DOI: 10.28919/ejma.2024.4.19

Eur. J. Math. Appl. (2024)4:19
URL: http://ejma.euap.org

EXISTENCE AND ULAM STABILITY RESULTS OF TEMPERED (κ, ψ)-HILFER FRACTIONAL TERMINAL DIFFERENTIAL PROBLEMS

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Abstract. The main objective of this paper is to investigate several aspects including the existence, uniqueness, and κ-Mittag-Leffler-Ulam-Hyers stability of a specific class of terminal value problems. These problems involve implicit nonlinear fractional differential equations and tempered (κ, ψ)-Hilfer fractional derivatives. To accomplish this, we employ several mathematical tools. These include the fixed point theorem of Banach, Schauder's fixed point theorem, and a generalization of the well-known Gronwall inequality. Additionally, we provide illustrative examples to demonstrate the practical effectiveness of our main findings.

1. Introduction

Fractional calculus, an extension of differentiation and integration to non-integer orders, has garnered significant attention in both theoretical studies and practical applications across diverse research domains. This versatility has elevated it to a pivotal tool in the field. Recent times have witnessed a substantial surge in research efforts dedicated to fractional calculus. This research delves into a spectrum of outcomes attained under varying conditions and manifestations of fractional differential equations and inclusions. For a more comprehensive exploration of the practical applications of fractional calculus, readers are encouraged to refer to works such as Herrmann [12], Hilfer [13], Kilbas *et al.* [15], and Samko *et al.* [40]. In this context, Agrawal [1] introduced certain generalizations of fractional integrals and derivatives, along with elucidating some of their inherent properties. Moreover, in [6, 7], Benchohra *et al.* embarked on demonstrating the existence, uniqueness, and stability outcomes for various classes of problems, each characterized by distinct conditions. This was achieved through an extension of the well-established Hilfer fractional derivative, which harmoniously unifies the Riemann-Liouville and Caputo fractional derivatives.

In a recent publication [10], Diaz introduced innovative definitions for the specialized functions κ-gamma and κ-beta. For those seeking further information, additional sources such as [9, 24, 25] are available. Sousa *et al.* presented the ψ-Hilfer fractional derivative in a separate

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Key words and phrases. ψ-Hilfer fractional derivative; κ-generalized ψ-Hilfer fractional derivative; tempered fractional operators; existence; uniqueness; generalized Gronwall inequality; Mittag-Leffler function; Ulam-Hyers stability; terminal condition.

Received 28/07/2024.

work [44], highlighting essential properties associated with this particular fractional operator. For deeper insights and outcomes based on this operator, one can delve into papers such as [4, 42, 43] and the references they provide. Motivated by the works mentioned above, we have introduced a novel extension of the renowned Hilfer fractional derivative [38]. This extension, termed the κ-generalized ψ-Hilfer fractional derivative, has empowered us to establish a more comprehensive rendition of Grönwall's lemma. Furthermore, it has enabled the exploration of diverse forms of Ulam stability. In tandem, we have conducted an extensive investigation into qualitative and quantitative results applicable to distinct categories of fractional differential problems [16, 32–37]. These advancements were facilitated by the introduction of this newly developed generalized fractional operator. Comprehensive details are available in [6, 7].

In [16], the authors considered the initial value problem with nonlinear implicit κ-generalized ψ-Hilfer type fractional differential equation:

$$\begin{cases} \left({}^{H}_{\kappa}\mathcal{D}^{\sigma,\varepsilon;\psi}_{\varrho_1+}\mathfrak{w}\right)(\delta) = \aleph\left(\delta, \mathfrak{w}(\delta), \left({}^{H}_{\kappa}\mathcal{D}^{\sigma,\varepsilon;\psi}_{\varrho_1+}\mathfrak{w}\right)(\delta)\right), \quad \delta \in (\varrho_1, \varrho_2], \\ \left(\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{w}\right)({\varrho_1}^+) = c_0, \end{cases}$$

where ${}^{H}_{\kappa}\mathcal{D}^{\sigma,\zeta;\psi}_{\varrho_1+}$, $\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}$ are the κ-generalized ψ-Hilfer fractional derivative of order $\sigma \in (0, \kappa)$ and type $\zeta \in [0, 1]$, and κ-generalized ψ-fractional integral of order $\kappa(1-\theta)$, where $\kappa > 0$, $\aleph \in C([\varrho_1, \varrho_2] \times \mathbb{R}^2, \mathbb{R})$ and $c_0 \in \mathbb{R}$.

In [35], we have considered the following problem:

$$\begin{cases} \left({}^{H}_{\kappa}\mathcal{D}^{\sigma,\zeta;\psi}_{\varrho_1+}\mathfrak{w}\right)(\delta) = \aleph\left(\delta, \mathfrak{w}(\delta), \left({}^{H}_{\kappa}\mathcal{D}^{\sigma,\zeta;\psi}_{\varrho_1+}\mathfrak{w}\right)(\delta)\right), \quad \delta \in (\varrho_1, \varrho_2], \\ c_1\left(\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{w}\right)({\varrho_1}^+) + c_2\left(\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{w}\right)(\varrho_2) = c_3, \end{cases}$$

where ${}^{H}_{\kappa}\mathcal{D}^{\sigma,\zeta;\psi}_{\varrho_1+}$, $\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}$ are the κ-generalized ψ-Hilfer fractional derivative of order $\sigma \in (0, \kappa)$ and type $\zeta \in [0, 1]$, and κ-generalized ψ-fractional integral of order $\kappa(1-\theta)$ respectively, $\theta = \frac{1}{\kappa}(\zeta(\kappa - \sigma) + \sigma)$, $\kappa > 0$, $\aleph \in C([\varrho_1, \varrho_2] \times \mathbb{R}^2, \mathbb{R})$ and $c_1, c_2, c_3 \in \mathbb{R}$ such that $c_1 + c_2 \neq 0$.

Tempered fractional calculus has emerged as a consequential subclass of fractional calculus operators in recent times. This category has the ability to generalize a multitude of fractional calculus forms, featuring analytic kernels. As a result, it represents an extension of fractional calculus capable of delineating the transition between typical and anomalous diffusion. Initially introduced by Buschman in [8], definitions pertaining to fractional integration, encompassing weak singular and exponential kernels, were subsequently expanded upon. To delve deeper into this subject, one can refer to [5, 19, 22, 23, 26, 30, 41]. While the Caputo tempered fractional derivative remains relatively underexplored within the literature, it possesses the potential to wield substantial influence within the domain. By scrutinizing this derivative, our objective is to comprehensively comprehend its characteristics and potential applications within this distinct mathematical concept, thus advancing the realm of fractional calculus. A notable contribution to the field of tempered fractional integrals and derivatives emerged in the work of Kucche *et al.* cited as [21]. Their endeavors yielded substantial progress as they introduced a novel framework for computing these integrals and derivatives. This was accompanied by a comprehensive suite of associated properties and outcomes. In a subsequent endeavor [18], the same team continued their exploration, extending the theory and delving into tempered fractional calculus with a focus on functions. This included the introduction of the tempered Hilfer-type operator. Building upon the foundation laid by prior research, and utilizing the κ-gamma, κ-beta, and κ-Mittag-Leffler functions along with their properties from [10], we propose an innovative definition for the tempered (κ, ψ)-Hilfer fractional operator. Moreover, we introduce a fresh, generalized version of the Gronwall inequality. Our study encompasses a comprehensive analysis of the inherent properties of this operator. We hold the view that this operator naturally extends the existing framework, augmenting it and offering valuable insights into the exploration of function-related applications within tempered fractional calculus. Additionally, it opens the door to promising avenues of future research in this vibrant and evolving domain.

While solving differential equations precisely is difficult or impossible in several situations, along with nonlinear analysis and optimization, we investigate approximate solutions. It is important to stress that only stable estimates are acceptable. As a result, numerous methodologies for stability analysis are employed such as Lyapunov and exponential stability. Ulam, a mathematician, first raised the stability issue in functional equations in a 1940 lecture at Wisconsin University. S.M. Ulam posed the question, "Under what conditions does an additive mapping exist near an approximately additive mapping?" [45]. The succeeding year, Hyers addressed Ulam's issue for additive functions defined on Banach spaces in [14]. Rassias [28] showed the existence of unique linear mappings close to approximation additive mappings in 1978, generalizing Hyers' results. In comparison to Lyapunov and exponential stability analysis, Ulam-Hyers stability analysis focuses on the behavior of a function under perturbations, rather than the stability of a dynamical system or equilibrium point. The authors of [2, 3, 35, 38] investigated the Ulam stabilities of fractional differential problems with different conditions. Considerable focus has been given to investigating the stability of various types of functional equations, specifically Ulam-Hyers and Ulam-Hyers-Rassias stability. This can be observed through the book by Benchohra *et al.* [6], as well as the research conducted by Luo *et al.* [20] and Rus [29], which delved into the stability of operatorial equations using the Ulam-Hyers approach.

In [17], the authors considered a class of problems for nonlinear Caputo tempered implicit fractional differential equations with boundary conditions and delay:

$$\begin{cases} {}_0^C D_\delta^{\beta,\gamma} \mathfrak{w}(\delta) = \Psi\left(\delta, \mathfrak{w}_\delta,\ {}_0^C D_\delta^{\beta,\gamma}\mathfrak{w}(\delta)\right), \quad \delta \in \Xi := [0, \varkappa], \\ \mathfrak{w}(\delta) = \Lambda(\delta), \quad \delta \in [-\kappa, 0], \\ \delta_1 \mathfrak{w}(0) + \delta_2 \mathfrak{w}(\varkappa) = \delta_3, \end{cases}$$

where $0 < \beta < 1$, $\gamma \geq 0$, ${}_0^C D_\delta^{\beta,\gamma}$ is the Caputo tempered fractional derivative, $\Psi : \Xi \times C([-\kappa, 0], \mathbb{R}) \times \mathbb{R}$ is a continuous function, $\mathfrak{w} \in C([-\kappa, 0], \mathbb{R})$, $0 < \varkappa < +\infty$, $\delta_1, \delta_2, \delta_3$ are real constants, and $\kappa > 0$ is the time delay. Their arguments are based on Banach, Schauder and Schaefer fixed point theorems.

In order to generalize our prior results, in this paper, we establish existence, uniqueness and stability results to the following tempered (κ,ψ)-Hilfer boundary value problem with nonlinear implicit fractional differential equation:

$$\left({}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}\mathfrak{w}\right)(\delta) = \aleph\left(\delta, \mathfrak{w}(\delta), \left({}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}\mathfrak{w}\right)(\delta)\right), \quad \delta \in (\varrho_1, \varrho_2], \tag{1}$$

$$\mathfrak{w}(\varrho_2) = \varkappa, \tag{2}$$

where ${}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}$, ${}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}$ are the tempered (κ,ψ)-Hilfer fractional derivative of order $\sigma \in (0,\kappa)$, $\varepsilon \in [0,1]$ and index $\lambda \in \mathbb{R}$, and tempered (κ,ψ)-fractional integral of order $\kappa(1-\theta)$ and index λ defined in Section 2 respectively, where $\theta = \frac{1}{\kappa}(\varepsilon(\kappa-\sigma)+\sigma)$, $\kappa > 0$, $\varkappa \in \mathbb{R}$, $\aleph : [\varrho_1, \varrho_2] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given appropriate function specified later.

The paper is structured as follows: Section 2 starts by introducing necessary notations and reviewing preliminaries related to κ-generalized ψ-Hilfer and tempered fractional operators, as well as functions like κ-Gamma, κ-Beta, κ-Mittag-Leffler, and several auxiliary results. Additionally, the definition of the tempered (κ,ψ)-Hilfer fractional derivative and some essential theorems and lemmas are presented. Section 3 contains an existence and uniqueness results for the problem (1)-(2), which relies on the Banach contraction principle and Schauder's fixed point theorem. Furthermore, in the Section 4, the definitions of κ-Mittag-Leffler-Ulam-Hyers stability and related remarks are provided, followed by the proof of the stability result for problem (1)-(2). The final section focuses on providing illustrative examples that effectively demonstrate the practical applicability of the main findings.

2. Preliminaries

First, we present the weighted spaces, notations, definitions, and preliminary facts which are used in this paper. Let $0 < \varrho_1 < \varrho_2 < \infty$, $J = [\varrho_1, \varrho_2]$, $\sigma \in (0,\kappa)$, $\varepsilon \in [0,1]$, $\lambda \in \mathbb{R}$, $\kappa > 0$ and $\theta = \frac{1}{\kappa}(\varepsilon(\kappa-\sigma)+\sigma)$. By $C(\nabla,\mathbb{R})$ we denote the Banach space of all continuous functions from ∇ into $\mathbb{R}$ with the norm

$$\|\mathfrak{w}\|_\infty = \sup\{|\mathfrak{w}(\delta)| : \delta \in \nabla\}.$$

$AC^{\jmath}(\nabla,\mathbb{R})$, $C^{\jmath}(\nabla,\mathbb{R})$ be the spaces of continuous functions, $\jmath$-times absolutely continuous and $\jmath$-times continuously differentiable functions on ∇, respectively.
Consider the weighted Banach space

$$C_{\theta;\psi}(\nabla) = \left\{\mathfrak{w} : (\varrho_1, \varrho_2] \to \mathbb{R} : \delta \to \Psi^{\psi}_{\theta}(\delta,\varrho_1)\mathfrak{w}(\delta) \in C(\nabla,\mathbb{R})\right\},$$

where $\Psi^{\psi}_{\theta}(\delta,\varrho_1) = (\psi(\delta)-\psi(\varrho_1))^{1-\theta}$, with the norm

$$\|\mathfrak{w}\|_{C_{\theta;\psi}} = \sup_{\delta\in[\varrho_1,\varrho_2]}\left|\Psi^{\psi}_{\theta}(\delta,\varrho_1)\mathfrak{w}(\delta)\right|,$$

and

$$C^{\jmath}_{\theta;\psi}(\nabla) = \left\{\mathfrak{w} \in C^{\jmath-1}(\nabla,\mathbb{R}) : \mathfrak{w}^{(\jmath)} \in C_{\theta;\psi}(\nabla)\right\}, \jmath \in \mathbb{N},$$
$$C^{0}_{\theta;\psi}(\nabla) = C_{\theta;\psi}(\nabla),$$

with the norm

$$\|\mathfrak{w}\|_{C^{\jmath}_{\theta;\psi}} = \sum_{\beta=0}^{\jmath-1} \|\mathfrak{w}^{(\beta)}\|_{\infty} + \|\mathfrak{w}^{(\jmath)}\|_{C_{\theta;\psi}}.$$

Consider the space $X^p_\psi(\varrho_1,\varrho_2)$, $(c\in\mathbb{R},\ 1\leq p\leq\infty)$ of those real-valued Lebesgue measurable functions $\widehat{\aleph}$ on $[\varrho_1,\varrho_2]$ for which $\|\widehat{\aleph}\|_{X^p_\psi}<\infty$, where the norm is defined by

$$\|\widehat{\aleph}\|_{X^p_\psi} = \left(\int_{\varrho_1}^{\varrho_2} \psi'(\delta)|\widehat{\aleph}(\delta)|^p d\delta\right)^{\frac{1}{p}},$$

where ψ is an increasing and positive function on $[\varrho_1,\varrho_2]$ such that ψ' is continuous on $[\varrho_1,\varrho_2]$ with $\psi(0)=0$. In particular, when $\psi(\mathfrak{w})=\mathfrak{w}$, the space $X^p_\psi(\varrho_1,\varrho_2)$ coincides with the $L_p(\varrho_1,\varrho_2)$ space.

In what follows, and to keep it concise, we will take into account the following:

$$\widehat{\lambda} := \max_{(\delta,\gamma)\in[\varrho_1,\varrho_2]\times[\varrho_1,\delta]} e^{-\lambda(\psi(\delta)-\psi(\gamma))} = \begin{cases} 1, & \text{if } \lambda\geq 0,\\ e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}, & \text{if } \lambda<0.\end{cases}$$

Definition 2.1 ([10]). *The κ-gamma function is defined by*

$$\Gamma_\kappa(\varsigma) = \int_0^\infty \delta^{\varsigma-1} e^{-\frac{\delta^\kappa}{\kappa}} d\delta, \varsigma>0.$$

When $\kappa\to 1$ then $\Gamma_\kappa(\varsigma)\to\Gamma(\varsigma)$, we have also some useful following relations $\Gamma_\kappa(\varsigma) = \kappa^{\frac{\varsigma}{\kappa}-1}\Gamma\left(\frac{\varsigma}{\kappa}\right)$, $\Gamma_\kappa(\varsigma+\kappa)=\varsigma\Gamma_\kappa(\varsigma)$ and $\Gamma_\kappa(\kappa)=\Gamma(1)=1$. Furthermore κ-beta function is defined as follows

$$B_\kappa(\varsigma,\widehat{\varsigma}) = \frac{1}{\kappa}\int_0^1 \delta^{\frac{\varsigma}{\kappa}-1}(1-\delta)^{\frac{\widehat{\varsigma}}{\kappa}-1} d\delta$$

so that $B_\kappa(\varsigma,\widehat{\varsigma}) = \frac{1}{\kappa}B\left(\frac{\varsigma}{\kappa},\frac{\widehat{\varsigma}}{\kappa}\right)$ and $B_\kappa(\varsigma,\widehat{\varsigma}) = \frac{\Gamma_\kappa(\varsigma)\Gamma_\kappa(\widehat{\varsigma})}{\Gamma_\kappa(\varsigma+\widehat{\varsigma})}$. The Mittag-Leffler function can also be refined into the κ-Mittag-Leffler function defined as follows

$$\mathbb{E}_\kappa^{\varsigma,\widehat{\varsigma}}(\mathfrak{w}) = \sum_{\beta=0}^{\infty} \frac{\mathfrak{w}^\beta}{\Gamma_\kappa(\varsigma\beta+\widehat{\varsigma})}, \varsigma,\widehat{\varsigma}>0,$$

then, we can have

$$\mathbb{E}_\kappa^{\varsigma}(\mathfrak{w}) = \mathbb{E}_\kappa^{\varsigma,\kappa}(\mathfrak{w}) = \sum_{\beta=0}^{\infty} \frac{\mathfrak{w}^\beta}{\Gamma_\kappa(\varsigma\beta+\kappa)}, \varsigma>0.$$

2.1. **Fractional Integrals.** Now, we give all the definitions to the different fractional integrals used throughout this paper.

Definition 2.2 (κ-Generalized ψ-fractional Integral [27]). *Let $\widehat{\aleph}\in X^p_\psi(\varrho_1,\varrho_2)$ and $[\varrho_1,\varrho_2]$ be a finite or infinite interval on the real axis $\mathbb{R}=(-\infty,\infty)$, $\psi(\delta)>0$ be an increasing function on $(\varrho_1,\varrho_2]$ and $\psi'(\delta)>0$ be continuous on (ϱ_1,ϱ_2) and $\sigma>0$. The generalized κ-fractional integral operators of a function $\widehat{\aleph}$ of order σ are defined by*

$$\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\widehat{\aleph}(\delta) = \int_{\varrho_1}^{\delta} \bar{\Psi}^{\kappa,\psi}_{\sigma}(\delta,\gamma)\psi'(\gamma)\widehat{\aleph}(\gamma)d\gamma,$$
$$\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_2-}\widehat{\aleph}(\delta) = \int_{\delta}^{\varrho_2} \bar{\Psi}^{\kappa,\psi}_{\sigma}(\gamma,\delta)\psi'(\gamma)\widehat{\aleph}(\gamma)d\gamma,$$

with $\kappa>0$ and $\bar{\Psi}^{\kappa,\psi}_{\sigma}(\delta,\gamma) = \dfrac{(\psi(\delta)-\psi(\gamma))^{\frac{\sigma}{\kappa}-1}}{\kappa\Gamma_\kappa(\sigma)}$.

Also in [25], Nápoles Valdés gave a more generalized fractional integral operators defined by

$$\mathcal{J}^{\sigma,\kappa;\psi}_{G,\varrho_1+}\widehat{\aleph}(\delta) = \frac{1}{\kappa\Gamma_\kappa(\sigma)}\int_{\varrho_1}^{\delta}\frac{\psi'(\gamma)\widehat{\aleph}(\gamma)d\gamma}{G(\psi(\delta)-\psi(\gamma),\frac{\sigma}{\kappa})},$$
$$\mathcal{J}^{\sigma,\kappa;\psi}_{G,\varrho_2-}\widehat{\aleph}(\delta) = \frac{1}{\kappa\Gamma_\kappa(\sigma)}\int_{\delta}^{\varrho_2}\frac{\psi'(\gamma)\widehat{\aleph}(\gamma)d\gamma}{G(\psi(\gamma)-\psi(\delta),\frac{\sigma}{\kappa})},$$

where $G(z,\sigma)\in AC[\varrho_1,\varrho_2]$; the space of absolutely continuous functions defined on $[\varrho_1,\varrho_2]$.

Definition 2.3 (The ψ-tempered fractional Integral [21]). *Let $\widehat{\aleph}\in X^p_\psi(\varrho_1,\varrho_2)$ and $[\varrho_1,\varrho_2]$ be a finite or infinite interval on the real axis $\mathbb{R}=(-\infty,\infty)$, $\psi(\delta)>0$ be an increasing function on $(\varrho_1,\varrho_2]$ and $\psi'(\delta)>0$ be continuous on (ϱ_1,ϱ_2), $\lambda\in\mathbb{R}$ and $\sigma>0$. The ψ-tempered fractional integral operators of a function $\widehat{\aleph}$ of order σ and index λ are defined by*

$${}^{T}_{\lambda}\mathcal{J}^{\sigma;\psi}_{\varrho_1+}\widehat{\aleph}(\delta) = \int_{\varrho_1}^{\delta}\frac{(\psi(\delta)-\psi(\gamma))^{\sigma-1}}{\Gamma(\sigma)}e^{-\lambda(\psi(\delta)-\psi(\gamma))}\psi'(\gamma)\widehat{\aleph}(\gamma)d\gamma,$$
$${}^{T}_{\lambda}\mathcal{J}^{\sigma;\psi}_{\varrho_2-}\widehat{\aleph}(\delta) = \int_{\delta}^{\varrho_2}\frac{(\psi(\gamma)-\psi(\delta))^{\sigma-1}}{\Gamma(\sigma)}e^{-\lambda(\psi(\gamma)-\psi(\delta))}\psi'(\gamma)\widehat{\aleph}(\gamma)d\gamma.$$

Obviously, the ψ-tempered fractional integral ${}^{T}_{\lambda}\mathcal{J}^{\sigma;\psi}_{\varrho_1+}$ reduces to the ψ-Riemann-Liouville fractional integral [6, 7] if $\lambda=0$.

We can now present the subsequent definition of a broader fractional integral that encompasses both integrals as specific instances.

Definition 2.4 (The (κ,ψ)-tempered fractional Integral [31]). *Let $\widehat{\aleph}\in X^p_\psi(\varrho_1,\varrho_2)$ and $[\varrho_1,\varrho_2]$ be a finite or infinite interval on the real axis $\mathbb{R}=(-\infty,\infty)$, $\psi(\delta)>0$ be an increasing function on $(\varrho_1,\varrho_2]$ and $\psi'(\delta)>0$ be continuous on (ϱ_1,ϱ_2), $\lambda\in\mathbb{R}$, $\kappa>0$ and $\sigma>0$. The (κ,ψ)-tempered fractional integral operators of a function $\widehat{\aleph}$ of order σ and index λ are defined by*

$${}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\widehat{\aleph}(\delta) = e^{-\lambda\psi(\delta)}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\left(\widehat{\aleph}(\delta)e^{\lambda\psi(\delta)}\right) = \int_{\varrho_1}^{\delta}\bar{\Psi}^{\kappa,\psi}_{\sigma}(\delta,\gamma)e^{-\lambda(\psi(\delta)-\psi(\gamma))}\psi'(\gamma)\widehat{\aleph}(\gamma)d\gamma,$$
$${}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_2-}\widehat{\aleph}(\delta) = e^{\lambda\psi(\delta)}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_2-}\left(\widehat{\aleph}(\delta)e^{-\lambda\psi(\delta)}\right) = \int_{\delta}^{\varrho_2}\bar{\Psi}^{\kappa,\psi}_{\sigma}(\gamma,\delta)e^{-\lambda(\psi(\gamma)-\psi(\delta))}\psi'(\gamma)\widehat{\aleph}(\gamma)d\gamma,$$

with $\bar{\Psi}^{\kappa,\psi}_{\sigma}(\delta,\gamma)=\dfrac{(\psi(\delta)-\psi(\gamma))^{\frac{\sigma}{\kappa}-1}}{\kappa\Gamma_\kappa(\sigma)}$. Now, the (κ,ψ)-tempered fractional integral ${}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}$ reduces to the ψ-tempered fractional integral ${}^{T}_{\lambda}\mathcal{J}^{\sigma;\psi}_{\varrho_1+}$ if $\kappa=1$.

2.2. **Fractional derivatives.** In this section, we present the definitions of various fractional derivatives that are utilized.

Definition 2.5 (κ-Generalized ψ-Hilfer Derivative [6, 7]). *Let $\jmath-1<\dfrac{\sigma}{\kappa}\leq\jmath$ with $\jmath\in\mathbb{N}$, $\nabla=[\varrho_1,\varrho_2]$ an interval such that $-\infty\leq\varrho_1<\varrho_2\leq\infty$ and $\widehat{\aleph},\psi\in C^\jmath([\varrho_1,\varrho_2],\mathbb{R})$ two functions such that ψ is increasing and $\psi'(\delta)\neq0$, for all $\delta\in\nabla$. The κ-generalized ψ-Hilfer fractional derivatives (left-sided and right-sided) ${}^{H}_{\kappa}\mathcal{D}^{\sigma,\varepsilon;\psi}_{\varrho_1+}(\cdot)$ and ${}^{H}_{\kappa}\mathcal{D}^{\sigma,\varepsilon;\psi}_{\varrho_2-}(\cdot)$ of a function $\widehat{\aleph}$ of order σ and type $0\leq\varepsilon\leq1$, with $\kappa>0$ are defined by*

$$\begin{aligned}{}^{H}_{\kappa}\mathcal{D}^{\sigma,\varepsilon;\psi}_{\varrho_1+}\widehat{\aleph}(\delta) &= \left(\mathcal{J}^{\varepsilon(\kappa\jmath-\sigma),\kappa;\psi}_{\varrho_1+}\left(\frac{1}{\psi'(\delta)}\frac{d}{d\delta}\right)^{\jmath}\left(\kappa^{\jmath}\mathcal{J}^{(1-\varepsilon)(\kappa\jmath-\sigma),\kappa;\psi}_{\varrho_1+}\widehat{\aleph}\right)\right)(\delta)\\ &= \left(\mathcal{J}^{\varepsilon(\kappa\jmath-\sigma),\kappa;\psi}_{\varrho_1+}\delta^{\jmath}_{\psi}\left(\kappa^{\jmath}\mathcal{J}^{(1-\varepsilon)(\kappa\jmath-\sigma),\kappa;\psi}_{\varrho_1+}\widehat{\aleph}\right)\right)(\delta)\end{aligned}$$

and

$$\begin{aligned}{}_{\kappa}^{H}\mathcal{D}_{\varrho_2-}^{\sigma,\varepsilon;\psi}\widehat{\aleph}\left(\delta\right) &= \left(\mathcal{J}_{\varrho_2-}^{\varepsilon(\kappa\jmath-\sigma),\kappa;\psi}\left(-\frac{1}{\psi'\left(\delta\right)}\frac{d}{d\delta}\right)^{\jmath}\left(\kappa^{\jmath}\mathcal{J}_{\varrho_2-}^{(1-\varepsilon)(\kappa\jmath-\sigma),\kappa;\psi}\widehat{\aleph}\right)\right)\left(\delta\right)\\ &= \left(\mathcal{J}_{\varrho_2-}^{\varepsilon(\kappa\jmath-\sigma),\kappa;\psi}(-1)^{\jmath}\delta_{\psi}^{\jmath}\left(\kappa^{\jmath}\mathcal{J}_{\varrho_2-}^{(1-\varepsilon)(\kappa\jmath-\sigma),\kappa;\psi}\widehat{\aleph}\right)\right)\left(\delta\right),\end{aligned}$$

where $\delta_{\psi}^{\jmath} = \left(\frac{1}{\psi'\left(\delta\right)}\frac{d}{d\delta}\right)^{\jmath}$.

Definition 2.6 (The tempered ψ-Hilfer Derivative [18])**.** *Let $\jmath - 1 < \sigma \leq \jmath$ with $\jmath \in \mathbb{N}$, $\lambda \in \mathbb{R}$, $\nabla = [\varrho_1, \varrho_2]$ an interval such that $-\infty \leq \varrho_1 < \varrho_2 \leq \infty$ and $\widehat{\aleph}, \psi \in C^{\jmath}([\varrho_1, \varrho_2], \mathbb{R})$ two functions such that ψ is increasing and $\psi'(\delta) \neq 0$, for all $\delta \in \nabla$. The tempered ψ-Hilfer fractional derivatives (left-sided and right-sided) ${}^{TH}\mathcal{D}_{\varrho_1+}^{\sigma,\varepsilon,\lambda;\psi}(\cdot)$ and ${}^{TH}\mathcal{D}_{\varrho_2-}^{\sigma,\varepsilon,\lambda;\psi}(\cdot)$ of a function $\widehat{\aleph}$ of order σ, index λ and type $0 \leq \varepsilon \leq 1$, are defined by*

$$\begin{aligned}{}^{TH}\mathcal{D}_{\varrho_1+}^{\sigma,\varepsilon,\lambda;\psi}\widehat{\aleph}\left(\delta\right) &= \left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\varepsilon(\jmath-\sigma);\psi}\left(\frac{1}{\psi'\left(\delta\right)}\frac{d}{d\delta}+\lambda\right)^{\jmath}\left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{(1-\varepsilon)(\jmath-\sigma);\psi}\widehat{\aleph}\right)\right)\left(\delta\right)\\ &= \left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\varepsilon(\jmath-\sigma);\psi}\mho_{\psi}^{\jmath}\left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{(1-\varepsilon)(\jmath-\sigma);\psi}\widehat{\aleph}\right)\right)\left(\delta\right)\\ &= e^{-\lambda\psi(\delta)}\times{}^{H}\mathcal{D}_{\varrho_1+}^{\sigma,\varepsilon;\psi}\left(\widehat{\aleph}(\delta)e^{\lambda\psi(\delta)}\right)\end{aligned}$$

and

$$\begin{aligned}{}^{TH}\mathcal{D}_{\varrho_2-}^{\sigma,\varepsilon,\lambda;\psi}\widehat{\aleph}\left(\delta\right) &= \left({}_{\lambda}^{T}\mathcal{J}_{\varrho_2-}^{\varepsilon(\jmath-\sigma);\psi}\left(-\frac{1}{\psi'\left(\delta\right)}\frac{d}{d\delta}+\lambda\right)^{\jmath}\left({}_{\lambda}^{T}\mathcal{J}_{\varrho_2-}^{(1-\varepsilon)(\jmath-\sigma);\psi}\widehat{\aleph}\right)\right)\left(\delta\right)\\ &= \left({}_{\lambda}^{T}\mathcal{J}_{\varrho_2-}^{\varepsilon(\jmath-\sigma);\psi}(-1)^{\jmath}\mho_{\psi}^{\jmath}\left({}_{\lambda}^{T}\mathcal{J}_{\varrho_2-}^{(1-\varepsilon)(\jmath-\sigma);\psi}\widehat{\aleph}\right)\right)\left(\delta\right)\\ &= e^{\lambda\psi(\delta)}\times{}^{H}\mathcal{D}_{\varrho_2-}^{\sigma,\varepsilon;\psi}\left(\widehat{\aleph}(\delta)e^{-\lambda\psi(\delta)}\right),\end{aligned}$$

where $\mho_{\psi}^{\jmath} = \left(\frac{1}{\psi'\left(\delta\right)}\frac{d}{d\delta}+\lambda\right)^{\jmath}$, ${}^{H}\mathcal{D}_{\varrho_1+}^{\sigma,\varepsilon;\psi}(\cdot)$ *and* ${}^{H}\mathcal{D}_{\varrho_2-}^{\sigma,\varepsilon;\psi}(\cdot)$ *are the left-sided and right-sided ψ-Hilfer fractional derivatives, defined in [44].*

By incorporating Definition 2.5 and Definition 2.6, we will now give the following definition of a more generalized fractional derivative that encompasses both tempered ψ-Hilfer derivative and κ-generalized ψ-Hilfer derivative as specific cases.

Definition 2.7 (The tempered (κ,ψ)-Hilfer Derivative [31])**.** *Let $\jmath - 1 < \frac{\sigma}{\kappa} \leq \jmath$ with $\jmath \in \mathbb{N}$, $\lambda \in \mathbb{R}$, $\kappa > 0$, $\nabla = [\varrho_1, \varrho_2]$ an interval such that $-\infty \leq \varrho_1 < \varrho_2 \leq \infty$ and $\widehat{\aleph}, \psi \in C^{\jmath}([\varrho_1, \varrho_2], \mathbb{R})$ two functions such that ψ is increasing and $\psi'(\delta) \neq 0$, for all $\delta \in \nabla$. The tempered (κ,ψ)-Hilfer fractional derivatives (left-sided and right-sided) ${}_{\kappa}^{TH}\mathcal{D}_{\varrho_1+}^{\sigma,\varepsilon,\lambda;\psi}(\cdot)$ and ${}_{\kappa}^{TH}\mathcal{D}_{\varrho_2-}^{\sigma,\varepsilon,\lambda;\psi}(\cdot)$ of a function $\widehat{\aleph}$ of order σ, index λ and type $0 \leq \varepsilon \leq 1$, are defined by*

$$\begin{aligned}{}_{\kappa}^{TH}\mathcal{D}_{\varrho_1+}^{\sigma,\varepsilon,\lambda;\psi}\widehat{\aleph}\left(\delta\right) &= \left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\varepsilon(\kappa\jmath-\sigma),\kappa;\psi}\left(\frac{1}{\psi'\left(\delta\right)}\frac{d}{d\delta}+\lambda\right)^{\jmath}\left(\kappa^{\jmath}\,{}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{(1-\varepsilon)(\kappa\jmath-\sigma),\kappa;\psi}\widehat{\aleph}\right)\right)\left(\delta\right)\\ &= \left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\varepsilon(\kappa\jmath-\sigma),\kappa;\psi}\mho_{\psi}^{\jmath}\left(\kappa^{\jmath}\,{}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{(1-\varepsilon)(\kappa\jmath-\sigma),\kappa;\psi}\widehat{\aleph}\right)\right)\left(\delta\right)\\ &= e^{-\lambda\psi(\delta)}\times{}_{\kappa}^{H}\mathcal{D}_{\varrho_1+}^{\sigma,\varepsilon;\psi}\left(\widehat{\aleph}(\delta)e^{\lambda\psi(\delta)}\right)\end{aligned}$$

and

$${}_{\kappa}^{TH}\mathcal{D}_{\varrho_2-}^{\sigma,\varepsilon,\lambda;\psi}\widehat{\aleph}\left(\delta\right) = \left({}_{\lambda}^{T}\mathcal{J}_{\varrho_2-}^{\varepsilon(\kappa\jmath-\sigma),\kappa;\psi}\left(-\frac{1}{\psi'\left(\delta\right)}\frac{d}{d\delta}+\lambda\right)^{\jmath}\left(\kappa^{\jmath}\,{}_{\lambda}^{T}\mathcal{J}_{\varrho_2-}^{(1-\varepsilon)(\kappa\jmath-\sigma),\kappa;\psi}\widehat{\aleph}\right)\right)\left(\delta\right)$$

$$= \left({}^{T}_{\lambda}\mathcal{J}^{\varepsilon(\kappa\jmath-\sigma),\kappa;\psi}_{\varrho_2-} (-1)^{\jmath} \mho^{\jmath}_{\psi} \left(\kappa^{\jmath}\, {}^{T}_{\lambda}\mathcal{J}^{(1-\varepsilon)(\kappa\jmath-\sigma),\kappa;\psi}_{\varrho_2-} \widehat{\aleph} \right) \right)(\delta)$$
$$= e^{\lambda\psi(\delta)} \times {}^{H}_{\kappa}\mathcal{D}^{\sigma,\varepsilon;\psi}_{\varrho_2-} \left(\widehat{\aleph}(\delta) e^{-\lambda\psi(\delta)} \right),$$

where $\mho^{\jmath}_{\psi} = \left(\frac{1}{\psi'(\delta)} \frac{d}{d\delta} + \lambda \right)^{\jmath}$. *The tempered* (κ, ψ)*-Hilfer fractional derivative* ${}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}$ *reduces to the tempered* ψ*-Hilfer fractional derivative* ${}^{TH}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}$ *if* $\kappa = 1$.

2.3. Necessary properties of fractional operators.

Theorem 2.8 ([31]). *Let* $\widehat{\aleph} : [\varrho_1, \varrho_2] \to \mathbb{R}$ *be an integrable function, and take* $\sigma > 0$, $\lambda \in \mathbb{R}$ *and* $\kappa > 0$. *Then* ${}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\widehat{\aleph}$ *exists for all* $\delta \in [\varrho_1, \varrho_2]$.

Theorem 2.9 ([31]). *Let* $\widehat{\aleph} \in X^p_{\psi}(\varrho_1, \varrho_2)$ *and take* $\sigma > 0$, $\lambda \in \mathbb{R}$ *and* $\kappa > 0$. *Then* ${}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\widehat{\aleph} \in C([\varrho_1, \varrho_2], \mathbb{R})$.

Lemma 2.10 ([31]). *Let* $\sigma > 0$, $\varepsilon > 0$, $\lambda \in \mathbb{R}$ *and* $\kappa > 0$. *Then, we have the following semigroup property given by*

$${}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\ {}^{T}_{\lambda}\mathcal{J}^{\varepsilon,\kappa;\psi}_{\varrho_1+}\aleph(\delta) = {}^{T}_{\lambda}\mathcal{J}^{\sigma+\varepsilon,\kappa;\psi}_{\varrho_1+}\aleph(\delta) = {}^{T}_{\lambda}\mathcal{J}^{\varepsilon,\kappa;\psi}_{\varrho_1+}\ {}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\aleph(\delta)$$

and

$${}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_2-}\ {}^{T}_{\lambda}\mathcal{J}^{\varepsilon,\kappa;\psi}_{\varrho_2-}\aleph(\delta) = {}^{T}_{\lambda}\mathcal{J}^{\sigma+\varepsilon,\kappa;\psi}_{\varrho_2-}\aleph(\delta) = {}^{T}_{\lambda}\mathcal{J}^{\varepsilon,\kappa;\psi}_{\varrho_2-}\ {}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_2-}\aleph(\delta).$$

Lemma 2.11 ([38]). *Let* $\sigma, \varepsilon > 0$, *and* $\kappa > 0$. *Then, we have*

$$\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\bar{\Psi}^{\kappa,\psi}_{\varepsilon}(\delta, \varrho_1) = \bar{\Psi}^{\kappa,\psi}_{\sigma+\varepsilon}(\delta, \varrho_1)$$

and

$$\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_2-}\bar{\Psi}^{\kappa,\psi}_{\varepsilon}(\varrho_2, \delta) = \bar{\Psi}^{\kappa,\psi}_{\sigma+\varepsilon}(\varrho_2, \delta).$$

Lemma 2.12 ([31]). *Let* $\sigma, \varepsilon > 0$, $\lambda \in \mathbb{R}$ *and* $\kappa > 0$. *Then, we have*

$${}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+} e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}\bar{\Psi}^{\kappa,\psi}_{\varepsilon}(\delta, \varrho_1) = e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}\ \bar{\Psi}^{\kappa,\psi}_{\sigma+\varepsilon}(\delta, \varrho_1)$$

and

$${}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_2-} e^{\lambda(\psi(\delta)-\psi(\varrho_1))}\bar{\Psi}^{\kappa,\psi}_{\varepsilon}(\varrho_2, \delta) = e^{\lambda(\psi(\delta)-\psi(\varrho_1))}\ \bar{\Psi}^{\kappa,\psi}_{\sigma+\varepsilon}(\varrho_2, \delta).$$

Theorem 2.13 ([31]). *Let* $0 < \varrho_1 < \varrho_2 < \infty, \sigma > 0, 0 \le \theta < 1$, $\lambda \in \mathbb{R}$, $\kappa > 0$ *and* $\mathfrak{w} \in C_{\theta;\psi}(\nabla)$. *If* $\frac{\sigma}{\kappa} > 1 - \theta$, *then*

$$\left({}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\mathfrak{w} \right)(\varrho_1) = \lim_{\delta \to \varrho_1+} \left({}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\mathfrak{w} \right)(\delta) = 0.$$

Lemma 2.14 ([31]). *Let* $\delta > \varrho_1$, $\sigma > 0, 0 \le \varepsilon \le 1, \lambda \in \mathbb{R}, \kappa > 0$. *Then for* $0 < \theta < 1; \theta = \frac{1}{\kappa}(\varepsilon(\kappa - \sigma) + \sigma)$, *we have*

$$\left[{}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+} \left(\Psi^{\psi}_{\theta}(\gamma, \varrho_1) \right)^{-1} e^{-\lambda(\psi(\gamma)-\psi(\varrho_1))} \right](\delta) = 0.$$

Theorem 2.15 ([31]). *If* $\aleph \in C^{\jmath}_{\theta;\psi}[\varrho_1, \varrho_2], \jmath - 1 < \frac{\sigma}{\kappa} < \jmath$, $0 \le \varepsilon \le 1$, $\lambda \in \mathbb{R}$, *where* $\jmath \in \mathbb{N}$ *and* $\kappa > 0$, *then*

$$\left({}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\ {}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}\aleph \right)(\delta)$$

$$= \aleph(\delta) - e^{-\lambda\psi(\delta)} \sum_{\beta=1}^{\jmath} \frac{(\psi(\delta)-\psi(\varrho_1))^{\theta-\beta}}{\kappa^{\beta-\jmath}\Gamma_\kappa(\kappa(\theta-\beta+1))} \left\{ \delta_\psi^{\jmath-\beta} \left(\mathcal{J}_{\varrho_1+}^{(1-\varepsilon)(\kappa\jmath-\sigma),\kappa;\psi} \aleph(\varrho_1) e^{\lambda\psi(\varrho_1)} \right) \right\},$$

where

$$\theta = \frac{1}{\kappa}\left(\varepsilon(\kappa\jmath - \sigma) + \sigma\right).$$

In particular, if $\jmath = 1$, we have

$$\left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}\ {}_{\kappa}^{TH}\mathcal{D}_{\varrho_1+}^{\sigma,\varepsilon,\lambda;\psi} \aleph \right)(\delta) = \aleph(\delta) - e^{-\lambda(\psi(\delta)-\psi(\varrho_1))} \frac{(\psi(\delta)-\psi(\varrho_1))^{\theta-1}}{\Gamma_\kappa(\varepsilon(\kappa-\sigma)+\sigma)}\ {}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\kappa(1-\theta),\kappa;\psi} \aleph(\varrho_1).$$

Lemma 2.16 ([31]). *Let $\sigma > 0, 0 \le \varepsilon \le 1$, $\lambda \in \mathbb{R}$, and $\mathfrak{w} \in C^1_{\theta;\psi}(\nabla)$, where $\kappa > 0$, then for $\delta \in (\varrho_1, \varrho_2]$, we have*

$$\left({}_{\kappa}^{TH}\mathcal{D}_{\varrho_1+}^{\sigma,\varepsilon,\lambda;\psi}\ {}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi} \mathfrak{w} \right)(\delta) = \mathfrak{w}(\delta).$$

Lemma 2.17 ([31]). *Let $\sigma > 0, 0 \le \varepsilon \le 1$, $\lambda \in \mathbb{R}$, and $\mathfrak{w} \in C^1_{\theta;\psi}(\nabla)$, where $\kappa > 0$, then for $\delta \in (\varrho_1, \varrho_2]$, we have*

$$\left({}_{\kappa}^{TH}\mathcal{D}_{\varrho_1+}^{\sigma,\varepsilon,\lambda;\psi}\ {}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi} \mathfrak{w} \right)(\delta) = \mathfrak{w}(\delta).$$

Lemma 2.18 ([39]). *Let $\sigma, \kappa > 0$ and $\lambda \in \mathbb{R}$. Then, we have*

$${}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi} e^{-\lambda(\psi(\delta)-\psi(\varrho_1))} \mathbb{E}_\kappa^\sigma \left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}} \right) = e^{-\lambda(\psi(\delta)-\psi(\varrho_1))} \left[\mathbb{E}_\kappa^\sigma \left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}} \right) - 1 \right],$$

and

$${}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi} \mathbb{E}_\kappa^\sigma \left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}} \right) \le \widehat{\lambda} \left[\mathbb{E}_\kappa^\sigma \left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}} \right) - 1 \right].$$

Now, we present a generalized Gronwall inequality that will play a crucial role in our Ulam stability results.

Theorem 2.19 ([39]). *Let $\mathfrak{w}$, $\mathfrak{q}$ be two integrable functions and $\mathfrak{p}$ continuous, with domain $[\varrho_1, \varrho_2]$. Let $\psi \in C^1[\varrho_1, \varrho_2]$ an increasing function such that $\psi'(\delta) \neq 0$, $\delta \in [\varrho_1, \varrho_2]$, $\sigma > 0$, $\kappa > 0$ and $\lambda \in \mathbb{R}$. Assume that:*

(1) *$\mathfrak{w}$ and $\mathfrak{q}$ are nonnegative;*
(2) *$\mathfrak{p}$ is nonnegative and nondecreasing.*

If

$$\mathfrak{w}(\delta) \le \mathfrak{q}(\delta) + \mathfrak{p}(\delta)\Gamma_\kappa(\sigma) \int_{\varrho_1}^{\delta} \psi'(\gamma) e^{-\lambda(\psi(\delta)-\psi(\gamma))} \bar{\Psi}_\sigma^{\kappa,\psi}(\delta,\gamma) \mathfrak{w}(\gamma)\, d\gamma,$$

then

$$\mathfrak{w}(\delta) \le \mathfrak{q}(\delta) + \int_{\varrho_1}^{\delta} \sum_{\beta=1}^{\infty} \left[\widehat{\lambda}\mathfrak{p}(\delta)\Gamma_\kappa(\sigma) \right]^\beta \psi'(\gamma) \bar{\Psi}_{\beta\sigma}^{\kappa,\psi}(\delta,\gamma) \mathfrak{q}(\gamma)\, d\gamma, \tag{3}$$

for all $\delta \in [\varrho_1, \varrho_2]$, where

$$\widehat{\lambda} := \max_{(\delta,\gamma)\in[\varrho_1,\varrho_2]\times[\varrho_1,\delta]} e^{-\lambda(\psi(\delta)-\psi(\gamma))} = \begin{cases} 1, & \text{if } \lambda \ge 0, \\ e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}, & \text{if } \lambda < 0. \end{cases}$$

And if $\mathfrak{q}$ is a nondecreasing function on $[\varrho_1, \varrho_2]$, then we have

$$\mathfrak{w}(\delta) \le \mathfrak{q}(\delta)\, \mathbb{E}_\kappa^\sigma \left(\widehat{\lambda}\mathfrak{p}(\delta)\Gamma_\kappa(\sigma)(\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}} \right).$$

3. Existence of Solutions

We consider the following fractional differential equation

$$\left({}_{\kappa}^{TH}\mathcal{D}_{\varrho_1+}^{\sigma,\varepsilon,\lambda;\psi}\mathfrak{w}\right)(\delta)=\xi(\delta),\quad \delta\in(\varrho_1,\varrho_2], \tag{4}$$

where $0<\sigma<\kappa, 0\leq\varepsilon\leq 1, \lambda\in\mathbb{R}$ with the condition

$$\mathfrak{w}(\varrho_2)=\varkappa, \tag{5}$$

where $\theta=\frac{\varepsilon(\kappa-\sigma)+\sigma}{\kappa}$, $\kappa>0$, $\xi(\cdot)\in C(\nabla,\mathbb{R})$, $\varkappa\in\mathbb{R}$.

The following theorem shows that the problem (4)-(5) have a unique solution.

Theorem 3.1. *Let $0<\sigma<\kappa, 0\leq\varepsilon\leq 1, \lambda\in\mathbb{R}$, $\kappa>0$, $\xi(\cdot)\in C(\nabla,\mathbb{R})$. The problem (4)-(5) has a unique solution given by:*

$$\mathfrak{w}(\delta)=\frac{e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}}{\Psi_\theta^\psi(\delta,\varrho_1)}\left[\Psi_\theta^\psi(\varrho_2,\varrho_1)e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\left(\varkappa-\left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}\xi\right)(\varrho_2)\right)\right]+\left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}\xi\right)(\delta). \tag{6}$$

Proof. Assume $\mathfrak{w}$ satisfies (4)-(5). By applying the fractional integral operator ${}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}(\cdot)$ on both sides of the fractional equation (4) and using Theorem 2.15, we obtain

$$\mathfrak{w}(\delta)=\frac{{}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\kappa(1-\theta),\kappa;\psi}\mathfrak{w}(\varrho_1)}{\Psi_\theta^\psi(\delta,\varrho_1)\Gamma_\kappa(\kappa\theta)}e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}+\left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}\xi\right)(\delta). \tag{7}$$

Using condition (5), we obtain

$$\varkappa=\frac{{}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\kappa(1-\theta),\kappa;\psi}\mathfrak{w}(\varrho_1)}{\Psi_\theta^\psi(\varrho_2,\varrho_1)\Gamma_\kappa(\kappa\theta)}e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}+\left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}\xi\right)(\varrho_2).$$

Thus,

$$\mathfrak{w}(\delta)=\frac{\Psi_\theta^\psi(\varrho_2,\varrho_1)e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\left(\varkappa-\left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}\xi\right)(\varrho_2)\right)}{\Psi_\theta^\psi(\delta,\varrho_1)}e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}+\left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}\xi\right)(\delta).$$

Reciprocally, apply operator ${}_{\kappa}^{TH}\mathcal{D}_{\varrho_1+}^{\sigma,\varepsilon,\lambda;\psi}(\cdot)$ on both sides of (6). Then, from Lemma 2.14 and Lemma 2.17 we obtain equation (4). This completes the proof. □

As a consequence of Theorem 3.1, we have the following result.

Lemma 3.2. *Let $\theta=\dfrac{\varepsilon(\kappa-\sigma)+\sigma}{\kappa}$ where $0<\sigma<\kappa$ and $0\leq\varepsilon\leq 1, \lambda\in\mathbb{R}$, let $\aleph:\nabla\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Then, the problem (1)-(2) is equivalent to the following integral equation:*

$$\mathfrak{w}(\delta)=\frac{e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}}{\Psi_\theta^\psi(\delta,\varrho_1)}\left[\Psi_\theta^\psi(\varrho_2,\varrho_1)e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\left(\varkappa-\left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}\xi\right)(\varrho_2)\right)\right]+\left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}\xi\right)(\delta), \tag{8}$$

where ξ be a function satisfying the functional equation

$$\xi(\delta)=\aleph(\delta,\mathfrak{w}(\delta),\xi(\delta)).$$

The following hypotheses will be used in the sequel :

(*Ax1*) The function $\aleph:\nabla\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous.

$(Ax2)$ There exist constants $\zeta_1 > 0$ and $0 < \zeta_2 < 1$ such that

$$|\aleph(\delta, \mathfrak{w}_1, \mathfrak{q}_1) - \aleph(\delta, \mathfrak{w}_2, \mathfrak{q}_2)| \leq \zeta_1|\mathfrak{w}_1 - \mathfrak{w}_2| + \zeta_2|\mathfrak{q}_1 - \mathfrak{q}_2|$$

for any $\mathfrak{w}_1, \mathfrak{w}_2, \mathfrak{q}_1, \mathfrak{q}_2 \in \mathbb{R}$ and $\delta \in \nabla$.

We are now in a position to state and prove our existence result for the problem (1)-(2) based on based on Banach's fixed point theorem [11].

Theorem 3.3. *Assume* (Ax1)-(Ax2) *hold. If*

$$\mathcal{L} = \frac{\zeta_1\widehat{\lambda}\Gamma_\kappa(\kappa\theta)\left(\psi(\varrho_2) - \psi(\varrho_1)\right)^{\frac{\sigma}{\kappa}}}{\Gamma_\kappa(\sigma + \kappa\theta)(1 - \zeta_2)}\left[e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\widehat{\lambda} + 1\right] < 1, \tag{9}$$

then the problem (1)-(2) *has a unique solution in* $C_{\theta;\psi}(\nabla)$.

Proof. Transform problem (1)-(2) into a fixed point problem by considering the operator $\Bbbk : C_{\theta;\psi}(\nabla) \to C_{\theta;\psi}(\nabla)$ by

(10)
$$(\Bbbk\mathfrak{w})(\delta) = \frac{e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}}{\Psi_\theta^\psi(\delta, \varrho_1)}\left[\Psi_\theta^\psi(\varrho_2, \varrho_1)e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\left(\varkappa - \left({}^T_\lambda\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\xi\right)(\varrho_2)\right)\right] + \left({}^T_\lambda\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\xi\right)(\delta),$$

where ξ be a function satisfying the functional equation

$$\xi(\delta) = \aleph(\delta, \mathfrak{w}(\delta), \xi(\delta)).$$

By Theorem 2.9, we have $\Bbbk\mathfrak{w} \in C_{\theta;\psi}(\nabla)$. We show that the operator $\Bbbk$ has a unique fixed point in $C_{\theta;\psi}(\nabla)$.

Let $\mathfrak{w}, \mathfrak{q} \in C_{\theta;\psi}(\nabla)$. Then for any for $\delta \in \nabla$, we have

$$\begin{aligned}|\Bbbk\mathfrak{w}(\delta) - \Bbbk\mathfrak{q}(\delta)| \leq{}& \frac{e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}}{\Psi_\theta^\psi(\delta, \varrho_1)}\left[\Psi_\theta^\psi(\varrho_2, \varrho_1)e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\left({}^T_\lambda\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}|\xi_1(\gamma) - \xi_2(\gamma)|\right)(\varrho_2)\right] \\ &+ \left({}^T_\lambda\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}|\xi_1(\gamma) - \xi_2(\gamma)|\right)(\delta),\end{aligned}$$

where ξ_1 and ξ_1 be functions satisfying the functional equations

$$\xi_1(\delta) = \aleph(\delta, \mathfrak{w}(\delta), \xi_1(\delta)),$$
$$\xi_2(\delta) = \aleph(\delta, \mathfrak{q}(\delta), \xi_2(\delta)).$$

By $(Ax2)$, we have

$$\begin{aligned}|\xi_1(\delta) - \xi_2(\delta)| &= |\aleph(\delta, \mathfrak{w}(\delta), \xi_1(\delta)) - \aleph(\delta, \mathfrak{q}(\delta), \xi_2(\delta))| \\ &\leq \zeta_1|\mathfrak{w}(\delta) - \mathfrak{q}(\delta)| + \zeta_2|\xi_1(\delta) - \xi_2(\delta)|.\end{aligned}$$

Then,

$$|\xi_1(\delta) - \xi_2(\delta)| \leq \frac{\zeta_1}{1 - \zeta_2}|\mathfrak{w}(\delta) - \mathfrak{q}(\delta)|.$$

Therefore, for each $\delta \in \nabla$ we get

$$\begin{aligned}|\Bbbk\mathfrak{w}(\delta) - \Bbbk\mathfrak{q}(\delta)| \leq{}& \frac{e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}}{\Psi_\theta^\psi(\delta, \varrho_1)}\left[\frac{\zeta_1\Psi_\theta^\psi(\varrho_2, \varrho_1)e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\left({}^T_\lambda\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}|\mathfrak{w}(\gamma) - \mathfrak{q}(\gamma)|\right)(\varrho_2)}{1 - \zeta_2}\right] \\ &+ \frac{\zeta_1}{1 - \zeta_2}\left({}^T_\lambda\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}|\mathfrak{w}(\gamma) - \mathfrak{q}(\gamma)|\right)(\delta).\end{aligned}$$

Thus,

$$
\begin{aligned}
&|\Bbbk\mathfrak{w}(\delta)-\Bbbk\mathfrak{q}(\delta)| \\
&\leq \Bigg[\frac{e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}\zeta_1\Psi_\theta^\psi(\varrho_2,\varrho_1)e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\left({}^T_\lambda\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\left(\psi(\gamma)-\psi(\varrho_1)\right)^{\theta-1}\right)(\varrho_2)}{\Psi_\theta^\psi(\delta,\varrho_1)(1-\zeta_2)} \\
&\quad+\frac{\zeta_1}{1-\zeta_2}\left({}^T_\lambda\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1^+}\left(\psi(\gamma)-\psi(\varrho_1)\right)^{\theta-1}\right)(\delta)\Bigg]\|\mathfrak{w}-\mathfrak{q}\|_{C_{\theta;\psi}} \\
&\leq \Bigg[\frac{e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}\zeta_1\Psi_\theta^\psi(\varrho_2,\varrho_1)e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\left(\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}e^{-\lambda(\psi(\delta)-\psi(\gamma))}\left(\psi(\gamma)-\psi(\varrho_1)\right)^{\theta-1}\right)(\varrho_2)}{\Psi_\theta^\psi(\delta,\varrho_1)(1-\zeta_2)} \\
&\quad+\frac{\zeta_1}{1-\zeta_2}\left(\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1^+}e^{-\lambda(\psi(\delta)-\psi(\gamma))}\left(\psi(\gamma)-\psi(\varrho_1)\right)^{\theta-1}\right)(\delta)\Bigg]\|\mathfrak{w}-\mathfrak{q}\|_{C_{\theta;\psi}}.
\end{aligned}
$$

By Lemma 2.11, we have

$$
\begin{aligned}
|\Bbbk\mathfrak{w}(\delta)-\Bbbk\mathfrak{q}(\delta)| &\leq \Bigg[\frac{e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}\zeta_1\Psi_\theta^\psi(\varrho_2,\varrho_1)e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\widehat{\lambda}\left(\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\left(\psi(\gamma)-\psi(\varrho_1)\right)^{\theta-1}\right)(\varrho_2)}{\Psi_\theta^\psi(\delta,\varrho_1)(1-\zeta_2)} \\
&\quad+\frac{\zeta_1\widehat{\lambda}}{1-\zeta_2}\left(\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1^+}\left(\psi(\gamma)-\psi(\varrho_1)\right)^{\theta-1}\right)(\delta)\Bigg]\|\mathfrak{w}-\mathfrak{q}\|_{C_{\theta;\psi}} \\
&\leq \Bigg[\frac{\zeta_1\Psi_\theta^\psi(\varrho_2,\varrho_1)e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\widehat{\lambda}^2\Gamma_\kappa(\kappa\theta)\left(\psi(\varrho_2)-\psi(\varrho_1)\right)^{\frac{\sigma+\kappa\theta}{\kappa}-1}}{\Psi_\theta^\psi(\delta,\varrho_1)\Gamma_\kappa(\sigma+\kappa\theta)(1-\zeta_2)} \\
&\quad+\frac{\zeta_1\widehat{\lambda}\Gamma_\kappa(\kappa\theta)\left(\psi(\delta)-\psi(\varrho_1)\right)^{\frac{\sigma+\kappa\theta}{\kappa}-1}}{\Gamma_\kappa(\sigma+\kappa\theta)(1-\zeta_2)}\Bigg]\|\mathfrak{w}-\mathfrak{q}\|_{C_{\theta;\psi}}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\left|\Psi_\theta^\psi(\delta,\varrho_1)\left(\Bbbk\mathfrak{w}(\delta)-\Bbbk\mathfrak{q}(\delta)\right)\right| &\leq \Bigg[\frac{\zeta_1 e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\widehat{\lambda}^2\Gamma_\kappa(\kappa\theta)\left(\psi(\varrho_2)-\psi(\varrho_1)\right)^{\frac{\sigma}{\kappa}}}{\Gamma_\kappa(\sigma+\kappa\theta)(1-\zeta_2)} \\
&\quad+\frac{\zeta_1\widehat{\lambda}\Gamma_\kappa(\kappa\theta)\left(\psi(\delta)-\psi(\varrho_1)\right)^{\frac{\sigma}{\kappa}}}{\Gamma_\kappa(\sigma+\kappa\theta)(1-\zeta_2)}\Bigg]\|\mathfrak{w}-\mathfrak{q}\|_{C_{\theta;\psi}} \\
&\leq \frac{\zeta_1\widehat{\lambda}\Gamma_\kappa(\kappa\theta)\left(\psi(\varrho_2)-\psi(\varrho_1)\right)^{\frac{\sigma}{\kappa}}}{\Gamma_\kappa(\sigma+\kappa\theta)(1-\zeta_2)}\|\mathfrak{w}-\mathfrak{q}\|_{C_{\theta;\psi}} \\
&\quad\times\left[e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\widehat{\lambda}+1\right].
\end{aligned}
$$

Thus,

$$
\|\Bbbk\mathfrak{w}-\Bbbk\mathfrak{q}\|_{C_{\theta;\psi}} \leq \mathcal{L}\|\mathfrak{w}-\mathfrak{q}\|_{C_{\theta;\psi}}.
$$

By (9), the operator $\Bbbk$ is a contraction on $C_{\theta;\psi}(\nabla)$. Hence, by Banach's contraction principle, $\Bbbk$ has a unique fixed point $\mathfrak{w}\in C_{\theta;\psi}(\nabla)$, which is a solution to our problem (1)-(2). □

Our next existence result for the problem (1)-(2) is based on based on Schauder's fixed point theorem [11].

The following hypothesis will be used in the sequel :

($Ax3$) There exist functions $\varpi_2, \varpi_3 \in C(\nabla, \mathbb{R}_+)$ with

$$\varpi_2^* = \sup_{\delta\in\nabla} \varpi_2(\delta), \ \ \varpi_3^* = \sup_{\delta\in\nabla} \varpi_3(\delta) < 1,$$

such that

$$|\aleph(\delta, \mathfrak{w}_1, \mathfrak{q}_1) - \aleph(\delta, \mathfrak{w}_2, \mathfrak{q}_2)| \leq \varpi_2(\delta)\Psi_\theta^\psi(\delta, \varrho_1)|\mathfrak{w}_1 - \mathfrak{w}_2| + \varpi_3(\delta)|\mathfrak{q}_1 - \mathfrak{q}_2|$$

for any $\mathfrak{w}_1, \mathfrak{w}_2, \mathfrak{q}_1, \mathfrak{q}_2 \in \mathbb{R}$ and $\delta \in \nabla$.

Theorem 3.4. *Assume* (Ax1) *and* (Ax3) *hold. If*

$$\ell = \frac{\varpi_2^*\widehat{\lambda}\left(\psi(\varrho_2) - \psi(\varrho_1)\right)^{1-\theta+\frac{\sigma}{\kappa}}}{(1-\varpi_3^*)\Gamma_\kappa(\sigma+\kappa)}\left(1 + \widehat{\lambda}e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\right) < 1, \tag{11}$$

then the problem (1)-(2) *has at least one solution in* $C_{\theta;\psi}(\nabla)$.

Proof. In several steps, we will use Schauder's fixed point theorem to prove that the operator $\Bbbk$ defined in (10) has a fixed point.

Step 1: The operator $\Bbbk$ is continuous.
Let $\{\mathfrak{w}_\jmath\}$ be a sequence such that $\mathfrak{w}_\jmath \longrightarrow \mathfrak{w}$ in $C_{\theta;\psi}(\nabla)$. For each $\delta \in (\varrho_1, \varrho_2]$, we have

$$\begin{aligned} |\Bbbk\mathfrak{w}_\jmath(\delta) - \Bbbk\mathfrak{w}(\delta)| \leq \frac{e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}}{\Psi_\theta^\psi(\delta,\varrho_1)} &\left[\Psi_\theta^\psi(\varrho_2,\varrho_1)e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\left({}_\lambda^T\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}|\xi(\gamma) - \xi_\jmath(\gamma)|\right)(\varrho_2)\right] \\ &+ \left({}_\lambda^T\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}|\xi(\gamma) - \xi_\jmath(\gamma)|\right)(\delta), \end{aligned}$$

where ξ and $\xi_\jmath$ be functions satisfying the functional equations

$$\begin{aligned} \xi(\delta) &= \aleph(\delta, \mathfrak{w}(\cdot), \xi(\delta)), \\ \xi_\jmath(\delta) &= \aleph(\delta, \mathfrak{w}_\jmath(\cdot), \xi_\jmath(\delta)). \end{aligned}$$

Since $\mathfrak{w}_\jmath \to \mathfrak{w}$, then we get $\xi_\jmath(\delta) \to \xi(\delta)$ as $\jmath \to \infty$ for each $\delta \in (\varrho_1, \varrho_2]$, and since $\aleph$ is continuous, then we have

$$\|\Bbbk\mathfrak{w}_\jmath - \Bbbk\mathfrak{w}\|_{C_{\theta;\psi}} \to 0 \text{ as } \jmath \to \infty.$$

Step 2: $\Bbbk(B_M) \subset B_M$.
Let M a positive constant such that

$$M \geq \frac{\widehat{\lambda}e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\Psi_\theta^\psi(\varrho_2,\varrho_1)|\varkappa| + \dfrac{\varpi_1^*\widehat{\lambda}\left(\psi(\varrho_2)-\psi(\varrho_1)\right)^{1-\theta+\frac{\sigma}{\kappa}}}{(1-\varpi_3^*)\Gamma_\kappa(\sigma+\kappa)}\left[1 + \widehat{\lambda}e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\right]}{\left[1 - \dfrac{\varpi_2^*\widehat{\lambda}\left(\psi(\varrho_2)-\psi(\varrho_1)\right)^{1-\theta+\frac{\sigma}{\kappa}}}{(1-\varpi_3^*)\Gamma_\kappa(\sigma+\kappa)}\left(1 + \widehat{\lambda}e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\right)\right]}.$$

We define the following bounded closed set

$$B_M = \left\{\mathfrak{w} \in C_{\theta;\psi}(\nabla) : \|\mathfrak{w}\|_{C_{\theta;\psi}} \leq M\right\}.$$

For each $\delta \in (\varrho_1, \varrho_2]$, (10) implies that

$$\begin{aligned} |\Bbbk\mathfrak{w}(\delta)| \leq \frac{e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}}{\Psi_\theta^\psi(\delta,\varrho_1)} &\left[\Psi_\theta^\psi(\varrho_2,\varrho_1)e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\left(|\varkappa| + \left({}_\lambda^T\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}|\xi(\gamma)|\right)(\varrho_2)\right)\right] \\ &+ \left({}_\lambda^T\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}|\xi(\gamma)|\right)(\delta). \end{aligned} \tag{12}$$

By the hypothesis (*Ax3*), for $\delta \in (\varrho_1, \varrho_2]$, we have

$$\begin{aligned}|\xi(\delta)| &= |\aleph(\delta, \mathfrak{w}(\delta), \xi(\delta)) - \aleph(\delta, 0, 0)| + |\aleph(\delta, 0, 0)| \\ &\leq \varpi_1(\delta) + \varpi_2(\delta)\Psi_\theta^\psi(\delta, \varrho_1)|\mathfrak{w}(\delta)| + \varpi_3(\delta)|\xi(\delta)|,\end{aligned}$$

where $\varpi_1(\delta) = |\aleph(\delta, 0, 0)|$ and $\varpi_1^* = \sup_{\delta\in\nabla} \varpi_1(\delta)$, which implies that

$$|\xi(\delta)| \leq \varpi_1^* + \varpi_2^* M + \varpi_3^* |\xi(\delta)|,$$

then

$$|\xi(\delta)| \leq \frac{\varpi_1^* + \varpi_2^* M}{1 - \varpi_3^*} := \Delta.$$

Thus for $\delta \in (\varrho_1, \varrho_2]$, from (12) we get

$$\begin{aligned}|\Psi_\theta^\psi(\delta, \varrho_1)\Bbbk\mathfrak{w}(\delta)| &\leq e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}\left[\Psi_\theta^\psi(\varrho_2, \varrho_1)e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\left(|\varkappa| + \Delta\left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}(1)\right)(\varrho_2)\right)\right] \\ &\quad + \Delta\Psi_\theta^\psi(\delta, \varrho_1)\left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}(1)\right)(\delta).\end{aligned}$$

Thus,

$$\begin{aligned}&|\Psi_\theta^\psi(\delta, \varrho_1)\Bbbk\mathfrak{w}(\delta)| \\ &\leq e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}\left[\Psi_\theta^\psi(\varrho_2, \varrho_1)e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\left(|\varkappa| + \Delta\left(\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}e^{-\lambda(\psi(\delta)-\psi(\gamma))}\right)(\varrho_2)\right)\right] \\ &\quad + \Delta\Psi_\theta^\psi(\delta, \varrho_1)\left(\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}e^{-\lambda(\psi(\delta)-\psi(\gamma))}\right)(\delta).\end{aligned}$$

By Lemma 2.11, we have

$$\begin{aligned}|\Psi_\theta^\psi(\delta, \varrho_1)\Bbbk\mathfrak{w}(\delta)| &\leq \widehat{\lambda}\left[e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\left(\Psi_\theta^\psi(\varrho_2, \varrho_1)|\varkappa| + \frac{\Delta\widehat{\lambda}\left(\psi(\varrho_2) - \psi(\varrho_1)\right)^{1-\theta+\frac{\sigma}{\kappa}}}{\Gamma_\kappa(\sigma+\kappa)}\right)\right] \\ &\quad + \frac{\Delta\widehat{\lambda}\left(\psi(\delta) - \psi(\varrho_1)\right)^{1-\theta+\frac{\sigma}{\kappa}}}{\Gamma_\kappa(\sigma+\kappa)}.\end{aligned}$$

Thus,

$$\begin{aligned}|\Psi_\theta^\psi(\delta, \varrho_1)\Bbbk\mathfrak{w}(\delta)| &\leq \widehat{\lambda}e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\Psi_\theta^\psi(\varrho_2, \varrho_1)|\varkappa| \\ &\quad + \frac{\Delta\widehat{\lambda}\left(\psi(\varrho_2) - \psi(\varrho_1)\right)^{1-\theta+\frac{\sigma}{\kappa}}}{\Gamma_\kappa(\sigma+\kappa)}\left[1 + \widehat{\lambda}e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\right] \\ &\leq M.\end{aligned}$$

Then, for each $\delta \in (\varrho_1, \varrho_2]$ we obtain

$$\|\Bbbk\mathfrak{w}\|_{C_{\theta;\psi}} \leq M.$$

Step 3: $\Bbbk(B_M)$ is relatively compact.
Let $\vartheta_1, \vartheta_2 \in (\varrho_1, \varrho_2]$, $\vartheta_1 < \vartheta_2$ and let $\mathfrak{w} \in B_M$. Then

$$\begin{aligned}&\left|\Psi_\theta^\psi(\vartheta_1, \varrho_1)\Bbbk\mathfrak{w}(\vartheta_1) - \Psi_\theta^\psi(\vartheta_2, \varrho_1)\Bbbk\mathfrak{w}(\vartheta_2)\right| \\ &\leq \left(\frac{e^{-\lambda(\psi(\vartheta_1)-\psi(\varrho_1))}}{\Psi_\theta^\psi(\vartheta_1, \varrho_1)} - \frac{e^{-\lambda(\psi(\vartheta_2)-\psi(\varrho_1))}}{\Psi_\theta^\psi(\vartheta_2, \varrho_1)}\right) \\ &\quad \times \left[\Psi_\theta^\psi(\varrho_2, \varrho_1)e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\left(\varkappa - \left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}\xi\right)(\varrho_2)\right)\right]\end{aligned}$$

$$
\begin{aligned}
&+\left|\Psi_\theta^\psi(\vartheta_1,\varrho_1)\left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1^+}^{\sigma,\kappa;\psi}|\xi(\gamma)|\right)(\vartheta_1)-\Psi_\theta^\psi(\vartheta_2,\varrho_1)\left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1^+}^{\sigma,\kappa;\psi}|\xi(\gamma)|\right)(\vartheta_2)\right|\\
\le&\left(\frac{e^{-\lambda(\psi(\vartheta_1)-\psi(\varrho_1))}}{\Psi_\theta^\psi(\vartheta_1,\varrho_1)}-\frac{e^{-\lambda(\psi(\vartheta_2)-\psi(\varrho_1))}}{\Psi_\theta^\psi(\vartheta_2,\varrho_1)}\right)\\
&\times\left[\Psi_\theta^\psi(\varrho_2,\varrho_1)e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\left(\varkappa-\left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}\xi\right)(\varrho_2)\right)\right]\\
&+\int_{\varrho_1}^{\vartheta_1}\Big|\Psi_\theta^\psi(\vartheta_1,\varrho_1)\bar{\Psi}_\sigma^{\kappa,\psi}(\vartheta_1,\gamma)e^{-\lambda(\psi(\vartheta_1)-\psi(\gamma))}\\
&-\Psi_\theta^\psi(\vartheta_2,\varrho_1)\bar{\Psi}_\sigma^{\kappa,\psi}(\vartheta_2,\gamma)e^{-\lambda(\psi(\vartheta_2)-\psi(\gamma))}\Big||\psi'(\gamma)\xi(\gamma)|d\gamma\\
&+\left|\Psi_\theta^\psi(\vartheta_2,\varrho_1)\left({}_{\lambda}^{T}\mathcal{J}_{\vartheta_1^+}^{\sigma,\kappa;\psi}|\xi(\gamma)|\right)(\vartheta_2)\right|.
\end{aligned}
$$

By Lemma 2.12, we get

$$
\begin{aligned}
&\left|\Psi_\theta^\psi(\vartheta_1,\varrho_1)\Bbbk\mathfrak{w}(\vartheta_1)-\Psi_\theta^\psi(\vartheta_2,\varrho_1)\Bbbk\mathfrak{w}(\vartheta_2)\right|\\
&\le\left(\frac{e^{-\lambda(\psi(\vartheta_1)-\psi(\varrho_1))}}{\Psi_\theta^\psi(\vartheta_1,\varrho_1)}-\frac{e^{-\lambda(\psi(\vartheta_2)-\psi(\varrho_1))}}{\Psi_\theta^\psi(\vartheta_2,\varrho_1)}\right)\\
&\times\left[\Psi_\theta^\psi(\varrho_2,\varrho_1)e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\left(\varkappa-\left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}\xi\right)(\varrho_2)\right)\right]\\
&+\Delta\int_{\varrho_1}^{\vartheta_1}\Big|\Psi_\theta^\psi(\vartheta_1,\varrho_1)\bar{\Psi}_\sigma^{\kappa,\psi}(\vartheta_1,\gamma)e^{-\lambda(\psi(\vartheta_1)-\psi(\gamma))}\\
&-\Psi_\theta^\psi(\vartheta_2,\varrho_1)\bar{\Psi}_\sigma^{\kappa,\psi}(\vartheta_2,\gamma)e^{-\lambda(\psi(\vartheta_2)-\psi(\gamma))}\Big||\psi'(\gamma)|d\gamma\\
&+\frac{\Delta\widehat{\lambda}\Psi_\theta^\psi(\vartheta_2,\varrho_1)\left(\psi(\vartheta_2)-\psi(\vartheta_1)\right)^{\frac{\sigma}{\kappa}}}{\Gamma_\kappa(\sigma+\kappa)}.
\end{aligned}
$$

As $\vartheta_1\to\vartheta_2$, the right-hand side of the above inequality tends to zero. The equicontinuity for the other cases is obvious, thus we omit the details. From Step 1 to Step 3, along with the Arzela-Ascoli theorem, we conclude that $\Bbbk: C_{\theta;\psi}(\nabla)\to C_{\theta;\psi}(\nabla)$ continuous and compact. As a consequence of Schauder's fixed point theorem, we deduce that $\Bbbk$ has a fixed point which is a solution of the problem (1)-(2). □

4. κ-Mittag-Leffler-Ulam-Hyers stability

In this Section, we consider the κ-Mittag-Leffler-Ulam-Hyers stability for our problem (1)-(2). Let $\mathfrak{w}\in C_{\theta;\psi}(\nabla)$, $\epsilon>0$. We consider the following inequality :

$$
\left|\left({}_{\kappa}^{TH}\mathcal{D}_{\varrho_1+}^{\sigma,\varepsilon,\lambda;\psi}\mathfrak{w}\right)(\delta)-\aleph\left(\delta,\mathfrak{w}(\delta),\left({}_{\kappa}^{TH}\mathcal{D}_{\varrho_1+}^{\sigma,\varepsilon,\lambda;\psi}\mathfrak{w}\right)(\delta)\right)\right|\le\epsilon\mathbb{E}_\kappa^\sigma\left(\left(\psi(\delta)-\psi(\varrho_1)\right)^{\frac{\sigma}{\kappa}}\right). \tag{13}
$$

Definition 4.1 ([35]). *Problem (1)-(2) is said to be κ-Mittag-Leffler-Ulam-Hyers stable with respect to $\mathbb{E}_\kappa^\sigma\left(\left(\psi(\delta)-\psi(\varrho_1)\right)^{\frac{\sigma}{\kappa}}\right)$ if there exists a real number $a_{\mathbb{E}_\kappa^\sigma}>0$ such that for each $\epsilon>0$ and for each solution $\mathfrak{w}\in C_{\theta;\psi}(\nabla)$ of inequality (13) there exists a solution $\mathfrak{q}\in C_{\theta;\psi}(\nabla)$ of (1)-(2) with*

$$
|\mathfrak{w}(\delta)-\mathfrak{q}(\delta)|\le a_{\mathbb{E}_\kappa^\sigma}\epsilon\mathbb{E}_\kappa^\sigma\left(\left(\psi(\delta)-\psi(\varrho_1)\right)^{\frac{\sigma}{\kappa}}\right),\qquad\delta\in\nabla.
$$

Definition 4.2 ([35]). *Problem (1)-(2) is generalized κ-Mittag-Leffler-Ulam-Hyers stable with respect to $\mathbb{E}_\kappa^\sigma\left(\left(\psi(\delta)-\psi(\varrho_1)\right)^{\frac{\sigma}{\kappa}}\right)$ if there exists $v: C([0,\infty),[0,\infty))$ with $v(0)=0$ such that* *for each $\epsilon > 0$ and for each solution $\mathfrak{w} \in C_{\theta;\psi}(\nabla)$ of inequality (13) there exists a solution $\mathfrak{q} \in C_{\theta;\psi}(\nabla)$ of (1)-(2) with*

$$|\mathfrak{w}(\delta) - \mathfrak{q}(\delta)| \leq v(\epsilon)\mathbb{E}_{\kappa}^{\sigma}\left((\psi(\delta) - \psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right), \qquad \delta \in \nabla.$$

Remark 4.3. *Its clear that : Definition 4.1 $\Longrightarrow$ Definition 4.2.*

Remark 4.4. *A function $\mathfrak{w} \in C_{\theta;\psi}(\nabla)$ is a solution of inequality (13) if and only if there exist $\widetilde{\aleph} \in C_{\theta,\kappa;\psi}(\nabla)$ such that*

(1) $|\widetilde{\aleph}(\delta)| \leq \epsilon \mathbb{E}_{\kappa}^{\sigma}\left((\psi(\delta) - \psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)$, $\delta \in (\varrho_1, \varrho_2]$,

(2) $\left({}_{\kappa}^{TH}\mathcal{D}_{\varrho_1+}^{\sigma,\varepsilon,\lambda;\psi}\mathfrak{w}\right)(\delta) = \aleph\left(\delta, \mathfrak{w}(\delta), \left({}_{\kappa}^{TH}\mathcal{D}_{\varrho_1+}^{\sigma,\varepsilon,\lambda;\psi}\mathfrak{w}\right)(\delta)\right) + \widetilde{\aleph}(\delta)$, $\delta \in (\varrho_1, \varrho_2]$.

Theorem 4.5. *Assume that the hypotheses* (Ax1)-(Ax2) *and the condition (9) hold. Then,* (1)-(2) *is κ-Mittag-Leffler-Ulam-Hyers stable with respect to $\mathbb{E}_{\kappa}^{\sigma}\left((\psi(\delta) - \psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)$ and consequently generalized κ-Mittag-Leffler-Ulam-Hyers stable.*

Proof. Let $\mathfrak{w} \in C_{\theta;\psi}(\nabla)$ be a solution if inequality (13), and let us assume that $\mathfrak{q}$ is the unique solution of the problem

$$\begin{cases} \left({}_{\kappa}^{TH}\mathcal{D}_{\varrho_1+}^{\sigma,\varepsilon,\lambda;\psi}\mathfrak{q}\right)(\delta) = \aleph\left(\delta, \mathfrak{q}(\delta), \left({}_{\kappa}^{TH}\mathcal{D}_{\varrho_1+}^{\sigma,\varepsilon,\lambda;\psi}\mathfrak{q}\right)(\delta)\right); \ \delta \in (\varrho_1, \varrho_2], \\ \mathfrak{q}(\varrho_2) = \mathfrak{w}(\varrho_2) = \varkappa, \\ \left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\kappa(1-\theta),\kappa;\psi}\mathfrak{q}\right)(\varrho_1{}^{+}) = \left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\kappa(1-\theta),\kappa;\psi}\mathfrak{w}\right)(\varrho_1{}^{+}). \end{cases}$$

By Lemma 3.2, we obtain for each $\delta \in (\varrho_1, \varrho_2]$

$$\mathfrak{w}(\delta) = \frac{e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}}{\Psi_{\theta}^{\psi}(\delta, \varrho_1)}\left[\Psi_{\theta}^{\psi}(\varrho_2, \varrho_1)e^{\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\left(\varkappa - \left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}\widehat{\xi}\right)(\varrho_2)\right)\right] + \left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}\widehat{\xi}\right)(\delta),$$

where $\widehat{\xi} \in C_{\theta;\psi}(\nabla)$, be a function satisfying the functional equation

$$\widehat{\xi}(\delta) = \aleph(\delta, \mathfrak{q}(\delta), \widehat{\xi}(\delta)).$$

Since $\mathfrak{w}$ is a solution of the inequality (13), by Remark 4.4, we have

$$\left({}_{\kappa}^{TH}\mathcal{D}_{\varrho_1+}^{\sigma,\varepsilon,\lambda;\psi}\mathfrak{w}\right)(\delta) = \aleph\left(\delta, \mathfrak{w}(\delta), \left({}_{\kappa}^{TH}\mathcal{D}_{\varrho_1+}^{\sigma,\varepsilon,\lambda;\psi}\mathfrak{w}\right)(\delta)\right) + \widetilde{\aleph}(\delta), \quad \delta \in (\varrho_1, \varrho_2]. \tag{14}$$

Clearly, the solution of (14) is given by

$$\mathfrak{w}(\delta) = \frac{{}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\kappa(1-\theta),\kappa;\psi}\mathfrak{w}(\varrho_1)}{\Psi_{\theta}^{\psi}(\delta, \varrho_1)\Gamma_{\kappa}(\kappa\theta)}e^{-\lambda(\psi(\delta)-\psi(\varrho_1))} + \left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}(\xi + \widetilde{\aleph})\right)(\delta).$$

where ξ be a function satisfying the functional equation

$$\xi(\delta) = \aleph(\delta, \mathfrak{w}(\delta), \xi(\delta)).$$

Hence, for each $\delta \in (\varrho_1, \varrho_2]$, we have

$$\begin{aligned} |\mathfrak{w}(\delta) - \mathfrak{q}(\delta)| &\leq \left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}|\xi(\gamma) - \widehat{\xi}(\gamma)|\right)(\delta) + \left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}\widetilde{\aleph}\right)(\delta) \\ &\leq \epsilon\, {}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}\mathbb{E}_{\kappa}^{\sigma}\left((\psi(\delta) - \psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right) + \frac{\zeta_1}{1-\zeta_2}\left({}_{\lambda}^{T}\mathcal{J}_{\varrho_1{}^{+}}^{\sigma,\kappa;\psi}|\mathfrak{w}(\gamma) - \mathfrak{q}(\gamma)|\right)(\delta). \end{aligned}$$

Using Lemma 2.12 and Lemma 2.18, we get

$$|\mathfrak{w}(\delta)-\mathfrak{q}(\delta)| \leq \epsilon\widehat{\lambda}\left[\mathbb{E}_{\kappa}^{\sigma}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)-1\right] + \frac{\eta_1}{1-\eta_2}\int_{\varrho_1}^{\delta}\psi'(\gamma)\, e^{-\lambda(\psi(\delta)-\psi(\gamma))}\bar{\Psi}_{\sigma}^{\kappa,\psi}(\delta,\gamma)|\mathfrak{w}(\gamma)-\mathfrak{q}(\gamma)|d\gamma.$$

By applying Theorem 2.19, we obtain

$$\begin{aligned}|\mathfrak{w}(\delta)-\mathfrak{q}(\delta)| &\leq \epsilon\widehat{\lambda}\mathbb{E}_{\kappa}^{\sigma}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right) \\ &\quad+\int_{\varrho_1}^{\delta}\sum_{\beta=1}^{\infty}\left(\frac{\eta_1\widehat{\lambda}}{1-\eta_2}\right)^{\beta}\psi'(\gamma)\,\bar{\Psi}_{\beta\sigma}^{\kappa,\psi}(\delta,\gamma)\epsilon\widehat{\lambda}\mathbb{E}_{\kappa}^{\sigma}\left((\psi(\gamma)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)d\gamma \\ &\leq \epsilon\widehat{\lambda}\mathbb{E}_{\kappa}^{\sigma}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)\mathbb{E}_{\kappa}^{\sigma}\left[\frac{\eta_1\widehat{\lambda}}{1-\eta_2}(\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right] \\ &\leq \epsilon\widehat{\lambda}\mathbb{E}_{\kappa}^{\sigma}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)\mathbb{E}_{\kappa}^{\sigma}\left[\frac{\eta_1\widehat{\lambda}}{1-\eta_2}(\psi(\varrho_2)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right].\end{aligned}$$

Then for each $\delta\in(\varrho_1,\varrho_2]$, we have

$$|\mathfrak{w}(\delta)-\mathfrak{q}(\delta)| \leq a_{\mathbb{E}_{\kappa}^{\sigma}}\epsilon\mathbb{E}_{\kappa}^{\sigma}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right),$$

where

$$a_{\mathbb{E}_{\kappa}^{\sigma}} = \widehat{\lambda}\mathbb{E}_{\kappa}^{\sigma}\left[\frac{\eta_1\widehat{\lambda}}{1-\eta_2}(\psi(\varrho_2)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right].$$

Hence, the problem (1)-(2) is κ-Mittag-Leffler-Ulam-Hyers stable with respect to $\mathbb{E}_{\kappa}^{\sigma}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)$. If we set $v(\epsilon)=a_{\mathbb{E}_{\kappa}^{\sigma}}\epsilon$, then the problem (1)-(2) is also generalized κ-Mittag-Leffler-Ulam-Hyers stable. □

5. Examples

In this segment, we are going to provide practical examples that showcase the fulfillment of the conditions outlined in the theorems of the existence and stability results. We will initially present the general case of our problem (1)-(2).

Example 5.1. *By taking $\varepsilon=\sigma=\frac{1}{2}$, $\lambda=3$, $\kappa=\frac{3}{2}$, $\psi(\delta)=e^{\delta}$, $\varrho_1=1$, $\varrho_2=\pi$ and $\varkappa=e$, from the problem (1)-(2), we obtain the following terminal value problem with (κ,ψ)-Hilfer nonlinear implicit fractional differential equation:*

$$\left({}_{\frac{3}{2}}^{TH}\mathcal{D}_{1+}^{\frac{1}{2},\frac{1}{2},3;\psi}\mathfrak{w}\right)(\delta)=\aleph\left(\delta,\mathfrak{w}(\delta),\left({}_{\frac{3}{2}}^{TH}\mathcal{D}_{1+}^{\frac{1}{2},\frac{1}{2},3;\psi}\mathfrak{w}\right)(\delta)\right),\quad \delta\in(1,\pi], \tag{15}$$

$$\mathfrak{w}(\pi)=e, \tag{16}$$

where $\nabla=[1,\pi]$, $\theta=\frac{1}{\kappa}(\varepsilon(\kappa-\sigma)+\sigma)=\frac{2}{3}$ and

$$\aleph(\delta,\mathfrak{w},\mathfrak{q})=\frac{\sqrt{e^{\delta}-e}(|\cos(\delta)|+\mathfrak{w}+\mathfrak{q})}{673e^{-\delta+\pi}},\ \delta\in\nabla,\ \mathfrak{w},\mathfrak{q}\in\mathbb{R}.$$

We have

$$C_{\theta;\psi}(\nabla)=C_{\frac{2}{3};\psi}(\nabla)=\left\{\mathfrak{w}:(1,\pi]\to\mathbb{R}:(\sqrt[3]{e^{\delta}-e})\mathfrak{w}\in C(\nabla,\mathbb{R})\right\}.$$

It is clear that the function $\aleph$ *is continuous on* ∇*. Then, the condition (*Ax1*) is satisfied. For each* $\mathfrak{w}, \bar{\mathfrak{w}}, \mathfrak{q}, \bar{\mathfrak{q}} \in \mathbb{R}$ *and* $\delta \in \nabla$*, we have*

$$|\aleph(\delta, \mathfrak{w}, \bar{\mathfrak{w}}) - \aleph(\delta, \mathfrak{q}, \bar{\mathfrak{q}})| \leq \frac{\sqrt{e^{\delta} - e}}{673e^{-\delta+5\pi}} \left(|\mathfrak{w} - \bar{\mathfrak{w}}| + |\mathfrak{q} - \bar{\mathfrak{q}}|\right), \ \delta \in \nabla,$$

*and so the condition (*Ax2*) is satisfied with* $\zeta_1 = \zeta_2 = \dfrac{\sqrt{e^{\pi} - e}}{673e^{4\pi}}$*. Also, the condition (9) of Theorem 3.3 is satisfied. Indeed, we have*

$$\begin{aligned}
\mathcal{L} &= \frac{\zeta_1 \widehat{\lambda} \Gamma_\kappa(\kappa\theta) \left(\psi(\varrho_2) - \psi(\varrho_1)\right)^{\frac{\sigma}{\kappa}}}{\Gamma_\kappa(\sigma + \kappa\theta)(1 - \zeta_2)} \left[e^{\lambda(\psi(\varrho_2) - \psi(\varrho_1))} \widehat{\lambda} + 1\right] \\
&= \frac{\sqrt{e^{\pi} - e}\sqrt[3]{e^{\pi} - e}}{(673e^{4\pi} - \sqrt{e^{\pi} - e})} \sqrt[3]{\frac{2}{3}} \Gamma\left(\frac{2}{3}\right) \left[e^{3(e^{\pi} - e)} + 1\right] \\
&\approx 8.32341334978677 \cdot 10^{-22} \\
&< 1.
\end{aligned}$$

Then the problem (15)*-*(16) *has a unique solution in* $C_{\frac{2}{3};\psi}([1, \pi])$ *and is* κ*-Mittag-Leffler-Ulam-Hyers stable with respect to* $\mathbb{E}^{\frac{1}{3}}_{\frac{3}{2}}\left(\sqrt[3]{e^{\delta} - e}\right)$*.*

Example 5.2. *Taking* $\varepsilon \to 0$*,* $\sigma = \frac{1}{2}$*,* $\lambda = 0$*,* $\kappa = 1$*,* $\psi(\delta) = \delta$*,* $\varrho_1 = 1$*,* $\varrho_2 = e$ *and* $\varkappa = \pi$*, we get a particular case of problem* (1)*-*(2) *using the Riemann–Liouville fractional derivative, given by*

$$\left({}^{TH}_{1}\mathcal{D}^{\frac{1}{2},0,0;\psi}_{1+} \mathfrak{w}\right)(\delta) = \left({}^{RL}\mathbb{D}^{\frac{1}{2}}_{1+} \mathfrak{w}\right)(\delta) = \aleph\left(\delta, \mathfrak{w}(\delta), \left({}^{RL}\mathbb{D}^{\frac{1}{2}}_{1+} \mathfrak{w}\right)(\delta)\right), \quad \delta \in (1, e], \tag{17}$$

$$\mathfrak{w}(e) = \pi, \tag{18}$$

where $\nabla = [1, e]$*,* $\theta = \frac{1}{\kappa}(\varepsilon(\kappa - \sigma) + \sigma) = \frac{1}{2}$*,*

$$\aleph(\delta, \mathfrak{w}, \mathfrak{q}) = \frac{11 + 2t + \mathfrak{w}\sqrt{\delta - 1} + \mathfrak{q}}{111e^{2t}}, \ \ \delta \in \nabla, \ \ \mathfrak{w}, \mathfrak{q} \in \mathbb{R}.$$

We have

$$C_{\theta;\psi}(\nabla) = C_{\frac{1}{2};\psi}(\nabla) = \left\{\mathfrak{w} : (1, e] \to \mathbb{R} : (\sqrt{\delta - 1})\mathfrak{w} \in C(\nabla, \mathbb{R})\right\}.$$

Clearly, the continuous function $\aleph$ *is continuous. Hence, the condition* (Ax1) *is satisfied. For each* $\mathfrak{w}, \bar{\mathfrak{w}}, \mathfrak{q}, \bar{\mathfrak{q}} \in \mathbb{R}$ *and* $\delta \in \nabla$*, we have*

$$|\aleph(\delta, \mathfrak{w}, \bar{\mathfrak{w}}) - \aleph(\delta, \mathfrak{q}, \bar{\mathfrak{q}})| \leq \frac{\sqrt{\delta - 1}}{111e^{2t}} |\mathfrak{w} - \bar{\mathfrak{w}}| + \frac{1}{111e^{2t}} |\mathfrak{q} - \bar{\mathfrak{q}}|, \ \delta \in \nabla,$$

and so the condition (Ax3) *is satisfied with* $\varpi_2(\delta) = \varpi_3(\delta) = \dfrac{1}{111e^{2t}}$*.*
Also, we have

$$\begin{aligned}
\ell &= \frac{\varpi_2^* \widehat{\lambda} \left(\psi(\varrho_2) - \psi(\varrho_1)\right)^{1 - \theta + \frac{\sigma}{\kappa}}}{(1 - \varpi_3^*)\Gamma_\kappa(\sigma + \kappa)} \left(1 + \widehat{\lambda} e^{\lambda(\psi(\varrho_2) - \psi(\varrho_1))}\right) \\
&= \frac{4\,(e - 1)}{(111e^2 - 1)\sqrt{\pi}} \\
&\approx 0.00473366304608964 \\
&< 1.
\end{aligned}$$

Since it is clear that all the conditions of Theorem 3.4 and Theorem 4.5 are satisfied, then the problem (15)-(16) has at least one solution in $C_{\frac{1}{2};\psi}([1,e])$ *and is Mittag-Leffler-Ulam-Hyers stable with respect to* $\mathbb{E}_1^{\frac{1}{2}}\left(\sqrt{\delta-1}\right)$.

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