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EXPLICIT FORMULAS IN RENEWAL THEORY

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ABSTRACT. Let $\{X_i\}$ be a sequence of independent identically distributed random variables with $\mathsf{E}X_1 > 0$. Let $\{S_k\}$ be the sequence of their partial sums and $U(\cdot) = \sum_{k=0}^{\infty} \mathsf{P}(S_k \in \cdot)$ be the corresponding renewal measure. We study the problem of representing in explicit form the restrictions of U to the positive and negative half-axes. A similar problem for generalized renewal measures of special type is also considered.

1. Introduction

Let $\{X_i\}$ be a sequence of independent identically distributed random variables with a common distribution F and expectation $\mu = \mathsf{E}X_1 > 0$, and let $\{S_k\}$ be the sequence of their partial sums: $S_k = \sum_{i=1}^{k} X_i$, $k \geq 1$, $S_0 = 0$. Denote by U the corresponding *renewal measure*, that is,

$$U(A) = \sum_{k=0}^{\infty} \mathsf{P}(S_k \in A) = \sum_{k=0}^{\infty} F^{k*}(A), \qquad A \in \mathscr{B},$$

where F^{k*} stands for the k-fold convolution of F and $\mathscr{B}$ is the σ-algebra of Borel sets of $\mathbb{R}$. If ν is a complex-valued measure defined on the σ-algebra $\mathscr{B}$, then we shall denote by $\widehat{\nu}(s)$ its Laplace transform: $\widehat{\nu}(s) = \int_{\mathbb{R}} \exp(sx)\, \nu(dx)$. It is well defined for all $s \in \mathbb{C}$ such that the integral $\int_{\mathbb{R}} \exp(\Re sx)\, |\nu|(dx)$ converges; here $|\nu|$ stands for the total variation of ν. It has been shown in [1, Section 4.3] that if the random variables X_i are nonnegative and have the Erlang distribution, that is, its Laplace transform $\widehat{F}(s)$ is a rational function, then the renewal function $U(t) := U([\,0, t\,])$ admits an explicit expression as a sum of "exponential polynomials"; namely, the function $\widehat{U}(s) = 1/[1 - \widehat{F}(s)]$ is split into a sum of partial fractions, and to each partial fraction $1/(s - s_0)^k$ the inversion procedure is applied, which leads to the measure with density $(-1)^k x^{k-1} \exp(-s_0 x)/(k-1)!$, $x > 0$.

The aim of the present paper is to derive similar explicit formulas in the general case for the restrictions $U^+ := U\big|_{(0,\infty)}$ and $U^- := U\big|_{(-\infty,0)}$ of the renewal measure to the positive and negative half-axes. However, we can practically without any additional costs solve a similar problem for the following more general objects than the renewal measure U:

$$\Phi_n(A) = \sum_{k=0}^{\infty} \frac{n \cdot (n+k-1)!}{k!} F^{k*}(A), \qquad A \in \mathscr{B}, \tag{1}$$

where $n \geq 1$ is an arbitrary integer. Put $x^- = \max(0, -x)$. The measure Φ_n is σ-finite (in the sense that $\Phi_n(A) < \infty$ for bounded $A \in \mathscr{B}$) if and only if $\mathsf{E}(X_1^-)^n < \infty$ [5, Proposition]. The measure Φ_1 coincides with the renewal measure U. There exists a close connection between

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the measures Φ_n and higher renewal moments: let $X_i \geq 0$ almost surely (a.s.) and let $N(t) = \sup\{k \geq 0 : S_k \leq t\}$ be the number of renewals on the interval $(0, t]$; then the measure Φ_n is generated by the nondecreasing function

$$\Phi_n(t) := \mathsf{E}\{[N(t)+1]\,[N(t)+2]\cdots[N(t)+n]\}, \qquad n = 1, 2, \ldots$$

As pointed out in [10], when studying the higher renewal moments $\mathsf{E}N(t)^n$, it is preferable to investigate the functions $\Phi_n(t)$, since their Laplace-Stieltjes transforms have a rather simple form:

$$\widehat{\Phi}_n(s) := \int_0^\infty \exp(sx)\, d\Phi_n(x) = \frac{n!}{[1-\widehat{F}(s)]^n}, \qquad \Re s < 0.$$

The higher renewal moment $\mathsf{E}N(t)^n$ can be expressed as a linear combination of the $\Phi_k(t)$, $k = 0, \ldots, n$, where $\Phi_0(t) \equiv 1$, and vice versa. So our task will be to derive explicit formulas under suitable conditions for the restrictions $\Phi_n^+ := \Phi_n\big|_{(0,\infty)}$ and $\Phi_n^- := \Phi_n\big|_{(-\infty,0)}$ of the measure Φ_n defined by formula (1) (in particular, for the corresponding restrictions U^+ and U^- of the renewal measure U). We shall always assume that the underlying distribution F is *nonarithmetic* (recall: a distribution F on $\mathbb{R}$ is called *arithmetic* if it is concentrated on a set of points of the form $0, \pm\lambda, \pm 2\lambda, \ldots$ [2, Section V.2]). However, it is well known [9] that "there is usually a theorem about discrete renewal processes which is exactly parallel to any theorem about continuous renewal processes" and so arithmetic analogues of Theorems 2 and 3 of Section 3 are also valid.

2. Preliminaries

Denote $\eta = \inf\{n \geq 1 : S_n > 0\}$ and $\eta_- = \inf\{n \geq 1 : S_n < 0\}$. Let F_+ and F_- be the distributions of the *first strict ascending* and of the *first strict descending* heights of the random walk $\{S_n\}$:

$$F_+(A) = \mathsf{P}(S_\eta \in A,\, \eta < \infty), \quad F_-(A) = \mathsf{P}(S_{\eta_-} \in A,\, \eta_- < \infty), \qquad A \in \mathscr{B}.$$

We shall need the symmetric form of the Wiener–Hopf factorization [2, Section XVIII.6]:

$$1 - \widehat{F}(s) = a_0[1-\widehat{F}_-(s)]\cdot[1-\widehat{F}_+(s)], \qquad \Re s = 0, \tag{2}$$

where $a_0 = \exp\{-\sum_{n=1}^\infty \mathsf{P}(S_n = 0)/n\}$. Further,

$$1 - \widehat{F}(s) = [1-\widehat{F}_+(s)]\cdot[a\widehat{D}(s)]^{-1}, \qquad \Re s = 0, \tag{3}$$

where $a = \exp\{\sum_{n=1}^\infty \mathsf{P}(S_n \leq 0)/n\}$ and D denotes the distribution of the random variable $\inf_{k\geq 0} S_k > -\infty$ a.s.

Let us prove the following statement about the component $1 - \widehat{F}_+(s)$ of the factorization (2), which will be needed in the proof of Theorem 2 about an explicit formula for Φ_n^+.

Theorem 1. *Let $\{X_i\}$ be a sequence of independent identically distributed random variables with distribution F and a finite expectation $\mathsf{E}X_1 > 0$. Then the function $1 - \widehat{F}_+(s)$ is rational if and only if the function $f(s) := \int_{0+}^\infty \exp(sx)\, F(dx)$ is rational. Moreover, if $f(s) = p(s)/q(s)$ is an uncancellable ratio of polynomials, $\deg q(s) = m$, and the coefficient at s^m of $q(s)$ is equal to 1, then $\deg p(s) < \deg q(s)$ and*

$$1 - \widehat{F}_+(s) = \frac{s\prod_{j=1}^{l}(s-s_j)^{m_j}}{q(s)}, \tag{4}$$

where s_j, $j = 1, \ldots, l$, *are the roots of the characteristic equation* $1 - \widehat{F}(s) = 0$ *lying in the half-plane* $\{s \in \mathbb{C} : \Re s > 0\}$ *with multiplicities* m_j *(this means that* $1 - \widehat{F}(s) = (s - s_j)^{m_j} F_j(s)$*, where* $F_j(s_j) \neq 0$*); moreover,* $\sum_{j=1}^{l} m_j = m - 1$.

Proof. First, let us prove that $\deg p(s) < \deg q(s)$. We argue by contradiction. Suppose the contrary, that is, that $\deg p(s) \geq \deg q(s)$. Then

$$f(s) \to \begin{cases} \infty, & \text{if } \deg p(s) > \deg q(s), \\ c \neq 0, & \text{if } \deg p(s) = \deg q(s) \end{cases} \qquad \text{as } s \to -\infty,$$

which is impossible. Now we prove that the rationality of $f(s)$ implies the same property for $1-\widehat{F_+}(s)$. Denote by σ_j, $j = 1, \ldots, k$, the roots of the polynomial $q(s)$ and by n_j their respective multiplicities such that $\sum_{j=1}^{k} n_j = m$. We have $q(x) = (s - \sigma_1)^{n_1} \cdots (s - \sigma_1)^{n_k}$. Next, let us show that all $\Re\sigma_j$ are positive. Decomposing the function $f(s)$ into a linear combination of partial fractions, we get

$$f(s) = \sum_{j=1}^{k} \sum_{p=1}^{n_j} \frac{c_{jp}}{(s - \sigma_j)^p}.$$

Denote by $\mathscr{E}_j$ the measure on $(0, \infty)$ with density

$$e_j(x) := -\mathbf{1}_{(0,\infty)}(x) \exp(-\sigma_j x), \qquad x > 0;$$

here $\mathbf{1}_{(0,\infty)}(x)$ is the indicator of $(0, \infty)$. The fraction $1/(s - \sigma_j)^p$ is the Laplace transform $\widehat{\mathscr{E}_j^{p*}}(s)$ of the measure $\mathscr{E}_j^{p*}$ with density

$$e_j^{p*}(x) := (-1)^p \mathbf{1}_{(0,\infty)}(x) \frac{x^{p-1} \exp(-\sigma_j x)}{(p-1)!}, \qquad x > 0.$$

This measure cannot be finite if $\Re\sigma_j \leq 0$, since it would contradict the finiteness of F. Hence all $\Re\sigma_j$ must be positive. Let U_- be the renewal measure generated by F_-:

$$U_-(A) = \sum_{k=0}^{\infty} F_-^{k*}(A), \qquad A \in \mathscr{B}.$$

Since $\mu > 0$ implies that the random walk $\{S_k\}$ drifts to $+\infty$, the distribution F_- is defective ($F_-(\mathbb{R}) < 1$) [2, Chapter XVIII, § 4, (a)]. It follows that U_- is a *finite* measure. We have

$$1 - \widehat{F_+}(s) = a\widehat{U_-}(s) \left[1 - \int_{-\infty}^{0+} \exp(sx)\, F(dx)\right] - a\widehat{U_-}(s) f(s), \tag{5}$$

where $a\widehat{U_-}(s) f(s)$ is the Laplace transform of $V := aU_- * F|_{(0,\infty)}$ (the convolution of aU_- with the restriction of F to $(0, \infty)$). We have

$$\widehat{V}(s) = a\widehat{U_-}(s) \sum_{j=1}^{k} \sum_{p=1}^{n_j} \frac{c_{jp}}{(s - \sigma_j)^p}. \tag{6}$$

Let us prove that $\widehat{F_+}(s)$ is a rational function, which is equivalent to the rationality of $1-\widehat{F_+}(s)$. The distribution F_+ is concentrated on $(0, \infty)$. It follows from (5) that $F_+ = V|_{(0,\infty)}$. The double sum in (6) is the Laplace transform of the absolutely continuous measure

$$\sum_{j=1}^{k} \sum_{p=1}^{n_j} c_{jp} \mathscr{E}_j^{p*}.$$

We have

$$F_+ = a\sum_{j=1}^{k}\sum_{p=1}^{n_j} c_{jp}(U_- * \mathscr{E}_j^{p*})\big|_{(0,\infty)}$$

with density

$$f_+(x) = a\sum_{j=1}^{k}\sum_{p=1}^{n_j} c_{jp}U_- * e_j^{p*}(x), \qquad x > 0,$$

where

$$\begin{aligned} U_- * e_j^{p*}(x) &= \frac{(-1)^p}{(p-1)!}\int_{-\infty}^{0+}(x-y)^{p-1}e^{-\sigma_j(x-y)}\,U_-(dy) \\ &= \frac{(-1)^p}{(p-1)!}\sum_{q=0}^{p-1}\binom{p-1}{q}x^q e^{-\sigma_j x}\int_{-\infty}^{0+}|y|^{p-1-q}e^{\sigma_j y}\,U_-(dy) \\ &=: \sum_{r=1}^{p} d_{jpr}e_j^{r*}(x), \end{aligned}$$

exact values of the coefficients d_{jpr} being irrelevant. Hence

$$f_+(x) = a\sum_{j=1}^{k}\sum_{p=1}^{n_j} c_{jp}\sum_{r=1}^{p} d_{jpr}e_j^{r*}(x) =: a\sum_{j=1}^{k}\sum_{r=1}^{n_j} D_{jr}e_j^{r*}(x), \qquad x > 0,$$

exact values of the coefficients D_{jr} being irrelevant again. As a result, we have

$$\widehat{F_+}(s) = a\sum_{j=1}^{k}\sum_{r=1}^{n_j}\frac{D_{jr}}{(s-\sigma_j)^p} =: \frac{p_1(s)}{q(s)},$$

where $p_1(s)$ is polynomial and $\deg p_1(s) < \deg q(s)$. Finally, we have

$$1 - \widehat{F_+}(s) = \frac{q(s) - p_1(s)}{q(s)},$$

where $\deg[q(s) - p_1(s)] = \deg q(s) = m$. The polynomial $q(s) - p_1(s)$ has the roots 0, $s_1, \ldots, s_k$ with multiplicities 1, $m_1, \ldots, m_k$, respectively. It remains to prove the converse statement. We suppose that (4) holds and prove that $f(s)$ is a rational function. It follows from (2) that $F = a_0(F_- + F_+ - F_- * F_+)$. We have $F|_{(0,\infty)} = a_0\big[F_+ - (F_- * F_+)|_{(0,\infty)}\big]$, which implies $f(s) = a_0\big[\widehat{F_+}(s) - g(s)\big]$, where $g(s)$ denotes the Laplace transform of $(F_- * F_+)|_{(0,\infty)}$. So, in order to prove the rationality of $f(s)$, it suffices to show that $g(s)$ is a rational function. This is done exactly as above, one needs only to interchange the roles as follows: instead of $f(s)$ and U_-, take $\widehat{F_+}(s)$ and F_-, respectively. After some quite similar calculations, one arrives to the desired conclusion about rationality of $f(s)$. The proof of the theorem is complete. □

The following lemma has been proved in [8].

Lemma 1. *Let D^{n*} be the n-fold convolution of the distribution D of the random variable $\inf_{k\geq 0} S_k$. Then for each $A \in \mathscr{B}$*

$$\Phi_n(A) = a^n\int_{-\infty}^{0}\Psi_n(A - x)\,D^{n*}(dx),$$

where Ψ_n is the measure defined by formula (1) with F replaced by the distribution F_+ of the first strict ascending height S_η.

Let $s_0 \in \mathbb{C}$. Define an operator $T(s_0)$ as follows. Let ν be a σ-finite measure such that the measure $\int_A \exp(s_0 x)\nu(dx)$, $A \in \mathscr{B}$, is finite. We set

$$v(x; s_0) = \begin{cases} +\displaystyle\int_x^\infty e^{-s_0(x-y)}\,\nu(dy) & \text{for } x \geq 0, \\ -\displaystyle\int_{-\infty}^x e^{-s_0(x-y)}\,\nu(dy) & \text{for } x < 0. \end{cases}$$

Define the measure $T(s_0)\nu$ by the formula

$$T(s_0)\nu(A) = \int_A v(x; s_0)\,dx, \qquad A \in \mathscr{B}.$$

If $\int_{\mathbb{R}} |x| \exp(\Re s_0 x)\,|\nu|(dx) < \infty$, then

$$\widehat{T(s_0)\nu}(s) = \frac{\widehat{\nu}(s) - \widehat{\nu}(s_0)}{s - s_0}, \qquad \Re s = \Re s_0.$$

Note the following property of the operator $T(s_0)$. If a measure ν is concentrated on $(-\infty, 0]$ or $[0, \infty)$, then the measure $T(s_0)\nu$ will also be concentrated on $(-\infty, 0]$ or $[0, \infty)$, respectively. Denote, for the sake of brevity, $T := T(s_0)$ in case $s_0 = 0$.

3. Main Results

Fix an integer $n \geq 1$. We formulate the assertion about an explicit formula for the restriction $\Phi_n^+ = \Phi_n\big|_{(0,\infty)}$ of the measure Φ_n defined by formula (1).

Theorem 2. *Let $\{X_i\}$ be a sequence of independent identically distributed random variables with distribution F and a finite expectation $\mathsf{E}X_1 > 0$. Then the Laplace transform $\widehat{U}^+(s)$ of the restriction $U^+ = U\big|_{(0,\infty)}$ of the renewal measure U is a rational function if and only if the function $f(s) = \int_{0+}^\infty \exp(sx)\,F(dx)$ is rational. Suppose that $\mathsf{E}|X_1|^n < \infty$, $f(s) = p(s)/q(s)$ is an uncancellable ratio of polynomials, $\deg q(s) = m$, and the coefficient of s^m of the polynomial $q(s)$ is equal to 1. Let s_j, $j = 1, \ldots, l$, be the roots of the characteristic equation $1 - \widehat{F}(s) = 0$ lying in the half-plane $\{s \in \mathbb{C} : \Re s > 0\}$, and let m_j be their multiplicities. Then $\sum_{j=1}^l m_j = m - 1$ and*

$$\widehat{\Phi}_n^+(s) = \sum_{k=1}^n (-1)^k \frac{\gamma_k^{(n)}}{s^k} + \sum_{j=1}^l \sum_{k=1}^{nm_j} (-1)^k \frac{A_{jk}^{(n)}}{(s - s_j)^k}, \qquad \Re s < 0, \tag{7}$$

where the coefficients $\gamma_k^{(n)}$ are defined by the asymptotic expansion

$$\frac{n!}{[1 - \widehat{F}(s)]^n} = \sum_{k=1}^n (-1)^k \frac{\gamma_k^{(n)}}{s^k} + o\left(\frac{1}{s}\right) \qquad \text{as } s \to 0, \tag{8}$$

and

$$\sum_{k=1}^{nm_j} (-1)^k \frac{A_{jk}^{(n)}}{(s - s_j)^k}$$

is the principal part of the Laurent expansion of the analytic function $n!/[1 - \widehat{F}(s)]^n$ in a neighborhood of the isolated singular point $s = s_j$, $j = 1, \ldots, l$. In other words, relation (7) means that Φ_n^+ is an absolutely continuous σ-finite measure with the density

$$\varphi_n^+(t) = \mathbf{1}_{(0,\infty)}(t) \left[\sum_{k=1}^n \gamma_k^{(n)} \frac{t^{k-1}}{(k-1)!} + \sum_{j=1}^l \sum_{k=1}^{nm_j} A_{jk}^{(n)} \frac{t^{k-1}}{(k-1)!} e^{-s_j t} \right]. \tag{9}$$

Proof. Let $f(s)$ be a rational function. By Theorem 1, the function $1-\widehat{F}_+(s)$ is rational. Denote, for the sake of brevity, $d(s) := n!\,a^n[\widehat{D}(s)]^n$. This is the Laplace transform of a positive measure concentrated on $(-\infty, 0]$ and having a finite moment of order $n-1$ [3, Theorem 5]. It follows from the relations (3) and (4) that

$$\frac{n!}{[1-\widehat{F}(s)]^n} = \frac{d(s)}{[1-\widehat{F}_+(s)]^n} = \frac{d(s)[q(s)]^n}{s^n\prod_{j=1}^{l}(s-s_j)^{nm_j}}. \tag{10}$$

Splitting the rational function in (10) into a sum of partial fractions. we obtain

$$\frac{n!}{[1-\widehat{F}(s)]^n} = d(s)\left[1+\sum_{k=1}^{n}\frac{a_k}{s^k}+\sum_{j=1}^{l}\sum_{k=1}^{nm_j}\frac{b_{jk}}{(s-s_j)^k}\right]. \tag{11}$$

Put $s_0 = 0$. Consider $d(s)/(s-s_j)^k$ for $j = 0, \ldots, l$. We have

$$\frac{d(s)}{(s-s_j)^k} = \frac{d(s_j)}{(s-s_j)^k} + \frac{d_{j,1}(s)}{(s-s_j)^{k-1}} = \sum_{i=1}^{k}\frac{d_{j,k-i}(s_j)}{(s-s_j)^i} + d_{j,k}(s), \tag{12}$$

where

$$d_{j,0}(s) := d(s), \quad d_{j,i}(s) := \frac{d_{j,i-1}(s)-d_{j,i-1}(s_j)}{s-s_j}, \quad i = 1, \ldots, k,$$

are the Laplace transforms of the measures $n!\,a^nT(s_j)^i(D^{n*})$ concentrated on $(-\infty, 0]$. Substituting (12) into (11) and collecting similar terms, we obtain, by the uniqueness of both the asymptotic relation (8) and the Laurent expansion, that

$$\frac{n!}{[1-\widehat{F}(s)]^n} = a_n d_{0,n}(s) + \widehat{\beta}(s) + \sum_{k=1}^{n}(-1)^k\frac{\gamma_k^{(n)}}{s^k} + \sum_{j=1}^{l}\sum_{k=1}^{nm_j}(-1)^k\frac{A_{jk}^{(n)}}{(s-s_j)^k}, \tag{13}$$

where, by the properties of the operators T and $T(s_j)$,

$$\widehat{\beta}(s) := d(s) + \sum_{k=1}^{n-1}a_k d_{0,k}(s) + \sum_{j=1}^{l}\sum_{k=1}^{nm_j}b_{jk}d_{j,k}(s)$$

is the Laplace transform of some finite measure β concentrated on $(-\infty, 0]$, and $d_{0,n}(s)$, $\Re s > 0$, is the Laplace transform of the σ-finite measure $n!\,a^nT^n(D^{n*})$, which is also concentrated on $(-\infty, 0]$.

We cannot claim that $n!/[1-\widehat{F}(s)]^n$ is the Laplace transform of the measure Φ_n, since in our case Φ_n is a σ-finite measure on the *whole* line $\mathbb{R}$. Therefore, we cannot immediately pass from (13) to the desired relation among measures. However, applying the theory of generalized functions in just the same way as in the proof of Theorem 1 in [6], we arrive at the following equality:

$$\Phi_n = a_n n!\,a^nT^n(D^{n*}) + \beta + \sum_{k=1}^{n}\gamma_k^{(n)}L^{k*} + \sum_{j=1}^{l}\sum_{k=1}^{nm_j}A_{jk}^{(n)}\mathscr{E}_j^{k*},$$

where L is the restriction of Lebesgue measure to the set $(0, \infty)$ and $\mathscr{E}_j$ is the measure with the density $\mathbf{1}_{(0,\infty)}(x)\exp(-s_jx)$, $j = 1, \ldots, l$. Thus, for the restriction $\Phi_n^+ = \Phi_n\big|_{(0,\infty)}$ the following representation holds:

$$\Phi_n^+ = \sum_{k=1}^{n}\gamma_k^{(n)}L^{k*} + \sum_{j=1}^{l}\sum_{k=1}^{nm_j}A_{jk}^{(n)}\mathscr{E}_j^{k*},$$

which is equivalent to (7) and (9).

Let now $\widehat{U}^+(s) = \widehat{\Phi}_1^+(s)$ be a rational function. There exists a real-valued measure κ concentrated on $(-\infty, 0]$ such that for each bounded Borel set $A \subset (0, \infty)$

$$a\Psi_1(A) = \int_{-\infty}^{0} \Phi_1(A - y)\, \kappa(dy). \tag{14}$$

Actually, as shown in [7], the distribution D has an inverse D^{-1} in the Banach algebra of finite measures, that is, $D * D^{-1} = \delta_0$ is the Dirac measure concentrated at the origin. Put $\kappa = D^{-1}$. To obtain (14), we now use Lemma 1 with $n = 1$ and $A - y := \{u \in \mathbb{R} : u + y \in A\}$ in place of A and integrate the resulting equality with respect to the measure κ. We have

$$\begin{aligned}\int_{-\infty}^{0} \Phi_1(A - y)\, \kappa(dy) &= \int_{-\infty}^{0} a \int_{-\infty}^{0} \Psi_1(A - x - y)\, D(dx)\, \kappa(dy)\\ &= a \iiint\limits_{\{(x,y,u):x+y+u\in A\}} \Psi_1(du)\, D(dx)\, \kappa(dy)\\ &= a \int_{0}^{\infty} \delta_0(A - u)\Psi_1(du) = a\Psi_1(A),\end{aligned}$$

which proves equality (14).

Since the measure Φ_1^+ is absolutely continuous with the density $\varphi_1^+(t)$ given by (9) for $n = 1$, relation (14) implies that the measure Ψ_1 is also absolutely continuous with the density

$$\psi_1(t) = a^{-1} \int_{-\infty}^{0} \varphi_1^+(t - x)\, \kappa(dx), \qquad t > 0. \tag{15}$$

Substituting (9) into (15) and writing

$$(t - x)^k = \sum_{i=0}^{k} \binom{k}{i} t^i (-x)^{k-i},$$

we arrive at the conclusion that the density $\psi_1(t)$ is of the same form as $\varphi_1^+(t)$, that is, the Laplace transform $\widehat{\Psi}_1(s)$ of the measure Ψ_1 is a rational function. Further, the equality $\widehat{\Psi}_1(s) = 1/[1 - \widehat{F}_+(s)]$ implies that the function $1 - \widehat{F}_+(s)$ is rational. By Theorem 1, the function $f(s)$ is also rational. This completes the proof of the theorem. □

Theorem 3. *Let $\{X_i\}$ be a sequence of independent identically distributed random variables with distribution F and a finite expectation $\mathsf{E}X_1 > 0$. Suppose that $F((-\infty, 0)) > 0$. Then the Laplace transform $\widehat{U}^-(s)$ of the restriction $U^- = U\big|_{(-\infty,0)}$ of the renewal measure U is a rational function if and only if the function $f_-(s) := \int_{-\infty}^{0-} \exp(sx)\, F(dx)$ is rational. Suppose that $f_-(s) = P(s)/Q(s)$ is an uncancellable ratio of polynomials, $\deg Q(s) = m$, and the coefficient of s^m of the polynomial $Q(s)$ is equal to 1. Let s_j, $j = 1, \ldots, l$, be the roots of the characteristic equation $1 - \widehat{F}(s) = 0$ lying in the half-plane $\{s \in \mathbb{C} : \Re s < 0\}$, and let m_j be their multiplicities. Then $\sum_{j=1}^{l} m_j = m$ and*

$$\widehat{\Phi}_n^-(s) = \sum_{j=1}^{l} \sum_{k=1}^{nm_j} \frac{B_{jk}^{(n)}}{(s - s_j)^k}, \qquad \Re s \geq 0, \tag{16}$$

where $\sum_{k=1}^{nm_j} B_{jk}^{(n)}/(s - s_j)^k$ is the principal part of the Laurent expansion of the analytic function $n!/[1 - \widehat{F}(s)]^n$ in a neighborhood of the isolated singular point $s = s_j$, $j = 1, \ldots, l$. In other *words, relation* (16) *means that* Φ_n^- *is an absolutely continuous measure with the density*

$$\varphi_n^-(t) = \mathbf{1}_{(-\infty,0)}(t) \sum_{j=1}^{l} \sum_{k=1}^{nm_j} B_{jk}^{(n)} \frac{|t|^{k-1}}{(k-1)!} e^{-s_j t}.$$

Proof. Suppose $f_-(s)$ is a rational function and $f_-(s) = P(s)/Q(s)$ is an uncancellable ratio of polynomials as in the statement of the theorem. Obviously, $\widehat{F}(s) = f_-(s) + \int_{0-}^{\infty} \exp(sx)\, F(dx)$ is an analytic function in the half-plane $\{s \in \mathbb{C} : \Re s < 0\}$ with the exception of a finite number of isolated singular points which are the roots of the polynomial $Q(s)$. Further, we shall need the following lemma.

Lemma 2. *Let the hypotheses of Theorem* 3 *be satisfied. Then among the roots of the characteristic equation* $1 - \widehat{F}(s) = 0$ *lying in the half-plane* $\{s \in \mathbb{C} : \Re s < 0\}$*, there exists a real root, say,* $s_1 = \alpha < 0$*, such that the real parts of the remaining roots are strictly less than* α*; the multiplicity of* $s_1 = \alpha$ *is equal to* 1.

Proof. The function $\widehat{F}(t)$ is defined in a left neighborhood of the point $t = 0$ since $\widehat{F}(s) = f_-(s) + \int_{0-}^{\infty} \exp(sx)\, F(dx)$, where the second summand is an analytic function in the half-plane

$$\Pi_- := \{s \in \mathbb{C} \colon \Re s < 0\},$$

and $f_-(s)$ is a rational function. Since $\big[\widehat{F}(t)\big]'_{t=0} > 0$, we have $\widehat{F}(t) < 1$ in a left neighborhood of the point $t = 0$. Let $\mathscr{Z} = \{s_0, s_1, \dots, s_l\}$ be the set of the roots of the characteristic equation lying in Π_- and let $\Re s_0 = \max_j \Re s_j$. We claim that $s_0 \in \mathbb{R}$. Suppose the contrary: $\Im s_0 \neq 0$. We prove that this is impossible as follows. First, we observe that $\widehat{F}(\Re s_0) = 1$ since otherwise $\widehat{F}(\Re s_0) > 1$ and, by continuity of $\widehat{F}(t)$, there would exist an $\alpha \in (\Re s_0, 0)$ such that $\widehat{F}(\alpha) = 1$ which means that $\alpha \in \mathscr{Z}$ and $\alpha = \Re\alpha > \Re s_0$, a contradiction. Denote by G the probability distribution defined by

$$G(A) := \int_A \exp(\Re s_0)\, F(dx), \qquad A \in \mathscr{B}.$$

The distribution G is absolutely continuous with respect to F and vice versa:

$$G(dx) = \exp(\Re s_0)\, F(dx), \qquad F(dx) = \exp(-\Re s_0)\, G(dx).$$

Hence G is nonarithmetic as well F, the latter being so by hypothesis. The characteristic function of G is equal to $\widehat{G}(it) = \widehat{F}(\Re s_0 + i\Im s_0 t)$, $t \in \mathbb{R}$. Under the assumption $\Im s_0 \neq 0$, we would have $\widehat{G}(\Im s_0 t) = 1$. Perusing the proof of Theorem 2.1.4 in [4], we come to conclusion that the distribution G is purely discrete and that the points of the distribution function $G(t) := G((-\infty, t])$ necessarily have the form $2\pi k/\Im s_0$ (k integer). This means that G is an arithmetic distribution, which is a contradiction. Hence $\Im s_0 = 0$. The proof of the lemma is complete. □

We prolong the proof of Theorem 3. Among the roots of the equation $1 - \widehat{F}(s) = 0$ lying in the half-plane $\{s \in \mathbb{C} : \Re s < 0\}$, there exists a real root, say $s_1 = \alpha < 0$, such that the real parts of the remaining roots are strictly less than α; the multiplicity of $s_1 = \alpha$ is equal to 1. Consider the random walk $\{T_n\}$ generated by a sequence $\{Y_k\}$ of independent identically distributed random variables with distribution $G(A) := \int_A \exp(\alpha x)\, F(dx)$, $A \in \mathscr{B}$. We have

$$g_-(u) := \int_{-\infty}^{0-} e^{ux} G(dx) = \frac{P(u+\alpha)}{Q(u+\alpha)},$$

that is, $g_-(u)$ is a rational function.

It is clear that the expectation of Y_1 is *negative* and *finite*: $\mathsf{E}Y_1 = [\widehat{F}(\xi)]'_{\xi=\alpha} < 0$. Denote

$$\Xi_n(A) = \sum_{k=0}^{\infty} \frac{n \cdot (n+k-1)!}{k!} \mathsf{P}(T_k \in A), \qquad A \in \mathscr{B},$$

and $\Xi_n^- = \Xi_n\big|_{(-\infty,0)}$. It is not difficult to see that the situation with the measure Ξ_n is quite symmetric with the measure Φ_n in Theorem 2. Moreover, $\widehat{\Xi}_n^-(u) = \widehat{\Phi}_n^-(u+\alpha)$. The only difference consists in the sign of the expectation $\mathsf{E}Y_1$. Hence, applying Theorem 2, we come to the conclusion that the characteristic equation $1 - \widehat{G}(u) = 0$ has roots $\sigma_2, \dots, \sigma_l$ lying in the half-plane $\{u \in \mathbb{C} : \Re u < 0\}$ with cumulative multiplicity $\sum_{j=2}^{l} m_j = m - 1$ and, moreover, the following representation holds:

$$\widehat{\Xi}_n^-(s) = \sum_{k=1}^{n} \frac{\delta_k^{(n)}}{u^k} + \sum_{j=2}^{l} \sum_{k=1}^{nm_j} \frac{B_{jk}^{(n)}}{(u-\sigma_j)^k}, \qquad \Re u > 0, \tag{17}$$

where $\sum_{k=1}^{n} \delta_k^{(n)}/u^k$ and $\sum_{k=1}^{nm_j} B_{jk}^{(n)}/(u-\sigma_j)^k$ are the Laurent expansions of the analytic function $n!/[1-\widehat{G}(u)]^n$ in neighborhoods of the isolated singular points $u = 0$ and $u = \sigma_j$, $j = 2, \dots, l$, respectively. Returning to the variable $s = u + \alpha$, we obtain the desired equality (16) from (17); here we set $s_1 = \alpha$, $s_j = \sigma_j + \alpha$, $j = 2, \dots, l$; these are the roots of the characteristic equation $1 - \widehat{F}(s) = 0$ lying in the half-plane $\{s \in \mathbb{C} : \Re s < 0\}$.

Now let $\widehat{U}^-(s)$ be a rational function. Considering the renewal measure Ξ_1 generated by the random walk $\{T_k\}$, we verify that $\widehat{\Xi}_1^-(u)$ is a rational function, whence, by Theorem 2, the function $g_-(s)$ is rational. This is equivalent to the rationality of the function $f_-(s)$. The proof of the theorem is complete. □

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