_2).$$

$\mathbf{H}_6$ For $i \in [\![0, n]\!]$,

$$\begin{aligned}\Pi_i := \Bigg\{ &\max\{\mathsf{M}\delta_{h_i}(s_i); \mathsf{M}\delta_g\} + \mathsf{M}\delta_p(s_i) + \delta_p(t) + 2(t - s_i)\mathsf{N}_\alpha \sup_{s\in[0,\mathsf{T}]} \delta_\varsigma(s) + 2(t - s_i)\mathsf{N}_\alpha \delta_B \delta_\Theta(t) \times \\ &\times \left(1 + \max\{\mathsf{M}\delta_{h_i}(s_i); \mathsf{M}\delta_g\} + \mathsf{M}\delta_p(s_i) + \delta_p(t_{i+1}) + 2(t_{i+1} - s_i)\mathsf{N}_\alpha \sup_{s\in[0,\mathsf{T}]} \delta_\varsigma(s)\right)\Bigg\} < 1.\end{aligned}$$

We give the main result of this section.

Theorem 3.1. *Under the asumptions $\mathbf{H}_1 - \mathbf{H}_6$, the fractional order, non instantaneous impulsive integrodifferential system (1) admits at least a mild solution which is controllable on I, provided that*

$$\mathsf{M}\lambda + \frac{\mathsf{L}_\Theta\sqrt{t - s_i}\mathsf{N}_\alpha \left(\sup\limits_{s\in(s_i,t_{i+1}]} \varsigma(s) + \mathsf{L}_B\right)\{1 + \mathsf{M}\lambda\}}{1 - \sqrt{t_{i+1} - s_i}\mathsf{L}_\Theta \mathsf{N}_\alpha \sup\limits_{s\in(s_i,t_{i+1}]} \varsigma(s)} < 1, \quad i \in [\![0, n]\!]. \tag{6}$$

Proof. Using the hypothesis $\mathbf{H}_6$, for an arbitrary function v, let's define the control function $\overline{u}_v$, satisfying the following integral equation:

(7)

$$\overline{u}_v(t) = \begin{cases} \Theta_1^{-1}\Big\{\overline{v} - \mathsf{R}_\alpha(t_1)\left[g(\beta(v),v)(0) - p\left(0, v_{\rho(0,v_0)}\right)\right] - p\left(t_1, v_{\rho(t_1,v_{t_1})}\right) \\ -\displaystyle\int_0^{t_1}(t_1-s)^{\alpha-1}\mathsf{S}_\alpha(t_1-s)\left[\zeta\left(s, v_{\rho(s,v_s)}, \overline{u}_v(s)\right)\right]ds\Big\}(t), \quad t\in[0,t_1] \\ \\ \Theta_i^{-1}\Big\{\overline{v} - \mathsf{R}_\alpha(t_{i+1}-s_i)\left[h_i\left(t_{i+1}, v_{\rho(t_{i+1},v_{t_{i+1}})}\right) - p\left(s_i, v_{\rho(s_i,v_{s_i})}\right)\right] - p\left(t_{i+1}, v_{\rho(t_{i+1},v_{t_{i+1}})}\right) \\ +\displaystyle\int_{s_i}^{t_{i+1}}(t_{i+1}-s)^{\alpha-1}\mathsf{S}_\alpha(t_{i+1}-s)\left[\zeta\left(s, v_{\rho(s,v_s)}, \overline{u}_v(s)\right)\right]ds\Big\}(t), \quad t\in\bigcup_{i=1}^{n}(s_i,t_{i+1}]. \end{cases}$$

Using the control $\overline{u}_v$, we introduce the operator $\Lambda : \Xi_{PC} \to \Xi_{PC}$ defined by:

(8)

$$(\Lambda v)(t) = \begin{cases} g(\beta(v),v)(t), \;\; t\in[-\tau,0] \\ \mathsf{R}_\alpha(t)\left[g(\beta(v),v)(0) - p\left(0, v_{\rho(0,v_0)}\right)\right] + p\left(t, v_{\rho(t,v_t)}\right) + \displaystyle\int_0^t (t-s)^{\alpha-1}\mathsf{S}_\alpha(t-s)\times \\ \qquad\qquad \times\left[\zeta\left(s, v_{\rho(s,v_s)}, u(s)\right) + B\overline{u}_v(s)\right]ds, \quad t\in[0,t_1] \\ \\ h_i\left(t, v_{\rho(t,v_t)}\right), \quad t\in\bigcup_{i=1}^{n}(t_i,s_i] \\ \mathsf{R}_\alpha(t-s_i)\left[h_i\left(t, v_{\rho(t,v_t)}\right) - p\left(s_i, v_{\rho(s_i,v_{s_i})}\right)\right] + p\left(t, v_{\rho(t,v_t)}\right) + \displaystyle\int_{s_i}^t (t-s)^{\alpha-1}\mathsf{S}_\alpha(t-s)\times \\ \qquad\qquad \times\left[\zeta\left(s, v_{\rho(s,v_s)}, u(s)\right) + B\overline{u}_v(s)\right]ds, \quad t\in\bigcup_{i=1}^{n}(s_i,t_{i+1}]. \end{cases}$$

We will show that the operator Λ has a fixed point. We give the proof into the following four steps.

Step 1: We show that there is $r > 1$, such that $\Lambda(\mathsf{B}_r) \subset \mathsf{B}_r$, where $\mathsf{B}_r = \{v \in \Xi_{PC} : \|v\| \leq r\}$. If it is not true, for any $r > 1$, there exists $v^\star \in \mathsf{B}_r$, such that $\Lambda v^\star \notin \mathsf{B}_r$.

In fact, we have from $\mathbf{H}_1 - \mathbf{H}_5$ that:

For $t \in [0, t_1]$,

$$\begin{aligned} r &\leq \|\Lambda v^\star(t)\| \\ &\leq \|\mathsf{R}_\alpha(t)g(\beta(v^\star),v^\star)(0)\| + \left\|\mathsf{R}_\alpha(t)p\left(0, v^\star_{\rho(0,v^\star_0)}\right)\right\| + \left\|p\left(t, v^\star_{\rho(t,v^\star_t)}\right)\right\| \\ &+ \left\|\int_0^t (t-s)^{\alpha-1}\mathsf{S}_\alpha(t-s)\zeta\left(s, v^\star_{\rho(s,v^\star_s)}, \overline{u}_{v^\star}(s)\right)ds\right\| + \left\|\int_0^t (t-s)^{\alpha-1}\mathsf{S}_\alpha(t-s)B\overline{u}_{v^\star}(s)ds\right\| \\ &\leq \mathsf{M}\lambda(1+\|v^\star\|) + \mathsf{M}\eta(0)\phi(\|v^\star_0\|_{\Xi}) + \eta(t)\phi\left(\|v^\star_{\rho(t,v^\star_t)}\|_{\Xi}\right) \\ &+ \int_0^t \mathsf{N}_\alpha\varsigma(s)\left(\varpi(\|v^\star_{\rho(s,v^\star_s)}\|_{\Xi}) + \|\overline{u}_{v^\star}(s)\|_{\mathscr{U}}\right)ds + \int_0^t \mathsf{N}_\alpha\mathsf{L}_B\|\overline{u}_{v^\star}(s)\|_{\mathscr{U}}ds \\ &\leq \mathsf{M}\lambda(1+\|v^\star\|) + \mathsf{M}\eta(0)\phi(\|v^\star\|) + \eta(t)\phi\left(\|v^\star\|\right) + \mathsf{N}_\alpha\varpi(\|v^\star\|)\int_0^t\varsigma(s)ds \\ &+ \mathsf{N}_\alpha\sup_{s\in[0,t_1]}\varsigma(s)\left(\int_0^t ds\right)^{\frac{1}{2}}\left(\int_0^t\|\overline{u}_{v^\star}(s)\|^2_{\mathscr{U}}ds\right)^{\frac{1}{2}} + \mathsf{N}_\alpha\mathsf{L}_B\left(\int_0^t ds\right)^{\frac{1}{2}}\left(\int_0^t\|\overline{u}_{v^\star}(s)\|^2_{\mathscr{U}}ds\right)^{\frac{1}{2}} \end{aligned}$$

$$\le \mathsf{M}\lambda(1+r) + \mathsf{M}\eta(0)\phi(r) + \eta(t)\phi(r) + t\mathsf{N}_\alpha\varpi(r)\sup_{s\in[0,t_1]}\varsigma(s)$$

$$+ \mathsf{N}_\alpha \sup_{s\in[0,t_1]}\varsigma(s)\,(t)^{\frac{1}{2}}\,\|\overline{u}_{v^\star}\|_{\mathsf{L}^2} + \mathsf{N}_\alpha \mathsf{L}_B\,(t)^{\frac{1}{2}}\,\|\overline{u}_{v^\star}\|_{\mathsf{L}^2}$$

$$\le \mathsf{M}\lambda(1+r) + \mathsf{M}\eta(0)\phi(r) + \eta(t)\phi(r) + t\mathsf{N}_\alpha\varpi(r)\sup_{s\in[0,t_1]}\varsigma(s)$$

$$+ \sqrt{t}\mathsf{N}_\alpha\left(\sup_{s\in[0,t_1]}\varsigma(s) + \mathsf{L}_B\right)\|\overline{u}_{v^\star}\|_{\mathsf{L}^2} \tag{9}$$

Concerning the control $\overline{u}_{v^\star}$, we have:

$$\begin{aligned}
\|\overline{u}_{v^\star}\|_{\mathsf{L}^2} &= \Big\|\Theta_1^{-1}\big\{\overline{v} - \mathsf{R}_\alpha(t_1)\left[g(\beta(v),v)(0) - p\left(0, v_{\rho(0,v_0)}\right)\right] - p\left(t_1, v_{\rho(t_1,v_{t_1})}\right)\\
&\quad - \int_0^{t_1}(t_1-s)^{\alpha-1}\mathsf{S}_\alpha(t_1-s)\left[\zeta\left(s, v_{\rho(s,v_s)}, \overline{u}_v(s)\right)\right]ds\Big\}\Big\|_{\mathsf{L}^2}\\
&\le \mathsf{L}_\Theta\Big\{\|\overline{v}\| + \|\mathsf{R}_\alpha(t_1)\|_{\mathcal{L}(\mathbb{H})}\|g(\beta(v^\star),v^\star)(0)\| + \|\mathsf{R}_\alpha(t_1)\|_{\mathcal{L}(\mathbb{H})}\left\|p\left(0, v^\star_{\rho(0,v_0^\star)}\right)\right\|\\
&\quad + \left\|p\left(t_1, v^\star_{\rho(t_1,v^\star_{t_1})}\right)\right\| + \int_0^{t_1}\left\|(t_1-s)^{\alpha-1}\mathsf{S}_\alpha(t_1-s)\right\|_{\mathcal{L}(\mathbb{H})}\left\|\zeta\left(s, v^\star_{\rho(s,v_s^\star)}, \overline{u}_{v^\star}(s)\right)\right\|ds\Big\}\\
&\le \mathsf{L}_\Theta\Big\{\|\overline{v}\| + \mathsf{M}\lambda(1+\|v^\star\|) + \mathsf{M}\eta(0)\phi(\|v_0^\star\|_\Xi) + \eta(t_1)\phi(\|v^\star_{\rho(t_1,v^\star_{t_1})}\|_\Xi)\\
&\quad + \int_0^{t_1}\mathsf{N}_\alpha\varsigma(s)\left(\varpi(\|v^\star_{\rho(s,v_s^\star)}\|_\Xi) + \|\overline{u}_{v^\star}(s)\|_{\mathscr{U}}\right)ds\Big\}\\
&\le \mathsf{L}_\Theta\Bigg\{\|\overline{v}\| + \mathsf{M}\lambda(1+\|v^\star\|) + \mathsf{M}\eta(0)\phi(\|v^\star\|) + \eta(t_1)\phi(\|v^\star\|)\\
&\quad + t_1\mathsf{N}_\alpha\varpi(\|v^\star\|)\sup_{s\in[0,t_1]}\varsigma(s) + \mathsf{N}_\alpha\sup_{s\in[0,t_1]}\varsigma(s)\left(\int_0^{t_1}ds\right)^{\frac{1}{2}}\left(\int_0^{t_1}\|\overline{u}_{v^\star}(s)\|_{\mathscr{U}}ds\right)^{\frac{1}{2}}\Bigg\}\\
&\le \mathsf{L}_\Theta\Bigg\{r + \mathsf{M}\lambda(1+r) + \mathsf{M}\eta(0)\phi(r) + \eta(t_1)\phi(r) + t_1\mathsf{N}_\alpha\varpi(r)\sup_{s\in[0,t_1]}\varsigma(s)\\
&\quad + \sqrt{t_1}\mathsf{N}_\alpha\|\overline{u}_{v^\star}\|_{\mathsf{L}^2}\sup_{s\in[0,t_1]}\varsigma(s)\Bigg\}.
\end{aligned}$$

By this inequality, we deduce that

$$\|\overline{u}_{v^\star}\|_{\mathsf{L}^2} \le \frac{\mathsf{L}_\Theta\left\{r + \mathsf{M}\lambda(1+r) + \mathsf{M}\eta(0)\phi(r) + \eta(t_1)\phi(r) + t_1\mathsf{N}_\alpha\varpi(r)\sup_{s\in[0,t_1]}\varsigma(s)\right\}}{1 - \sqrt{t_1}\mathsf{L}_\Theta\mathsf{N}_\alpha\sup_{s\in[0,t_1]}\varsigma(s)}. \tag{10}$$

Using (9) and (10), we get:

For $t \in [0, t_1]$,

$$r \le \mathsf{M}\lambda(1+r) + \mathsf{M}\eta(0)\phi(r) + \eta(t)\phi(r) + t\mathsf{N}_\alpha\varpi(r)\sup_{s\in[0,t_1]}\varsigma(s)$$

$$+ \sqrt{t}\mathsf{N}_\alpha\left(\sup_{s\in[0,t_1]}\varsigma(s) + \mathsf{L}_B\right)\|\overline{u}_{v^\star}\|_{\mathsf{L}^2}$$

$$\leq \mathsf{M}\lambda(1+r)+\mathsf{M}\eta(0)\phi(r)+\eta(t)\phi\left(r\right)+t\mathsf{N}_{\alpha}\varpi(r)\sup_{s\in[0,t_1]}\varsigma(s)+\mathsf{L}_{\Theta}\sqrt{t}\mathsf{N}_{\alpha}\left(\sup_{s\in[0,t_1]}\varsigma(s)+\mathsf{L}_B\right)$$
$$+\frac{\left\{r+\mathsf{M}\lambda(1+r)+\mathsf{M}\eta(0)\phi(r)+\eta(t_1)\phi(r)+t_1\mathsf{N}_{\alpha}\varpi(r)\sup_{s\in[0,t_1]}\varsigma(s)\right\}}{1-\sqrt{t_1}\mathsf{L}_{\Theta}\mathsf{N}_{\alpha}\sup_{s\in[0,t_1]}\varsigma(s)}.$$

Dividing both sides by r, we obtain :

$$1\leq \mathsf{M}\lambda(\frac{1}{r}+1)+\mathsf{M}\eta(0)\frac{\phi(r)}{r}+\eta(t)\frac{\phi\left(r\right)}{r}+t\mathsf{N}_{\alpha}\frac{\varpi(r)}{r}\sup_{s\in[0,t_1]}\varsigma(s)+\mathsf{L}_{\Theta}\sqrt{t}\mathsf{N}_{\alpha}\left(\sup_{s\in[0,t_1]}\varsigma(s)+\mathsf{L}_B\right)$$
$$+\frac{\left\{1+\mathsf{M}\lambda(\frac{1}{r}+1)+\mathsf{M}\eta(0)\frac{\phi(r)}{r}+\eta(t_1)\frac{\phi(r)}{r}+t_1\mathsf{N}_{\alpha}\frac{\varpi(r)}{r}\sup_{s\in[0,t_1]}\varsigma(s)\right\}}{1-\sqrt{t_1}\mathsf{L}_{\Theta}\mathsf{N}_{\alpha}\sup_{s\in[0,t_1]}\varsigma(s)}.$$

Taking the limit as $r\to\infty$ and using the hypothesis $\mathbf{H}_1(b)$, $\mathbf{H}_2(b)$ and $\mathbf{H}_3(a)$, this contradicts with our assumption (6). Thus, there exists $r>1$, such that $\Lambda(\mathsf{B}_r)\subset\mathsf{B}_r$.

For $t\in(t_i,s_i]$, $\quad i\in[\![1,n]\!]$

$$\begin{aligned} r\leq \|\Lambda v^{\star}(t)\| &\leq \kappa_i(t)\psi_i\left(\left\|v^{\star}_{\rho(t,v^{\star}_t)}\right\|_{\Xi}\right)\\ &\leq \kappa_i(t)\psi_i\left(\|v^{\star}\|\right)\\ &\leq \kappa_i(t)\psi_i\left(r\right). \end{aligned} \tag{11}$$

Using the hypothesis $\mathbf{H}_3-(a)$ and taking the limit as $r\to\infty$, this contradicts with our assumptions. Thus, there exists $r>1$, such that $\Lambda(\mathsf{B}_r)\subset\mathsf{B}_r$.

For $t\in(s_i,t_{i+1}]$, $\quad i\in[\![1,n]\!]$

$$\begin{aligned} r&\leq\|\Lambda v^{\star}(t)\|\\ &\leq\left\|\mathsf{R}_{\alpha}(t-s_i)h_i(t,v^{\star}_{\rho(t,v^{\star}_t)})\right\|+\left\|\mathsf{R}_{\alpha}(t-s_i)p\left(s_i,v^{\star}_{\rho(s_i,v^{\star}_{s_i})}\right)\right\|+\left\|p\left(t,v^{\star}_{\rho(t,v^{\star}_t)}\right)\right\|\\ &+\left\|\int_{s_i}^{t}(t-s)^{\alpha-1}\mathsf{S}_{\alpha}(t-s)\zeta\left(s,v^{\star}_{\rho(s,v^{\star}_s)},\overline{u}_{v^{\star}}(s)\right)ds\right\|+\left\|\int_{s_i}^{t}(t-s)^{\alpha-1}\mathsf{S}_{\alpha}(t-s)B\overline{u}_{v^{\star}}(s)ds\right\|\\ &\leq\mathsf{M}\kappa_i(t)\psi_i(\|v^{\star}_{\rho(t,v^{\star}_t)}\|_{\Xi})+\mathsf{M}\eta(s_i)\phi(\|v^{\star}_{\rho(s_i,v^{\star}_{s_i})}\|_{\Xi})+\eta(t)\phi\left(\|v^{\star}_{\rho(t,v^{\star}_t)}\|_{\Xi}\right)\\ &+\int_{s_i}^{t}\mathsf{N}_{\alpha}\varsigma(s)\left(\varpi(\|v^{\star}_{\rho(s,v^{\star}_s)}\|_{\Xi})+\|\overline{u}_{v^{\star}}(s)\|_{\mathscr{U}}\right)ds+\int_{s_i}^{t}\mathsf{N}_{\alpha}\mathsf{L}_B\|\overline{u}_{v^{\star}}(s)\|_{\mathscr{U}}ds\\ &\leq\mathsf{M}\kappa_i(t)\psi_i(\|v^{\star}\|)+\mathsf{M}\eta(s_i)\phi(\|v^{\star}\|)+\eta(t)\phi\left(\|v^{\star}\|\right)+(t-s_i)\mathsf{N}_{\alpha}\varpi(\|v^{\star}\|)\sup_{s\in(s_i,t_{i+1}]}\varsigma(s)\\ &+\mathsf{N}_{\alpha}\sup_{s\in(s_i,t_{i+1}]}\varsigma(s)\int_{s_i}^{t}\|\overline{u}_{v^{\star}}(s)\|_{\mathscr{U}}ds+\mathsf{N}_{\alpha}\mathsf{L}_B\int_{s_i}^{t}\|\overline{u}_{v^{\star}}(s)\|_{\mathscr{U}}ds\\ &\leq\mathsf{M}\kappa_i(t)\psi_i(r)+\mathsf{M}\eta(s_i)\phi(r)+\eta(t)\phi\left(r\right)+(t-s_i)\mathsf{N}_{\alpha}\varpi(r)\sup_{s\in(s_i,t_{i+1}]}\varsigma(s)\\ &+\sqrt{t-s_i}\mathsf{N}_{\alpha}\left(\sup_{s\in(s_i,t_{i+1}]}\varsigma(s)+\mathsf{L}_B\right)\|\overline{u}_{v^{\star}}\|_{\mathsf{L}^2}. \end{aligned} \tag{12}$$

Concerning the control $\overline{u}_{v^\star}$, we have:

$$\begin{aligned}
\|\overline{u}_{v^\star}\|_{\mathsf{L}^2} &= \Big\| \Theta_i^{-1}\Big\{ \overline{v} - \mathsf{R}_\alpha(t_{i+1}-s_i)\Big[h_i\left(t_{i+1}, v_{\rho(t_{i+1},v_{t_{i+1}})}\right) - p\left(s_i, v_{\rho(s_i,v_{s_i})}\right)\Big] \\
&\quad -p\left(t_{i+1}, v_{\rho(t_{i+1},v_{t_{i+1}})}\right) + \int_{s_i}^{t_{i+1}} (t_{i+1}-s)^{\alpha-1}\mathsf{S}_\alpha(t_{i+1}-s)\left[\zeta\left(s, v_{\rho(s,v_s)}, \overline{u}_v(s)\right)\right] ds \Big\}\Big\|_{\mathsf{L}^2} \\
&\le \mathsf{L}_\Theta \Big\{ \|\overline{v}\| + \|\mathsf{R}_\alpha(t_{i+1}-s_i)\|_{\mathcal{L}(\mathbb{H})}\|h_i(t_{i+1}, v^\star_{\rho(t_{i+1},v^\star_{t_{i+1}})})\| \\
&\quad + \|\mathsf{R}_\alpha(t_{i+1}-s_i)\|_{\mathcal{L}(\mathbb{H})}\left\|p\left(s_i, v^\star_{\rho(s_i,v^\star_{s_i})}\right)\right\| + \left\|p\left(t_{i+1}, v^\star_{\rho(t_{i+1},v^\star_{t_{i+1}})}\right)\right\| \\
&\quad + \int_{s_i}^{t_{i+1}} \|(t_{i+1}-s)^{\alpha-1}\mathsf{S}_\alpha(t_{i+1}-s)\|_{\mathcal{L}(\mathbb{H})}\left\|\zeta\left(s, v^\star_{\rho(s,v^\star_s)}, \overline{u}_{v^\star}(s)\right)\right\| ds \Big\} \\
&\le \mathsf{L}_\Theta \Big\{ \|\overline{v}\| + \mathsf{M}\kappa_i(t_{i+1})\psi_i(\|v^\star_{\rho(t_{i+1},v^\star_{t_{i+1}})}\|_{\Xi}) + \mathsf{M}\eta(s_i)\phi(\|v^\star_{\rho(s_i,v^\star_{s_i})}\|_{\Xi}) \\
&\quad + \eta(t_{i+1})\phi(\|v^\star_{\rho(t,v^\star_t)}\|_{\Xi}) + \int_{s_i}^{t_{i+1}} \mathsf{N}_\alpha\varsigma(s)\left(\varpi(\|v^\star_{\rho(s,v^\star_s)}\|_{\Xi}) + \|\overline{u}_{v^\star}(s)\|_{\mathscr{U}}\right) ds \Big\} \\
&\le \mathsf{L}_\Theta \{ \|\overline{v}\| + \mathsf{M}\kappa_i(t_{i+1})\psi_i(\|v^\star\|) + \mathsf{M}\eta(s_i)\phi(\|v^\star\|) + \eta(t_{i+1})\phi(\|v^\star\|) \\
&\quad + (t_{i+1}-s_i)\mathsf{N}_\alpha\varpi(\|v^\star\|)\sup_{s\in(s_i,t_{i+1}]}\varsigma(s) + \mathsf{N}_\alpha \sup_{s\in(s_i,t_{i+1}]}\varsigma(s)\int_{s_i}^{t_{i+1}} \|\overline{u}_{v^\star}(s)\|_{\mathscr{U}} ds \Big\} \\
&\le \mathsf{L}_\Theta \Big\{ r + \mathsf{M}\kappa_i(t_{i+1})\psi_i(r) + \mathsf{M}\eta(s_i)\phi(r) + \eta(t_{i+1})\phi(r) \\
&\quad + (t_{i+1}-s_i)\mathsf{N}_\alpha\varpi(r)\sup_{s\in(s_i,t_{i+1}]}\varsigma(s) + \sqrt{t_{i+1}-s_i}\,\mathsf{N}_\alpha\|\overline{u}_{v^\star}\|_{\mathsf{L}^2}\sup_{s\in(s_i,t_{i+1}]}\varsigma(s) \Big\}.
\end{aligned}$$

By this inequality, we deduce that

$$\|\overline{u}_{v^\star}\|_{\mathsf{L}^2} \le \frac{\mathsf{L}_\Theta\left\{ r + \mathsf{M}\kappa_i(t_{i+1})\psi_i(r) + \mathsf{M}\eta(s_i)\phi(r) + \eta(t_{i+1})\phi(r) + (t_{i+1}-s_i)\mathsf{N}_\alpha\varpi(r)\sup_{s\in(s_i,t_{i+1}]}\varsigma(s)\right\}}{1 - \sqrt{t_{i+1}-s_i}\,\mathsf{L}_\Theta\mathsf{N}_\alpha\sup_{s\in(s_i,t_{i+1}]}\varsigma(s)}. \tag{13}$$

Using (12) and (13), we get:

For $t \in (s_i, t_{i+1}]$,

$$\begin{aligned}
r \le\, &\mathsf{M}\kappa_i(t)\psi_i(r) + \mathsf{M}\eta(s_i)\phi(r) + \eta(t)\phi(r) + (t-s_i)\mathsf{N}_\alpha\varpi(r)\sup_{s\in(s_i,t_{i+1}]}\varsigma(s) \\
&+ \sqrt{t-s_i}\,\mathsf{N}_\alpha\left(\sup_{s\in(s_i,t_{i+1}]}\varsigma(s) + \mathsf{L}_B\right)\|\overline{u}_{v^\star}\|_{\mathsf{L}^2} \\
\le\, &\mathsf{M}\kappa_i(t)\psi_i(r) + \mathsf{M}\eta(s_i)\phi(r) + \eta(t)\phi(r) + (t-s_i)\mathsf{N}_\alpha\varpi(r)\sup_{s\in(s_i,t_{i+1}]}\varsigma(s) \\
&+ \mathsf{L}_\Theta\sqrt{t-s_i}\,\mathsf{N}_\alpha\left(\sup_{s\in(s_i,t_{i+1}]}\varsigma(s) + \mathsf{L}_B\right)\times
\end{aligned}$$

$$\times \frac{\left\{r + \mathsf{M}\kappa_i(t_{i+1})\psi_i(r) + \mathsf{M}\eta(s_i)\phi(r) + \eta(t_{i+1})\phi(r) + (t_{i+1} - s_i)\mathsf{N}_\alpha \varpi(r) \sup_{s\in(s_i,t_{i+1}]} \varsigma(s)\right\}}{1 - \sqrt{t_{i+1} - s_i}\mathsf{L}_\Theta \mathsf{N}_\alpha \sup_{s\in(s_i,t_{i+1}]} \varsigma(s)}.$$

Dividing both sides by r, we obtain :

$$\begin{aligned}
1 \leq & \mathsf{M}\kappa_i(t)\frac{\psi_i(r)}{r} + \mathsf{M}\eta(s_i)\frac{\phi(r)}{r} + \eta(t)\frac{\phi(r)}{r} + (t - s_i)\mathsf{N}_\alpha \frac{\varpi(r)}{r} \sup_{s\in(s_i,t_{i+1}]} \varsigma(s) \\
& + \mathsf{L}_\Theta \sqrt{t - s_i}\mathsf{N}_\alpha \left(\sup_{s\in(s_i,t_{i+1}]} \varsigma(s) + \mathsf{L}_B\right) \times \\
& \times \frac{\left\{1 + \mathsf{M}\kappa_i(t_{i+1})\frac{\psi_i(r)}{r} + \mathsf{M}\eta(s_i)\frac{\phi(r)}{r} + \eta(t_{i+1})\frac{\phi(r)}{r} + (t_{i+1} - s_i)\mathsf{N}_\alpha \frac{\varpi(r)}{r} \sup_{s\in(s_i,t_{i+1}]} \varsigma(s)\right\}}{1 - \sqrt{t_{i+1} - s_i}\mathsf{L}_\Theta \mathsf{N}_\alpha \sup_{s\in(s_i,t_{i+1}]} \varsigma(s)}.
\end{aligned}$$

Taking the limit as $r \to \infty$ and using the hypothesis $\mathbf{H}_1(b)$, $\mathbf{H}_2(b)$ and $\mathbf{H}_3(a)$, this contradicts the condition (6). Thus, there exists $r > 1$, such that $\Lambda(\mathsf{B}_r) \subset \mathsf{B}_r$.

Step 2: We show that the operator $\Lambda : \mathsf{B}_r \to \mathsf{B}_r$ is continuous. Let $\{v^{(n)}\}_{n=1}^{\infty}$ be a sequence in B_r such that $v^{(n)} \xrightarrow[n\to\infty]{} v$ in B_r. Using the hypothesis $\mathbf{H}_1 - \mathbf{H}_5$ and the Hölder inequality, we have:

For $t \in [0, t_1]$,

$$\begin{aligned}
& \|(\Lambda v^{(n)})(t) - (\Lambda v)(t)\| \\
& \leq \|\mathsf{R}_\alpha(t)\|_{\mathcal{L}(\mathsf{X})} \left\|g(\beta(v^{(n)}), v^{(n)})(0) - g(\beta(v), v)(0)\right\| \\
& + \|\mathsf{R}_\alpha(t)\|_{\mathcal{L}(\mathsf{X})} \left\|p\left(0, v^{(n)}_{\rho(0,v_0^{(n)})}\right) - p\left(0, v_{\rho(0,v_0)}\right)\right\| + \left\|p\left(t, v^{(n)}_{\rho(t,v_t^{(n)})}\right) - p\left(t, v_{\rho(t,v_t)}\right)\right\| \\
& + \int_0^t \left\|(t-s)^{\alpha-1}\mathsf{S}_\alpha(t-s)\right\|_{\mathcal{L}(\mathsf{X})} \left\|\zeta\left(s, v^{(n)}_{\rho(s,v_s^{(n)})}, \overline{u}_{v^{(n)}}(s)\right) - \zeta\left(s, v_{\rho(s,v_s)}, \overline{u}_v(s)\right)\right\| ds \\
& + \int_0^t \left\|(t-s)^{\alpha-1}\mathsf{S}_\alpha(t-s)\right\|_{\mathcal{L}(\mathsf{X})} \|B\overline{u}_{v^{(n)}}(s) - B\overline{u}_v(s)\| \, ds \\
& \leq \mathsf{M} \left\|g(\beta(v^{(n)}), v^{(n)})(0) - g(\beta(v), v)(0)\right\| \\
& + \mathsf{M} \left\|p\left(0, v^{(n)}_{\rho(0,v_0^{(n)})}\right) - p\left(0, v_{\rho(0,v_0)}\right)\right\| + \left\|p\left(t, v^{(n)}_{\rho(t,v_t^{(n)})}\right) - p\left(t, v_{\rho(t,v_t)}\right)\right\| \\
& + \mathsf{N}_\alpha \int_0^t \varsigma(0) \left[\varpi\left(\left\|v^{(n)}_{\rho(s,v_s^{(n)})} - v_{\rho(s,v_s)}\right\|_{\varXi}\right) + \|\overline{u}_{v^{(n)}}(s) - \overline{u}_v(s)\|_{\mathscr{U}}\right] ds \\
& + \mathsf{N}_\alpha \mathsf{L}_B \int_0^t \|\overline{u}_{v^{(n)}}(s) - \overline{u}_v(s)\|_{\mathscr{U}} \, ds \\
& \leq \mathsf{M} \left\|g(\beta(v^{(n)}), v^{(n)})(0) - g(\beta(v), v)(0)\right\| \\
& + \mathsf{M} \left\|p\left(0, v^{(n)}_{\rho(0,v_0^{(n)})}\right) - p\left(0, v_{\rho(0,v_0)}\right)\right\| + \left\|p\left(t, v^{(n)}_{\rho(t,v_t^{(n)})}\right) - p\left(t, v_{\rho(t,v_t)}\right)\right\| \\
& + \mathsf{N}_\alpha \varsigma(0) \int_0^t \varpi\left(\left\|v^{(n)}_{\rho(s,v_s^{(n)})} - v_{\rho(s,v_s)}\right\|_{\varXi}\right) ds + \sqrt{t}\mathsf{N}_\alpha \left(\varsigma(0) + \mathsf{L}_B\right) \|\overline{u}_{v^{(n)}} - \overline{u}_v\|_{\mathsf{L}^2}. \qquad (14)
\end{aligned}$$

For the control function, using the hypothesis $\mathbf{H}_5$ and the Hölder inequality, we get:

$$
\begin{aligned}
&\|\overline{u}_{v^{(n)}} - \overline{u}_v\|_{\mathsf{L}^2}\\
&\leq \mathsf{L}_\Theta \Big\{\|\overline{v}^{(n)} - \overline{v}\| - \mathsf{M}\left\|g(\beta(v^{(n)}), v^{(n)})(0) - g(\beta(v), v)(0)\right\|\\
&+\mathsf{M}\left\|p\left(0, v^{(n)}_{\rho(0,v_0^{(n)})}\right) - p\left(0, v_{\rho(0,v_0)}\right)\right\| + \left\|p\left(t_1, v^{(n)}_{\rho(t_1,v_{t_1}^{(n)})}\right) - p\left(t_1, v_{\rho(t_1,v_{t_1})}\right)\right\|\\
&+\mathsf{N}_\alpha \int_0^{t_1} \varsigma(0)\left[\varpi\left(\left\|v^{(n)}_{\rho(s,v_s^{(n)})} - v_{\rho(s,v_s)}\right\|_{\varXi}\right) + \|\overline{u}_{v^{(n)}}(s) - \overline{u}_v(s)\|_{\mathscr{U}}\right] ds\\
&+\mathsf{N}_\alpha \mathsf{L}_B \int_0^{t_1} \|\overline{u}_{v^{(n)}}(s) - \overline{u}_v(s)\|_{\mathscr{U}}\, ds\Big\}\\
&\leq \mathsf{L}_\Theta \Big\{\|\overline{v}^{(n)} - \overline{v}\| - \mathsf{M}\left\|g(\beta(v^{(n)}), v^{(n)})(0) - g(\beta(v), v)(0)\right\|\\
&+\mathsf{M}\left\|p\left(0, v^{(n)}_{\rho(0,v_0^{(n)})}\right) - p\left(0, v_{\rho(0,v_0)}\right)\right\| + \left\|p\left(t_1, v^{(n)}_{\rho(t_1,v_{t_1}^{(n)})}\right) - p\left(t_1, v_{\rho(t_1,v_{t_1})}\right)\right\|\\
&+\mathsf{N}_\alpha \varsigma(0) \int_0^{t_1} \varpi\left(\left\|v^{(n)}_{\rho(s,v_s^{(n)})} - v_{\rho(s,v_s)}\right\|_{\varXi}\right) ds\\
&+\sqrt{t_1}\mathsf{N}_\alpha\left(\varsigma(0) + \mathsf{L}_B\right)\|\overline{u}_{v^{(n)}}(s) - \overline{u}_v(s)\|_{\mathsf{L}^2}\Big\}\\
&\leq \frac{\mathsf{L}_\Theta}{1 - \mathsf{L}_\Theta\sqrt{t_1}\mathsf{N}_\alpha\left(\varsigma(0) + \mathsf{L}_B\right)}\Big\{\|\overline{v}^{(n)} - \overline{v}\| - \mathsf{M}\left\|g(\beta(v^{(n)}), v^{(n)})(0) - g(\beta(v), v)(0)\right\|\\
&+\mathsf{M}\left\|p\left(0, v^{(n)}_{\rho(0,v_0^{(n)})}\right) - p\left(0, v_{\rho(0,v_0)}\right)\right\| + \left\|p\left(t_1, v^{(n)}_{\rho(t_1,v_{t_1}^{(n)})}\right) - p\left(t_1, v_{\rho(t_1,v_{t_1})}\right)\right\|\\
&+\mathsf{N}_\alpha \varsigma(0) \int_0^{t_1} \varpi\left(\left\|v^{(n)}_{\rho(s,v_s^{(n)})} - v_{\rho(s,v_s)}\right\|_{\varXi}\right) ds\Big\}
\end{aligned} \tag{15}
$$

Substituing (15) in (14), and using the Lebesgue dominated convergence theorem, we get $\|(\varLambda v^{(n)})(t) - (\varLambda v)(t)\| \xrightarrow[n\to\infty]{} 0$, for $t \in [0, t_1]$.

Similary, for $t \in (s_i, t_{i+1}]$, we get:

$$
\begin{aligned}
&\|(\varLambda v^{(n)})(t) - (\varLambda v)(t)\|\\
&\leq \|\mathsf{R}_\alpha(t - s_i)\|_{\mathcal{L}(\mathsf{X})}\left\|h_i\left(t, v^{(n)}_{\rho(t,v_t^{(n)})}\right) - h_i\left(t, v_{\rho(t,v_t)}\right)\right\|\\
&+\|\mathsf{R}_\alpha(t - s_i)\|_{\mathcal{L}(\mathsf{X})}\left\|p\left(s_i, v^{(n)}_{\rho(s_i,v_{s_i}^{(n)})}\right) - p\left(s_i, v_{\rho(s_i,v_{s_i})}\right)\right\| + \left\|p\left(t, v^{(n)}_{\rho(t,v_t^{(n)})}\right) - p\left(t, v_{\rho(t,v_t)}\right)\right\|\\
&+\int_{s_i}^t \left\|(t-s)^{\alpha-1}\mathsf{S}_\alpha(t-s)\right\|_{\mathcal{L}(\mathsf{X})}\left\|\zeta\left(s, v^{(n)}_{\rho(s,v_s^{(n)})}, \overline{u}_{v^{(n)}}(s)\right) - \zeta\left(s, v_{\rho(s,v_s)}, \overline{u}_v(s)\right)\right\| ds\\
&+\int_{s_i}^t \left\|(t-s)^{\alpha-1}\mathsf{S}_\alpha(t-s)\right\|_{\mathcal{L}(\mathsf{X})}\|B\overline{u}_{v^{(n)}}(s) - B\overline{u}_v(s)\|\, ds\\
&\leq \mathsf{M}\left\|h_i\left(t, v^{(n)}_{\rho(t,v_t^{(n)})}\right) - h_i\left(t, v_{\rho(t,v_t)}\right)\right\|\\
&+\mathsf{M}\left\|p\left(s_i, v^{(n)}_{\rho(s_i,v_{s_i}^{(n)})}\right) - p\left(s_i, v_{\rho(s_i,v_{s_i})}\right)\right\| + \left\|p\left(t, v^{(n)}_{\rho(t,v_t^{(n)})}\right) - p\left(t, v_{\rho(t,v_t)}\right)\right\|\\
&+\mathsf{N}_\alpha\varsigma(0)\int_{s_i}^t \varpi\left(\left\|v^{(n)}_{\rho(s,v_s^{(n)})} - v_{\rho(s,v_s)}\right\|_{\varXi}\right) ds + \mathsf{N}_\alpha\varsigma(0)\int_{s_i}^t \|\overline{u}_{v^{(n)}}(s) - \overline{u}_v(s)\|_{\mathscr{U}} ds\\
&+\mathsf{N}_\alpha\mathsf{L}_B\int_{s_i}^t (t-s)^{\alpha-1}\|\overline{u}_{v^{(n)}}(s) - \overline{u}_v(s)\|_{\mathscr{U}}\, ds
\end{aligned}
$$

$$\leq \mathsf{M}\left\|h_i\left(t, v^{(n)}_{\rho(t,v^{(n)}_t)}\right)-h_i\left(t, v_{\rho(t,v_t)}\right)\right\|+\mathsf{M}\left\|p\left(s_i, v^{(n)}_{\rho(s_i,v^{(n)}_{s_i})}\right)-p\left(s_i, v_{\rho(s_i,v_{s_i})}\right)\right\|$$

$$+\left\|p\left(t, v^{(n)}_{\rho(t,v^{(n)}_t)}\right)-p\left(t, v_{\rho(t,v_t)}\right)\right\|+\mathsf{N}_\alpha\varsigma(0)\int_{s_i}^{t}\varpi\left(\left\|v^{(n)}_{\rho(s,v^{(n)}_s)}-v_{\rho(s,v_s)}\right\|_{\varXi}\right)$$

$$+\sqrt{t-s_i}\,\mathsf{N}_\alpha\left(\varsigma(0)+\mathsf{L}_B\right)\|\overline{u}_{v^{(n)}}-\overline{u}_v\|_{\mathsf{L}^2}. \tag{16}$$

For the control function, using hypothesis $\mathbf{H}_5$ and the Hölder inequality, we get:

$$\begin{aligned}\|\overline{u}_{v^{(n)}}-\overline{u}_v\|_{\mathsf{L}^2} \leq \frac{\mathsf{L}_\Theta}{1-\mathsf{L}_\Theta\sqrt{t_{i+1}-s_i}\,\mathsf{N}_\alpha\left(\varsigma(0)+\mathsf{L}_B\right)}&\Bigg\{\|\overline{v}^{(n)}-\overline{v}\|\\
&-\mathsf{M}\left\|h_i\left(t_{i+1}, v^{(n)}_{\rho\left(t_{i+1},v^{(n)}_{t_{i+1}}\right)}\right)-h_i\left(t_{i+1}, v_{\rho\left(t_{i+1},v_{t_{i+1}}\right)}\right)\right\|\\
&+\mathsf{M}\left\|p\left(s_i, v^{(n)}_{\rho\left(s_i,v^{(n)}_{s_i}\right)}\right)-p\left(s_i, v_{\rho\left(s_i,v_{s_i}\right)}\right)\right\|\\
&+\left\|p\left(t_{i+1}, v^{(n)}_{\rho\left(t_{i+1},v^{(n)}_{t_{i+1}}\right)}\right)-p\left(t_{i+1}, v_{\rho\left(t_{i+1},v_{t_{i+1}}\right)}\right)\right\|\\
&+\mathsf{N}_\alpha\varsigma(0)\int_{s_i}^{t_{i+1}}\varpi\left(\left\|v^{(n)}_{\rho\left(s,v^{(n)}_s\right)}-v_{\rho(s,v_s)}\right\|_{\varXi}\right)ds\Bigg\}\end{aligned} \tag{17}$$

Substituing (17) in (16), and using the Lebesgue dominated convergence theorem, we have $\|(\varLambda v^{(n)})(t)-(\varLambda v)(t)\| \xrightarrow[n\to\infty]{} 0$, for $t\in(s_i,t_{i+1}]$,

The same conclusion hold for $t\in(t_i,s_i]$.

This mean the continuity of the operator $\varLambda$ on B_r.

Step 3: The operator $\varLambda$ is equicontinuous.

Let $\xi_1,\ \xi_2\in[0,t_1]$, such that $\xi_1<\xi_2$ and $v\in\mathsf{B}_r$.

$$\begin{aligned}&\|(\varLambda v)(\xi_2)-(\varLambda v)(\xi_1)\|\\
&\leq \|\mathsf{R}_\alpha(\xi_2)-\mathsf{R}_\alpha(\xi_1)\|_{\mathcal{L}(\mathsf{X})}\left\|g(\beta(v),v)(0)-p\left(0,v_{\rho(0,v_0)}\right)\right\|\\
&+\left\|p\left(\xi_2, v_{\rho(\xi_2,v_{\xi_2})}\right)-p\left(\xi_1, v_{\rho(\xi_1,v_{\xi_1})}\right)\right\|\\
&+\int_0^{\xi_1}\left\|(\xi_2-s)^{\alpha-1}\mathsf{S}_\alpha(\xi_2-s)-(\xi_1-s)^{\alpha-1}\mathsf{S}_\alpha(\xi_1-s)\right\|_{\mathcal{L}(\mathsf{X})}\left\|\zeta\left(s,v_{\rho(s,v_s)},\overline{u}_v(s)\right)\right\|ds\\
&+\int_0^{\xi_1}\left\|(\xi_2-s)^{\alpha-1}\mathsf{S}_\alpha(\xi_2-s)-(\xi_1-s)^{\alpha-1}\mathsf{S}_\alpha(\xi_1-s)\right\|_{\mathcal{L}(\mathsf{X})}\|B\overline{u}_v(s)\|\,ds\\
&+\int_{\xi_1}^{\xi_2}\left\|(\xi_2-s)^{\alpha-1}\mathsf{S}_\alpha(\xi_2-s)\right\|_{\mathcal{L}(\mathsf{X})}\left\|\zeta\left(s,v_{\rho(s,v_s)},\overline{u}_v(s)\right)\right\|ds\\
&+\int_{\xi_1}^{\xi_2}\left\|(\xi_2-s)^{\alpha-1}\mathsf{S}_\alpha(\xi_2-s)\right\|_{\mathcal{L}(\mathsf{X})}\|B\overline{u}_v(s)\|\,ds\end{aligned} \tag{18}$$

By the norm continuity of $(\mathsf{R}_\alpha(t))_{t\geq 0}$ and $(t^{\alpha-1}\mathsf{S}_\alpha(t))_{t\geq 0}$, we deduce that the right hand of the above inequality tends to 0 as $\xi_1\to\xi_2$.

For $\xi_1,\ \xi_2\in(t_i,s_i]$, such that $\xi_1<\xi_2$ and $v\in\mathsf{B}_r$, we have:

$$\|(\varLambda v)(\xi_2)-(\varLambda v)(\xi_1)\| \leq \left\|h_i\left(\xi_2, v_{\rho(\xi_2,v_{\xi_2})}\right)-h_i\left(\xi_1, v_{\rho(\xi_1,v_{\xi_1})}\right)\right\| \xrightarrow[\xi_1\to\xi_2]{} 0. \tag{19}$$

Let $\xi_1,\ \xi_2 \in (s_i, t_{i+1}],\ i \in [\![1, n]\!]$, such that $\xi_1 < \xi_2$ and $v \in \mathsf{B}_r$.

$$
\begin{aligned}
&\|(\varLambda v)(\xi_2) - (\varLambda v)(\xi_1)\| \\
&\leq \left\| \mathsf{R}_\alpha(\xi_2 - s_i) h_i\left(\xi_2, v_{\rho(\xi_2, v_{\xi_2})}\right) - \mathsf{R}_\alpha(\xi_1 - s_i) h_i\left(\xi_1, v_{\rho(\xi_1, v_{\xi_1})}\right) \right\| \\
&+ \|\mathsf{R}_\alpha(\xi_2 - s_i) - \mathsf{R}_\alpha(\xi_1 - s_i)\|_{\mathcal{L}(\mathsf{X})} \left\| p\left(s_i, v_{\rho(s_i, v_{s_i})}\right) \right\| \\
&+ \left\| p\left(\xi_2, v_{\rho(\xi_2, v_{\xi_2})}\right) - p\left(\xi_1, v_{\rho(\xi_1, v_{\xi_1})}\right) \right\| \\
&+ \int_{s_i}^{\xi_1} \left\| (\xi_2 - s)^{\alpha-1} \mathsf{S}_\alpha(\xi_2 - s) - (\xi_1 - s)^{\alpha-1} \mathsf{S}_\alpha(\xi_1 - s) \right\|_{\mathcal{L}(\mathsf{X})} \left\| \zeta\left(s, v_{\rho(s, v_s)}, \overline{u}_v(s)\right) \right\| ds \\
&+ \int_{s_i}^{\xi_1} \left\| (\xi_2 - s)^{\alpha-1} \mathsf{S}_\alpha(\xi_2 - s) - (\xi_1 - s)^{\alpha-1} \mathsf{S}_\alpha(\xi_1 - s) \right\|_{\mathcal{L}(\mathsf{X})} \| B\overline{u}_v(s) \| \, ds \\
&+ \int_{\xi_1}^{\xi_2} \left\| (\xi_2 - s)^{\alpha-1} \mathsf{S}_\alpha(\xi_2 - s) \right\|_{\mathcal{L}(\mathsf{X})} \left\| \zeta\left(s, v_{\rho(s, v_s)}, \overline{u}_v(s)\right) \right\| ds \\
&+ \int_{\xi_1}^{\xi_2} \left\| (\xi_2 - s)^{\alpha-1} \mathsf{S}_\alpha(\xi_2 - s) \right\|_{\mathcal{L}(\mathsf{X})} \| B\overline{u}_v(s) \| \, ds \qquad (20)
\end{aligned}
$$

By the norm continuity of $(\mathsf{R}_\alpha(t))_{t\geq 0}$ and $(t^{\alpha-1}\mathsf{S}_\alpha(t))_{t\geq 0}$, we deduce that the right hand of the above inequality tends to 0 as $\xi_1 \to \xi_2$.

Therefore, the operator $\varLambda$ is equicontinuous on $[0, \mathsf{T}]$.

Step 4: The conditions of Mönch hold. Let $\mathsf{D} \subseteq \mathsf{B}_r$ be a countable set such that $\mathsf{D} \subseteq \overline{conv}(\{0\} \cup \varLambda(\mathsf{D}))$. We will show that the set D is relatively compact. In fact, we only need to show that the Kuratowski measure of noncompactness of the set D is null; that is, $\mu(\mathsf{D}) = 0$.

Suppose that $\mathsf{D} = \{v^{(n)}\}_{n=1}^{\infty} \subseteq \mathsf{B}_r$ is a equicontinuous set. We have:

For $t \in [0, t_1]$,

$$
\begin{aligned}
&\mu_{\mathscr{U}}\left(\{\overline{u}_{v^{(n)}}\}_{n=1}^{\infty}(t)\right) \\
&= \mu\left(\varTheta_i^{-1}\left\{\overline{v} - \mathsf{R}_\alpha(t_1)\left[g(\beta(v^{(n)}), v^{(n)})(0) - p\left(0, v^{(n)}_{\rho(0, v_0^{(n)})}\right)\right] - p\left(t_1, v^{(n)}_{\rho(t_1, v_{t_1}^{(n)})}\right)\right.\right. \\
&\left.\left. - \int_0^{t_1} (t_1 - s)^{\alpha-1} \mathsf{S}_\alpha(t_1 - s)\left[\zeta\left(s, v^{(n)}_{\rho(s, v_s^{(n)})}, \overline{u}_{v^{(n)}}(s)\right)\right] ds\right\}(t)\right) \\
&\leq \delta_\varTheta(t)\left\{\mu(\{\overline{v}^{(n)}\}_{n=1}^{\infty}) + \mathsf{M}\delta_g \mu(\{v^{(n)}(0)\}_{n=1}^{\infty}) + \mathsf{M}\delta_p(0)\mu\left(\left\{v^{(n)}_{\rho(0, v_0^{(n)})}\right\}_{n=1}^{\infty}\right)\right. \\
&\left. + \delta_p(t_1)\mu\left(\left\{v^{(n)}_{\rho(t, v_t^{(n)})}\right\}_{n=1}^{\infty}\right) + 2\int_0^{t_1} \mathsf{N}_\alpha \delta_\zeta(s)\mu\left(\left\{v^{(n)}_{\rho(s, v_s^{(n)})}\right\}_{n=1}^{\infty}\right) ds\right\} \\
&\leq \delta_\varTheta(t)\left\{\mu(\{\overline{v}^{(n)}\}_{n=1}^{\infty}) + \mathsf{M}\delta_g \mu(\{v^{(n)}(0)\}_{n=1}^{\infty}) + \mathsf{M}\delta_p(0) \times\right. \\
&\times \sup_{\theta \in [-\tau, 0]} \mu\left(\left\{v^{(n)}\left(\rho(0, v_0^{(n)}) + \theta\right)\right\}_{n=1}^{\infty}\right) + \delta_p(t_1) \sup_{\theta \in [-\tau, 0]} \mu\left(\left\{v^{(n)}\left(\rho(t_1, v_{t_1}^{(n)}) + \theta\right)\right\}_{n=1}^{\infty}\right) \\
&\left. + 2\int_0^{t} \mathsf{N}_\alpha \delta_\zeta(s) \sup_{\theta \in [-\tau, 0]} \mu\left(\left\{v^{(n)}\left(\rho(s, v_s^{(n)}) + \theta\right)\right\}_{n=1}^{\infty}\right) ds\right\} \\
&\leq \delta_\varTheta(t)\left\{\mu(\{\overline{v}^{(n)}\}_{n=1}^{\infty}) + \mathsf{M}\delta_g \mu(\{v^{(n)}(0)\}_{n=1}^{\infty}) + \mathsf{M}\delta_p(0) \sup_{s \in [0, \mathsf{T}]} \mu\left(\left\{v^{(n)}(s)\right\}_{n=1}^{\infty}\right)\right.
\end{aligned}
$$

$$+\delta_p(t_1)\sup_{s\in[0,\mathsf{T}]}\mu\left(\left\{v^{(n)}(s)\right\}_{n=1}^{\infty}\right)+2\int_0^{t_1}\mathsf{N}_\alpha\delta_\zeta(s)\sup_{s\in[0,\mathsf{T}]}\mu\left(\left\{v^{(n)}(s)\right\}_{n=1}^{\infty}\right)ds\Bigg\}$$

$$\leq\delta_\Theta(t)\left\{1+\mathsf{M}\delta_g+\mathsf{M}\delta_p(0)+\delta_p(t_1)+2t_1\mathsf{N}_\alpha\sup_{s\in[0,\mathsf{T}]}\delta_\zeta(s)\right\}\sup_{s\in[0,\mathsf{T}]}\mu\left(\left\{v^{(n)}(s)\right\}_{n=1}^{\infty}\right)\tag{21}$$

Similary, for $t\in(s_i,t_{i+1}],\quad i\in[\![1,n]\!]$ we get:

$$\mu_{\mathscr{U}}\left(\{\overline{u}_{v^{(n)}}\}_{n=1}^{\infty}(t)\right)$$

$$=\mu\left(\Theta_i^{-1}\left\{\overline{v}^{(n)}-\mathsf{R}_\alpha(t_{i+1}-s_i)\left[h_i\left(t_{i+1},v^{(n)}_{\rho(t_{i+1},v^{(n)}_{t_{i+1}})}\right)-p\left(s_i,v^{(n)}_{\rho(s_i,v^{(n)}_{s_i})}\right)\right]\right.\right.$$

$$\left.\left.-p\left(t_{i+1},v^{(n)}_{\rho(t_{i+1},v^{(n)}_{t_{i+1}})}\right)-\int_{s_i}^{t_{i+1}}(t_{i+1}-s)^{\alpha-1}\mathsf{S}_\alpha(t_{i+1}-s)\left[\zeta\left(s,v^{(n)}_{\rho(s,v^{(n)}_s)},\overline{u}_v(s)\right)\right]ds\right\}(t)\right)$$

$$\leq\delta_\Theta(t)\left\{1+\mathsf{M}\delta_{h_i}(s_i)+\mathsf{M}\delta_p(s_i)+\delta_p(t_{i+1})+2(t_{i+1}-s_i)\mathsf{N}_\alpha\sup_{s\in[0,\mathsf{T}]}\delta_\zeta(s)\right\}\times$$

$$\times\sup_{s\in[0,\mathsf{T}]}\mu\left(\left\{v^{(n)}(s)\right\}_{n=1}^{\infty}\right)\tag{22}$$

Using (21), we have:

For $t\in[0,t_1]$,

$$\mu\left(\left\{\Lambda v^{(n)}\right\}_{n=1}^{\infty}(t)\right)$$

$$=\mu\left(\left\{\mathsf{R}_\alpha(t)\left[g(\beta(v^{(n)}),v^{(n)})(0)-p\left(0,v^{(n)}_{\rho(0,v^{(n)}_0)}\right)\right]+p\left(t,v^{(n)}_{\rho(t,v^{(n)}_t)}\right)\right.\right.$$

$$\left.\left.+\int_0^t(t-s)^{\alpha-1}\mathsf{S}_\alpha(t-s)\left[\zeta\left(s,v^{(n)}_{\rho(s,v^{(n)}_s)},\overline{u}_{v^{(n)}}(s)\right)+B\overline{u}_{v^{(n)}}(s)\right]ds\right\}_{n=1}^{\infty}\right)$$

$$\leq\mathsf{M}\delta_g\mu(\{v^{(n)}(0)\}_{n=1}^{\infty})+\mathsf{M}\delta_p(0)\mu\left(\left\{v^{(n)}_{\rho(0,v^{(n)}_0)}\right\}_{n=1}^{\infty}\right)$$

$$+\delta_p(t)\mu\left(\left\{v^{(n)}_{\rho(t,v^{(n)}_t)}\right\}_{n=1}^{\infty}\right)+2\int_0^t\mathsf{N}_\alpha\delta_\zeta(s)\mu\left(\left\{v^{(n)}_{\rho(s,v^{(n)}_s)}\right\}_{n=1}^{\infty}\right)ds$$

$$+2\int_0^t\mathsf{N}_\alpha\delta_B\mu_{\mathscr{U}}\left(\{\overline{u}_{v^{(n)}}(s)\}_{n=1}^{\infty}\right)ds$$

$$\leq\left\{\mathsf{M}\delta_g+\mathsf{M}\delta_p(0)+\delta_p(t)+2t\mathsf{N}_\alpha\sup_{s\in[0,\mathsf{T}]}\delta_\zeta(s)+2t\mathsf{N}_\alpha\delta_B\delta_\Theta(t)\Bigg(1+\mathsf{M}\delta_g+\mathsf{M}\delta_p(0)\right.$$

$$\left.+\delta_p(t_1)+2t_1\mathsf{N}_\alpha\sup_{s\in[0,\mathsf{T}]}\delta_\zeta(s)\Bigg)\right\}\sup_{s\in[0,\mathsf{T}]}\mu\left(\left\{v^{(n)}(s)\right\}_{n=1}^{\infty}\right)\tag{23}$$

Similary, using (22), we get for $t\in(s_i,t_{i+1}],\ i\in[\![1,n]\!]$,

$$\mu\left(\left\{\Lambda v^{(n)}\right\}_{n=1}^{\infty}(t)\right)$$

$$=\mu\left(\left\{\mathsf{R}_\alpha(t-s_i)\left[h_i\left(t,v^{(n)}_{\rho(t,v^{(n)}_t)}\right)-p\left(s_i,v^{(n)}_{\rho(s_i,v^{(n)}_{s_i})}\right)\right]+p\left(t,v^{(n)}_{\rho(t,v^{(n)}_t)}\right)\right.\right.$$

$$\left.\left.+\int_{s_i}^t(t-s)^{\alpha-1}\mathsf{S}_\alpha(t-s)\left[\zeta\left(s,v^{(n)}_{\rho(s,v^{(n)}_s)},\overline{u}_{v^{(n)}}(s)\right)+B\overline{u}_{v^{(n)}}(s)\right]ds\right\}_{n=1}^{\infty}\right)$$

$$\leq \Bigg\{ \mathsf{M}\delta_{h_i}(s_i) + \mathsf{M}\delta_p(s_i) + \delta_p(t) + 2(t-s_i)\mathsf{N}_\alpha \sup_{s\in[0,\mathsf{T}]} \delta_\zeta(s) + 2(t-s_i)\mathsf{N}_\alpha \delta_B \delta_\Theta(t) \Bigg(1 + \mathsf{M}\delta_{h_i}(s_i)$$

(24)

$$+\mathsf{M}\delta_p(s_i) + \delta_p(t_{i+1}) + 2(t_{i+1}-s_i)\mathsf{N}_\alpha \sup_{s\in[0,\mathsf{T}]} \delta_\zeta(s) \Bigg) \Bigg\} \sup_{s\in[0,\mathsf{T}]} \mu\left(\left\{ v^{(n)}(s) \right\}_{n=1}^{\infty} \right)$$

By (23) and (24) and Lemma 2.1, we conclude that for $t \in (s_i, t_{i+1}],\ i \in [\![0, n]\!]$,

$$\mu\left(\left\{ \Lambda v^{(n)} \right\}_{n=1}^{\infty}(t) \right)$$
$$\leq \Bigg\{ \max\left(\mathsf{M}\delta_{h_i}(s_i); \mathsf{M}\delta_g\right) + \mathsf{M}\delta_p(s_i) + \delta_p(t) + 2(t-s_i)\mathsf{N}_\alpha \sup_{s\in[0,\mathsf{T}]} \delta_\zeta(s) + 2(t-s_i)\mathsf{N}_\alpha \delta_B \delta_\Theta(t) \times$$
$$\times \left(1 + \max\left(\mathsf{M}\delta_{h_i}(s_i); \mathsf{M}\delta_g\right) + \mathsf{M}\delta_p(s_i) + \delta_p(t_{i+1}) + 2(t_{i+1}-s_i)\mathsf{N}_\alpha \sup_{s\in[0,\mathsf{T}]} \delta_\zeta(s) \right) \Bigg\}$$
$$\sup_{s\in[0,\mathsf{T}]} \mu\left(\left\{ v^{(n)}(s) \right\}_{n=1}^{\infty} \right)$$
$$\leq \Pi_i \mu\left(\left\{ v^{(n)} \right\}_{n=1}^{\infty} \right).$$

In the same way, for $t \in (t_i, s_i],\ i \in [\![1, n]\!]$, we get:

$$\mu\left(\left\{ \Lambda v^{(n)} \right\}_{n=1}^{\infty}(t) \right) \leq \delta_{h_i}(s_i) \sup_{s\in[0,\mathsf{T}]} \mu\left(\left\{ v^{(n)}(s) \right\}_{n=1}^{\infty} \right)$$
$$\leq \Pi_i \mu\left(\left\{ v^{(n)} \right\}_{n=1}^{\infty} \right). \tag{25}$$

Thus,

$$\mu(\mathsf{D}) \leq \overline{conv}\left(\{0\} \cup \Lambda(\mathsf{D})\right) = \mu\left(\Lambda(\mathsf{D})\right) \leq \Pi_i \mu(\mathsf{D}). \tag{26}$$

As $\Pi_i < 1$, for $i \in [\![0, n]\!]$ (Hypothesis $\mathbf{H}_6$), we conclude that $\mu(\mathsf{D}) = 0$.

By the Mönch fixed point theorem (Theorem 2.5), the operator Λ has at least one fixed on B_r and this fixed point is a mild solution for the state dependent delay, non-instantaneaous impulsive integrodifferential system (1). Clearly, for any $i \in [\![0, n]\!]$, $(\Lambda v)(t_{i+1}) = \overline{v}$ which implies that the system (1) is controllable on I. This completes the proof. □

4. Application

As an application, we consider the following state-dependent delay, fractional order, neutral integrodifferential system with non-instantaneous impulses and non-local conditions.

$$\begin{cases} +\dfrac{\partial^\alpha}{\partial t^\alpha}\left[x(t-\rho(t), z) - a\left(t, x(t-\rho(t), z)\right)\right] = \dfrac{\partial^2}{\partial z^2}\left[x(t-\rho(t), z) - a\left(t, x(t-\rho(t), z)\right)\right] \\ + \displaystyle\int_0^t e^{\frac{-(t-s)}{\tau}} \dfrac{\partial^2}{\partial z^2}\left[x(t-\rho(t), z) - a\left(t, x(t-\rho(t), z)\right)\right] ds + f\left(t, x(t-\rho(t), z), u(t, z)\right) \\ +m(z)u(t, z),\ t \in \mathsf{I} = \displaystyle\bigcup_{i=0}^{n} \mathsf{I}_i := \bigcup_{i=0}^{n} (s_i, t_{i+1}] \\ x(t, z) = H_i\left(t, x(t-\rho(t), z)\right),\ t \in \displaystyle\bigcup_{i=1}^{n} (t_i, s_i] \\ x_0(t, z) = l\left(x_{\eta(x)}(\theta, z)\right),\ \theta \in [-r, 0],\ z \in [0, \pi], \end{cases} \tag{27}$$

where $\alpha \in (0, 1)$.

To represent this system in the abstract form, let $\mathscr{U} = \mathbb{H} = \mathsf{L}^2([0,\pi])$. In the sequel, $\mathsf{A} : \mathsf{D}(\mathsf{A}) \subset \mathbb{H} \to \mathbb{H}$ is the operator given by: $\mathsf{A}v = \Delta = \dfrac{\partial^2}{\partial t^2}v$, with domain $\mathsf{D}(\mathsf{A}) = \{v \in \mathbb{H} : \dfrac{\partial^2}{\partial t^2}v \in \mathbb{H},\ v(0) = v(\mathsf{T}) = 0\}$.

It is well known that $\Delta v = v''$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on $\mathbb{H}$. Hence, A is sectorial of type and $\mathbf{A}_1$ is satisfied. We also consider the operator $\gamma(t) : \mathsf{D}(\mathsf{A}) \subseteq \mathbb{H} \to \mathbb{H},\ t \geq 0,\ \gamma(t)v = e^{-\frac{t}{\tau}}\Delta v$ for $v \in \mathsf{D}(\mathsf{A})$. Moreover, it is easy to see that conditions $\mathbf{A}_2$ and $\mathbf{A}_3$ are satisfied with $b(t) = e^{-\frac{t}{\tau}}$ and $\mathsf{D} = \mathscr{C}_0^\infty([0,\mathsf{T}])$, where $\mathscr{C}_0^\infty([0,\mathsf{T}])$ is the space of infinitely differentiable functions that vanish at $z = 0$ and $z = \mathsf{T}$.

Moreover, we set:

$$\begin{aligned} f\left(t, x(t-\rho(t), z), u(t,z)\right) &:= \sin\left(x(t-\rho(t))u(t)\right) \\ a\left(t, x(t-\rho(t), z)\right) &:= \cos\left(x(t-\rho(t))\right) \\ H_i\left(t, x(t-\rho(t), z)\right) &:= \cos\left(ix(t-\rho(t))\right) \\ (Bu)(z) &:= m(z)u(t,z),\ z \in [0,\pi] \end{aligned}$$

Under the above conditions we can represent the system (27) in the abstract form (1).

By these above functions, the assumptions $(\mathbf{H}_1)$-$(\mathbf{H}_5)$ are satisfied. Moreover, we choose the appropriate parameters to make $(\mathbf{H}_6)$ hold. Therefore, all the conditions in Theorem 3.1 have been satisfied. Thus, the fractional order, non-instantaneous impulsive integrodifferential system (27) admits at least one mild solution which is controllable on I.

5. Conclusion

In this paper, we investigate the controllability of a class of state-dependent delay fractional order, neutral integrodifferential system with non-instantaneous impulses and nonlocal conditions. It is very important to point out that in the considered integrodifferential system, the nonnlinear function depend on the control function. Clearly, we use the theory of the $\alpha-$resolvent operator developped by J. P. C. Dos Santos in [7] and the theory of measures of noncompactness, combinated with the fixed point theory, to derive a set of conditions that guarantee the existence and the controllability of mild solutions for the aformentionned system. At the end, we gave an application to illustrate our results.

For future works, it will be very interesting to consider approximate controllability and optimal control problem for this system. It will also be very good to consider the stochastic case.

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