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FINITE 2-GROUPS WHOSE THE SAME-ORDER TYPE IS ARITHMETIC PROGRESSIONS

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ABSTRACT. The same-order type $\tau_e(G)$ of a finite group G is a set formed from the sizes of the equivalence classes containing elements of the same order in G. Lazorec and Tărnăuceanu [2] posed an open question about the groups whose same-order type consists of an arithmetic progression with 3 elements in 2-group. In this paper, we prove that the same-order type of a metacyclic group can not be an arithmetic progression. We also discussed that the arithmetic progression of an abelian group and the extension of an abelian group.

1. INTRODUCTION

We denote the same-order type of G by $\tau_e(G)$, as defined in [1]. Lazorec and Tărnăuceanu have given some results of finite groups whose same-order types are arithmetic progressions formed of 3 or 4 elements. The study of same-order types formed of 4 elements are clearly, so we only concern about the groups whose same-order types with 3 elements. First, they proved that if G is not a finite 2-group having more than one cyclic subgroup of order 2 and $|\tau_e(G)| = 3$, then $\tau_e(G)$ is an arithmetic progression if and only if $G \cong S_3$. Next, we consider if G is such a group, then $\tau_e(G)$ can be an arithmetic progression. Lazorec and Tărnăuceanu [2] used GAP [6] to get that the first group whose same-order types satisfy an arithmetic progression is $C_8 \rtimes C_2^2$(SmallGroup(32, 43)). Its same-order types are $\{1, 8, 15\}$. They also checked some finite groups of orders 64, 128, and 256 whose same-order types form arithmetic progressions. Subsequently, they posed the following open question:

Open Problem 1.1. *Classify all finite 2-groups whose same order types are arithmetic progressions formed of 3 elements.*

Following, we will prove that the same-order type of a metacyclic 2-group can not be an arithmetic progression. Also, we prove that if G is a finite ablian 2-group or an entension of an ablian 2-group by Z_2, then $\tau_e(G)$ will not be an arithmetic progression. Moreover, if G is an entension of an ablian 2-group by $Z_2 \times Z_2$, and $\tau_e(G)$ consists of an arithmetic progressions. Suppose that G has a central subgroup ${Z_2}^m$. Then $G/{Z_2}^m$ is isomorphic to three groups: $C_8 \rtimes {C_2}^2$(SmallGroup(32,43)), ${C_2}^2 \rtimes D_{16}$ (SmallGroup(64,128)) and $D_8 \circ D_{16}$ (SmallGroup(64,257)).

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2. Some Lemmas

In this section, we give some lemmas. Let G be a finite group and let n be a positive integer such that $n||G|$. We denote by s_n the number of elements of order n of G. And the exponent of G is denoted by $\exp(G)$.

Lemma 2.1 (See [5]). *Let G be a finite metacyclic 2-group. Then*

(1) G contains exactly one involution if and only if G is either cyclic or generalized quaternion.

(2) G contains more than three involutions if and only if G is either a dihedral group or a semi-dihedral group.

(3) All other metacyclic 2-groups contain exactly three involutions.

Lemma 2.2. *Let G be a finite 2-group such that $\tau_e(G)$ forms an arithmetic progression of 3 elements. Then the common difference of this arithmetic progression must be odd.*

Proof. Let G be a finite 2-group such that $\tau_e(G) = \{1, a, b\}$ is an arithmetic progression of same-order type in G, with $1 < a < b$. We claim that the common difference of the arithmetic progression must be odd. Otherwise, if the common difference is even, it will leads to both a and b being odd. And this is not possible since the number of elements of G of given order $n > 2$ is always even, as they come in pairs $\{g, g^{-1}\}$, with $g \in G$. Hence the common difference of $\tau_e(G)$ is odd. Moreover, we deduce that a is even while b is odd, and b is the number of elements of involutions in G. □

Lemma 2.3. *Let G be a finite 2-group such that $\tau_e(G)$ forms an arithmetic progression of 3 elements. Set t is an positive integer, then we can assume that the number of involutions in G is $2^t - 1$, and the number of elements of any order greater than 2 is 2^{t-1}.*

Proof. Set $|G| = 2^n$ and $\exp(G) = 2^k$, then the set of element orders of G is $\{1, 2, 2^2, 2^3, \ldots, 2^k\}$. By Lemma 2.2, we can assume $\tau_e(G) = \{1, a, b\}$, with $1 < a < b$, and b is the number of involutions in G. Since the same-order type of G is an arithmetic progression, then $a = \frac{b+1}{2}$, and we have

$$\tau_e(G) = \{1, \frac{b+1}{2}, b\}.$$

It follows that $s_{2^i} = \frac{b+1}{2}$ for $i = 2, 3, 4, \ldots, k$. Therefore,

$$|G| = 1 + b + \frac{(b+1)(k-1)}{2} = \frac{(b+1)(k+1)}{2},$$

that is $(k+1)(b+1) = 2^{n+1}$. Thus we can assume that $b = 2^t - 1$ and $s_{2^i} = \frac{b+1}{2} = 2^{t-1}$, where t is a positive integer. □

Lemma 2.4. *Let G be a finite 2-group such that $\tau_e(G)$ forms an arithmetic progression of 3 elements. Then $8 \leqslant exp(G) \leqslant \frac{|G|}{4}$.*

Proof. Let $|G| = n$. Since the length of $\tau_e(G)$ is 3, then the exponent of G is at least 4. Then we consider that $exp(G) = 4$. In the light of the Lemma 2.3, we can assume that $s_2(G) = 2^t - 1$ and $s_4(G) = 2^{t-1}$. Then

$$|G| = 1 + s_2 + s_4 = 2^{t-1} + 2^t,$$

which shows that $|G|$ is not a power of 2, and this is contradict that G is a 2-group. Hence $exp(G)$ is at least 8. Next we will proof $exp(G)$ is at most $\frac{|G|}{4}$. Recall that the first group whose same-order types satisfy an arithmetic progression is $C_8 \rtimes C_2^2$(SmallGroup(32, 43)), so $n \geqslant 2^5$. Assume $\exp(G) = n$, then G is a cyclic group of order n, so that $s_1 = s_2 = 1$, $s_4 = \phi(4) = 2$, $s_8 = \phi(8) = 4$, this implies that $\tau_e(G) = \{1, 2, 4\}$ since $|\tau_e(G)| = 3$, contradicting the fact that $\tau_e(G)$ is an arithmetic progression. If $\exp(G) = n/2$, then G has a normal cyclic subgroup of order $n/2$, and so G is a metacyclic 2-group. We first assume that $s_2 = 1$, then $\tau_e(G) = \{1, a, b\}$ where a and b are even integers. Since $\tau_e(G)$ is an arithmetic progression, $a - 1 = b - a$, a contradiction. Next we assume that $s_2 = 3$, then $\tau_e(G) = \{1, 2, 3\}$ since there are only two odd integers in $\tau_e(G)$. This implies that $\exp(G) = 4$ because if G has elements of order 8, then G has at least $\phi(8) = 4$ elements of order 8, contradicting that $\tau_e(G) = \{1, 2, 3\}$. Recall that $\exp(G) = n/2$, thus $n = 8$, which contradicts the fact that $n \geqslant 2^5$. Suppose G has more than 3 involutions, then G is a dihedral group D_n or a semidihedral group SD_n by Lemma 2.1. If $G \cong D_n$ then $s_4 = 2$, this follows that $s_2 = 3$ since $\tau_e(G)$ is an arithmetic progression of 3 elements, a contradiction. Finally, it is easy to check that $|\tau_e(G)| > 3$ if $G \cong SD_n$. Therefore, $\exp(G) \leqslant n/4$ because $\exp(G)$ is a divisor of $|G|$. □

3. Finite meta-abelian 2-groups

Lazorec and Tărnăuceanu [2] proved that the same-order type with 4 elements of a finite p-group can not be an arithmetic progression. Moreover, S_3 is the only finite group G such that $|\tau_e(G)| = 3$, and G is not a 2-group with more than one cyclic subgroup of order 2. Then we only consider that G is a finite 2-group with $|\tau_e(G)| = 3$. In the following, we prove that the same-order type of a finite metacyclic 2-group can not consist of an arithmetic progression.

Theorem 3.1. *Let G be a finite metacyclic 2-group such that $|\tau_e(G)| = 3$. Then $\tau_e(G)$ is not an arithmetic progression.*

Proof. We will prove this result by contradiction. Assume G is a metacyclic 2-group such that $\tau_e(G)$ is an arithmetic progression of 3 elements. We first assume that $s_2 = 1$, then $\tau_e(G) = \{1, a, b\}$ where a and b are even integers. Since $\tau_e(G)$ is an arithmetic progression, $a-1 = b-a$, a contradiction. We next assume that $s_2 = 3$, we have $\tau_e(G) = \{1, 2, 3\}$ since there are only two odd integers in $\tau_e(G)$. This implies that $\exp(G) = 4$ and $s_4 = 2$ because if G has elements of order 8 then G has at least $\phi(8) = 4$ elements of order 8. Thus $|G| = s_1+s_2+s_4 = 6$, which leads to a contradiction since G is a 2-group. Finally, we suppose that G has more than 3 involutions, then G is a dihedral group D_{2n} or a semidihedral group SD_{2n} by Lemma 2.1. Obviously, if $G \cong SD_{2n}$ then $|\tau_e(G)| > 3$, it is impossible. If $G \cong D_{2n}$ then $s_4 = 2$, this follows that $s_2 = 3$ since $\tau_e(G)$ is an arithmetic progression of 3 elements, a contradiction. □

We know that if A is an abelian group 2-group, then A can be decomposed:

$$G \cong {Z_2}^{m_1} \times {Z_{2^2}}^{m_2} \times \ldots \times {Z_{2^k}}^{m_k},$$

where $m_1, m_2, \ldots, m_k \geqslant 0$, and k is the exponent of G. Then we can compute the number of elements of each orders in A. Next we focus on the same-order types of a finite abelian 2-group and the extension of an abelian 2-group.

Theorem 3.2. *Let G be a finite abelian 2-group such that $|\tau_e(G)| = 3$. Then $\tau_e(G)$ will not be an arithmetic progression.*

Proof. Since G is a finite abelian 2-group, then the number of elements of order 2 in G is

$$s_2(G) = 2^{m_1+m_2+\ldots+m_k} - 1. \quad (1)$$

Suppose that $m_1 + m_2 + \ldots + m_k = t$, with $t > 0$, then we have

$$s_2(G) = 2^t - 1.$$

And the number of elements of order 4 in G is

$$\begin{aligned} s_4(G) &= 2^{m_1+2m_2+\ldots+2m_k} - 2^{m_1+m_2+\ldots+m_k} \\ &= 2^{2(m_1+\ldots+m_k)-m_1} - 2^t \\ &= 2^t(2^{t-m_1} - 1). \end{aligned} \quad (2)$$

Next we will discuss the value of $t - m_1$. If $t - m_1 = 0$, then $s_4(G) = 0$, which implies that G is an elementary abelian 2-group. It shows that the length of $\tau_e(G)$ is 2, a contradiction. On the other hands, if $t - m_1 \neq 0$, we have $s_4(G) \geqslant 2^t$. This leads to $s_4(G) > s_2(G)$, then $\tau_e(G)$ can not be an arithmetic progression by Lemma 2.3. Hence we arrive at a contradiction and our proof is finished. □

Theorem 3.3. *Let G be an extension of an abelian 2-group by Z_2. Then $\tau_e(G)$ will not be an arithmetic progression.*

Proof. Set $|G| = 2^n$, $G = A.Z_2$, where A is an abelian 2-group. Assume the number of involutions in G is $2^t - 1$. First we consider the number of involutions in A. If the involutions of G are all in A, then suppose that

$$s_2(G) = s_2(A) = 2^{m_1+m_2+\ldots+m_k} - 1 = 2^t - 1.$$

So we have $s_4(A) = 2^t(2^{t-m_1} - 1)$ by equation (2). If $t - m_1 = 0$, then $A = {Z_2}^t$, it follows that $|A| = 2^t$. However, $|A| = 2^{n-1}$, then it leads to $t = n - 1$, a contradiction. If $t - m_1 > 0$, the number of elements of order 4 in G is more than 2^t, a contradiction. Thus the involutions of G are not all in A. Next we consider the number of involutions in the cosets of A in G. Set

$$B = \{a \in A | a^z = a^{-1}, z \in G \setminus A, o(z) = 2\}.$$

Obviously, B is a subgroup of A, so $|B|$ is a power of 2. Since $\{a \in A | a^z = a^{-1}\} = \{a \in A | (az)^2 = 1\}$, then the number of involutions in $G \setminus A$ is a power of 2. We can suppose that $s_2(A) = 2^{t_1} - 1$ and $s_2(G \setminus A) = 2^{t_2}$. then $s_2(A) + s_2(G \setminus A) = s_2(G)$, it follows that

$$2^{t_1} - 1 + 2^{t_2} = 2^t - 1.$$

This equation has solutions if and only if $t_1 = t_2 = t - 1$. Hence the number of involutions in A is

$$s_2(A) = 2^{t_1} - 1 = 2^{t-1} - 1.$$

Note that $A \cong {Z_2}^{m_1} \times {Z_{2^2}}^{m_2} \times \ldots \times {Z_{2^k}}^{m_k}$, and it implies that $m_1 + \ldots + m_k = t - 1$. According to equation (2), the number of elements of order 4 in A is

$$\begin{aligned} s_4(A) &= 2^{2(m_1+\ldots+m_k)-m_1} - 2^{m_1+\ldots+m_k} \\ &= 2^{t-1}(2^{t-1-m_1} - 1). \end{aligned}$$

If $t-1-m_1=0$. which means that $s_4(A)=0$, and $|A|=2^{t-1}$. Moreover, $|A|=2^{n-1}$, so we have $t=n$, a contradiction. If $t-1-m_1=1$, then $s_4(A)=2^{t-1}$. Recall that $s_2(G)=2^t-1$, and $\tau_e(G)$ consisit of an arithmetic progression, then the number of elements of order 4 in G is 2^{t-1}. Thus $s_4(G)=s_4(A)=2^{t-1}$, which shows that the cosets of A only have the elements of order 2. Furthermore, if there exist elements of order 8 in A, then we have

$$\begin{aligned} s_8(A) &= 2^{m_1+2m_2+3m_3+\ldots+3m_k} - 2^{m_1+2m_2+\ldots+2m_k} \\ &= 2^{3(t-1)-2m_1-m_2} - 2^{2(t-1)-m_1} \\ &= 2^t(2^{1-m_2}-1). \end{aligned} \tag{3}$$

Since $m_2 \geqslant 0$, this leads to $s_8(A) \geqslant 2^t$, then $\tau_e(G)$ can not be an arithmetic progression. Therefore, there will not be elements of order 8 in A. Following, $A \cong {Z_2}^{m_1} \times Z_4$. Then the exponent of G is 4, by 2.4 we know that $\tau_e(G)$ can not be an arithmetic progression. □

Theorem 3.4. *Let G is an extension of an abelian 2-group by $Z_2 \times Z_2$, and $\tau_e(G)$ consists of an arithmetic progression. Then G has a central subgroup Z which is isomorphic to ${Z_2}^m$ such that G/Z is isomorphic to one of the following groups:*

(1) $C_8 \rtimes {C_2}^2$ (SmallGroup(32,43));
(2) ${C_2}^2 \rtimes D_{16}$ (SmallGroup(64,128));
(3) $D_8 \circ D_{16}$ (SmallGroup(64,257)).

Proof. Set $|G|=2^n$, $G=A.(Z_2\times Z_2)$, where A is an abeian 2-group, and the number of involutions in G is 2^t-1. Then the number of elements of any order greater than 2 in G is 2^{t-1}. According to Theorem 3.3, the involutions in G will not all in A, and the number of involutions of each cosets of A is a power of 2. So we can set $s_2(G)=2^t-1$, $s_2(A)=2^{t_1}-1$, $s_2(Az_1)=2^{t_2}$, $s_2(Az_2)=2^{t_3}$, $s_2(Az_3)=2^{t_4}$, and z_1,z_2,z_3 are transversal of A in G. Consequently, we have

$$2^{t_1}+2^{t_2}+2^{t_3}+2^{t_4}=2^t.$$

By GAP [6] we get the solutions, $t_1=t_2=t_3=t_4=t-2$ or $t_{i_1}=t-3, t_{i_2}=t-3, t_{i_3}=t-2, t_{i_4}=t-1$, with $t_{i_1},t_{i_2},t_{i_3},t_{i_4}\in\{t_1,t_2,t_3,t_4\}$. It is necessary to discuss the value of t_1 in the following four cases.

Case 1. If $t_1=t_2=t_3=t_4=t-2$, then $s_2(A)=2^{t-2}-1$, and $s_2(Az_1)=s_2(Az_2)=s_2(Az_3)=2^{t-2}$. Since A is an abelian 2-group, then we have $s_2(A)=2^{m_1+\ldots+m_k}-1$, it follows that $m_1+\ldots+m_k=t-2$. Thus $s_4(A)=2^{t-2}(2^{t-2-m_1}-1)$ by equation (2). If $t-2-m_1=0$, it will leads to $|A|=2^{t-2}$, which implies that $n=t$, a contradiction. If $t-2-m_1\geqslant 2$, the number of elements of order 4 in G is more than 2^{t-1}, a contradiction. So $t-2-m_1=1$, thus $s_4(A)=2^{t-2}$. Then we will consider the number of elements of order 2 and order 4 in each cosets of A in G. Set

$$U=\{u\in A|u^z=u^{-1}a, a\in A\},$$

with $z\in Az_1$, and the order of a is 1 or 2. Obviously, U is a subgroup of A, which means that the sum of $s_2(Az_1)$ and $s_4(Az_1)$ is a factor of $|A|$. Note that A is a finite abelian 2-group, it follows that $s_2(Az_1)+s_4(Az_1)$ is a power of 2. Then we can assume that $s_2(Az_1)+s_4(Az_1)=2^{l_1}$, with $l_1>0$. Similarly, $s_2(Az_2)+s_4(Az_2)=2^{l_2}$, $s_2(Az_3)+s_4(Az_3)=2^{l_3}$, with $l_2,l_3>0$. Since

$s_4(A) = 2^{t-2}$, and $s_4(G) = 2^{t-1}$, then

$$\begin{aligned} s_4(Az_1) + s_4(Az_2) + s_4(Az_3) &= s_4(G) - s_4(A) \\ &= 2^{t-2}. \end{aligned}$$

Note that the number of elements of involutions in each cosets of A is 2^{t-2}, thus we have the equation

$$\begin{aligned} 2^{l_1} + 2^{l_2} + 2^{l_3} &= \sum_{i=1}^{3} s_2(Az_i) + \sum_{i=1}^{3} s_4(Az_i) \\ &= 2^{t-2} \cdot 3 + 2^{t-2} \\ &= 2^t \end{aligned}$$

By calculating, $l_1 = t-2, l_2 = t-2, l_3 = t-1$ is the only solution of this equation. That is $s_2(Az_1) + s_4(Az_1) = 2^{t-2}, s_2(Az_2) + s_4(Az_2) = 2^{t-2}, s_2(Az_3) + s_4(Az_3) = 2^{t-2}$. Recall that $s_2(Az_1) = s_2(Az_2) = s_2(Az_3) = 2^{t-2}$, then we have $s_4(Az_1) = 0, s_4(Az_2) = 0, s_4(Az_3) = 2^{t-2}$, Which shows that $|Az_1| = |Az_2| = 2^{t-2}$. Since $|Az_3| \geqslant (s_2(Az_3) + s_4(Az_3)) = 2^{t-1}$, then $|Az_3| > |Az_1|$, a contradiction. Hence $\tau_e(G)$ can not be an arithmetic progression in this case.

Case 2. Suppose that $s_2(A) = 2^{t-3} - 1$, $s_2(Az_1) = 2^{t-3}, s_2(Az_2) = 2^{t-2}, s_2(Az_3) = 2^{t-1}$. Since $s_2(A) = 2^{t-3} - 1$, then $m_1 + \ldots m_k = t-3$. Also, $s_4(A) = 2^{t-3}(2^{t-3-m_1} - 1)$ by equation (2). We claim that $t-3-m_1$ can not equal to 0. Otherwise, $|A| = 2^{t-3}$, and $|A| = 2^{n-2}$, then it will leads to $n < t$, a contradiction. If $t-3-m_1 > 2$, then $s_4(A) = 2^{t-3}(2^{t-3-m_1}) > 2^{t-1}$, a contradiction. So $t-3-m_1$ must equal to 1 or 2. If $t-3-m_1 = 1$, then $s_4(A) = 2^{t-3}$. To satisfy the sum of the number of elements of order 2 and order 4 in every cosets of A in G is a power of 2, and $s_4(G) = 2^{t-1}$, then by calculating we have $s_4(Az_1) = 2^{t-3}, s_4(Az_2) = 2^{t-2}, s_4(Az_3) = 0$. Since $s_2(Az_3) = 2^{t-1}$ and $s_4(Az_3) = 0$, then $|Az_3| = s_2(Az_3) = 2^{t-1}$. So we can get that $|A| = |Az_3| = 2^{t-1}$. Furthermore,

$$1 + s_2(A) + s_4(A) = 1 + 2^{t-3} + 2^{t-2} < 2^{t-1},$$

which means that there must be the elements of order 8 in A. And we have $s_8(A) = 2^{t-2}(2^{1-m_2} - 1)$ by equation (3). Thus $s_8(A) = 2^{t-2}$ and $m_2 = 0$. Similarity, $|Az_1| = |Az_2| = |Az_3| = 2^{t-1}$, and $s_2(Az_2) + s_4(Az_2) = 2^{t-1}$, then $s_8(Az_2) = 0$. Note that $s_8(Az_3) = 0$ and $s_8(G) = 2^{t-1}$, we can compute the number of elements of order 8 in Az_1

$$s_8(Az_1) = s_8(G) - s_8(A) - s_8(Az_2) - s_8(Az_3) = 2^{t-2}.$$

Obviously, the exponent of G is 8. Since $t-3-m_1 = 1$, then $m_1 = t-4$. Hence $A \cong {Z_2}^{t-4} \times Z_8$. And the number of elements of each cosets of A in G is shown in the following table:

TABLE 1. the number of elements in each cosets of ${Z_2}^{t-4} \times Z_8$ in G

	A	Az_1	Az_2	Az_3
s_2	2^{t-3}	2^{t-3}	2^{t-2}	2^{t-1}
s_4	2^{t-3}	2^{t-3}	2^{t-2}	0
s_8	2^{t-2}	2^{t-2}	0	0

Therefore, $G \cong ({Z_2}^{t-4} \times Z_8).(Z_2 \times Z_2)$. Then we assume that

$$ {I_A}^{-1}(u_2) = \{a \in A | a^{u_2} = a^{-1}, u_2 \in Az_2\}. $$

Then ${I_A}^{-1}(u_2)$ is a normal subgroup of A. If there are no elements of order 8 in ${I_A}^{-1}(u_2)$, then ${I_A}^{-1}(u_2) \cong {Z_2}^{t-4} \times Z_4$, it follows that

$$ G/{Z_2}^{t-4} \cong Z_8.(Z_2 \times Z_2). $$

Obviously, $Z \cong {Z_2}^{t-4}$ is contained in the center of G. Next we consider the number of elements of each order in G/Z. Note that the elements of order 2 in G/Z is divided into two parts. Then we can assume that $s_2(G/Z) = x_1 + x_2$, where x_1 is the involutions in G/Z that keeps the order constant in G, and the order of x_2 is changed. Similarly, we can suppose that $s_4(G/Z) = y_1 + y_2$ and $s_8(G) = z_1 + z_2$, where y_1, z_1 is the elements in G/Z that keeps the order constant after acting by Z, and the order of y_2, z_2 is changed. Since the exponent of G is 8, which shows that $z_2 = 0$, and the number of elements of order 8 in G/Z is $z_1 = s_8(G)/2^{t-4}$. Recall that $s_8(G) = 2^{t-1}$, so $s_8(G/Z) = 2^3$. Following, according to the relationship between the quantities of elements of each order in G we have

$$ 2^{t-4} \cdot x_1 + 2^{t-4} - 1 = s_2(G) = 2^t - 1 \tag{4} $$

$$ 2^{t-4} \cdot x_2 + 2^{t-4} \cdot y_1 = s_4(G) = 2^{t-1} \tag{5} $$

$$ 2^{t-4} \cdot y_2 + 2^{t-4} \cdot z = s_8(G) = 2^{t-1} \tag{6} $$

By equation (4), (5), (6), we obtain that $x_1 + 1 = 2^4$, $x_2 + y_1 = 2^3$, and $y_2 + z = 2^3$. Recall that $z_1 = 2^3$, so we deduce that $y_2 = 0$. Thus the number of elements of order 4 in G/Z that all keeps the order constant, and $s_4(G/Z) = y_1 = 2^3$. It implies that $x_2 = 0$, then $1 + s_2(G/Z) = 2^t/2^{t-4} = 2^4$. Consequently, the same-order type of G/Z is $\{1, 2^3, 2^4 - 1\}$. We find that $Z_8 \rtimes (Z_2 \times Z_2)$ (SmallGroup(32,43)) is the only group satisfying the condition. On the other hand, if there exist the elements of order 8 in ${I_A}^{-1}(u_2)$, we have ${I_A}^{-1}(u_2) \cong {Z_2}^{t-5} \times Z_8$. Also, ${Z_2}^{t-5}$ is contained in the center of G, then

$$ G/{Z_2}^{t-5} \cong ((Z_2 \times Z_8).(Z_2 \times Z_2)). $$

Similarly, from above we can get the same-order type of $G/{Z_2}^{t-5}$ is $\{1, 2^4, 2^5 - 1\}$. It is easy to check that the groups satisfying the conditions are $Z_8 \rtimes {Z_2}^2$ (SmallGroup(64,128)) and $D_8 \circ D_{16}$ (SmallGroup(64,257)). If $t - 3 - m_1 = 2$, we can get that $s_4(Az_1) = 2^{t-3}, s_4(Az_2) = 0, s_4(Az_3) = 0$. This leads to $|Az_2| = s_2(Az_2) = 2^{t-2}$, and $|Az_3| = s_2(Az_3) = 2^{t-1}$, which contradict to $|Az_2| = |Az_3|$.

Case 3. Suppose that $s_2(A) = 2^{t-2} - 1$, and $s_2(Az_1) = 2^{t-3}, s_2(Az_2) = 2^{t-3}, s_2(Az_3) = 2^{t-1}$. Since $s_2(A) = 2^{t-2} - 1$, it follows that $m_1 + \ldots + m_k = t - 2$. So we have $s_4(A) = 2^{t-2}(2^{t-2-m_1} - 1)$ by equation (2). Similarity, there must be the elements of order 4 in A. Otherwise, if A is an elementary abelian 2-group, then $|A| = 2^{t-2} = 2^{n-2}$, which shows that $n = t$, a contradiction. Then $t - 2 - m_1 > 0$. Moreover, if $t - 2 - m_1 \geqslant 2$, then $s_4(A) > 2^{t-1}$, which contradict to $s_4(G) = 2^{t-1}$. Thus $t - 2 - m_1 = 1$ and $s_4(A) = 2^{t-2}$. Recalled that the sum of the number of elements of order 2 and order 4 in every cosets of A in G is a power of 2. By computing, we have $s_4(Az_1) = 2^{t-3}, s_4(Az_2) = 2^{t-3}, s_4(Az_3) = 0$. Then $|Az_3| = 2^{t-1}$, it follows that

$|A| = |Az_3| = 2^{t-1}$. And $s_2(A) + s_4(A) = 2^{t-2} + 2^{t-2} = 2^{t-1}$, which shows that $s_8(A) = 0$, then $A \cong {Z_2}^{m_1} \times Z_4$. Obviously, the elements of order 8 only exist in cosets Az_1 and Az_2 since $s_8(Az_3) = 0$. Note that $s_8(G) = 2^{t-1}$, so

$$s_8(Az_1) + s_8(Az_2) = 2^{t-1}. \tag{7}$$

Since $|Az_1| = |Az_2| = |Az_3| = 2^{t-1}$, then

$$|Az_1| = s_2(Az_1) + s_4(Az_1) + s_8(Az_1) + \ldots + s_{2^k}(Az_1) = 2^{t-1}. \tag{8}$$

$$|Az_2| = s_2(Az_2) + s_4(Az_2) + s_8(Az_2) + \ldots + s_{2^k}(Az_2) = 2^{t-1}. \tag{9}$$

Where k is the exponent of G. Through equation (4), (5), (6), we can get the only solution, that is $s_8(Az_1) = s_8(Az_2) = 2^{t-2}$, and $s_{2^j}(Az_1) = s_{2^j}(Az_2) = 0$ if $j > 3$. Since $t - m_1 - 2 = 1$,then $m_1 = t - 3$. It follows that $A \cong {Z_2}^{t-3} \times Z_4$. Thus the number of elements of each cosets of A in G is shown in the following table

TABLE 2. the number of elements in each cosets of ${Z_2}^{t-3} \times Z_4$ in G

	A	Az_1	Az_2	Az_3
s_2	2^{t-2}	2^{t-3}	2^{t-3}	2^{t-1}
s_4	2^{t-2}	2^{t-3}	2^{t-3}	0
s_8	0	2^{t-2}	2^{t-2}	0

Hence $G \cong ({Z_2}^{t-3} \times Z_4).(Z_2 \times Z_2)$, and

$$G/{Z_2}^{t-3} \cong Z_4.(Z_2 \times Z_2).$$

Since $\tau_(G)$ is an arithmetic progression, it follows that the same-order type of $Z_4.(Z_2 \times Z_2)$ is $\{1, 2^2, 2^3 - 1\}$ from above. Obviously, no groups will satisfy such conditions.

Case 4. Suppose that $s_2(A) = 2^{t-1} - 1$, $s_2(Az_1) = 2^{t-3}, s_2(Az_2) = 2^{t-3}, s_2(Az_3) = 2^{t-2}$. Then $m_1 + \ldots m_k = t - 1$. And $s_4(A) = 2^{t-1}(2^{t-1-m_1} - 1)$ by equation (2). Obviously, $s_4(A) \leqslant s_4(G) = 2^{t-1}$, so $t - 1 - m_1 \leqslant 1$. If $t - 1 - m_1 = 1$, it follows that $s_4(A) = 2^{t-1}$. Thus the elements of order 4 in G are all in A. Then there must be the elements of order 8 in A because the exponent of G is at least 8. By the equation (3), we have $s_8(A) = 2^t(2^{1-m_2} - 1)$. This leads to $m_2 = 0$ and $s_8(A) = 2^t$, a contradiction. Hence $t - 1 - m_1 = 0$, and $A \cong {Z_2}^{t-1}$. Consequently,

$$G/{Z_2}^{t-1} \cong Z_2 \times Z_2.$$

Which shows that the exponent of G will less than 8, and this leads to the same-order type of G can not be an arithmetic progression, a contradiction. Therefore, the proof is complete. □

In what follows, we have proved that if $\tau_e(G)$ consists of an arithmetic progression with 3 elements, then the exponent of G is at most 8. Moreover, if the same-order type of G is an arithmetic progression with $exp(G) = 8$, then $1 + s_2(G) + s_4(G) + s_8(G) = |G|$. Recalled that $s_4(G) = s_8(G) = 2^{t-1}$, and $s_2(G) = 2^t - 1$. Then we have $2^{t+1} = |G|$ from above equation. It implies that $s_2(G) = 2^t - 1 = |G|/2 - 1$ and $s_4(G) = s_8(G) = 2^{t-1} = |G|/4$. Thus the same-order type of G is $\{1, |G|/4, |G|/2 - 1\}$. Using by the GAP [6], we know that if G is a 2-group with $|G| \leqslant 2^{10}$, and G contains an arithmetic progression of same-order type formed of 3 elements, then its type is $\{1, |G|/4, |G|/2 - 1\}$.

The above suggests us to do the following problem.

Problem. *Let G be a finite 2-group and $\tau_e(G)$ form an arithmetic progression of 3 elements. Is it true that $\tau_e(G) = \{1, |G|/4, |G|/2 - 1\}$?*

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