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KNESER PROPERTY FOR SOME PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY IN BANACH SPACES

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Abstract. In this work, using integrated semigroup theory, we establish that the set consisting of the integral solutions of some partial functional differential equations is connected in the space of continuous functions.

1. Introduction

In this work, we study the following partial functional equation with infinite delay

$$(1.1)\qquad \begin{cases} x'(t) = Ax(t) + f(t, x_t) \text{ for } t \geq 0 \\ \\ x_0 = \varphi \in \mathcal{B}, \end{cases}$$

where A is a closed linear operator on a Banach space X which satifies the Hille-Yosida condition, the phase space $\mathcal{B}$ is a linear space of functions mapping $]-\infty, 0]$ into X satisfying some axioms which will be described in the sequel. For every $t \geq 0$, $x_t \in \mathcal{B}$ denotes the history function defined by

$$x_t(\theta) = x(t+\theta) \text{ for } \theta \in]-\infty, 0]$$

and f is an X-valued appropriate function. In the literature devoted to equations with finite delay, the state space is the space of all continuous functions on $[-r, 0]$, $r > 0$, endowed with the uniform norm topology. When the delay is unbounded, the selection of the state space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory.
Concerning the case of infinite delay, an extensive theory is developed for equation (1.1). We refer the reader to Hale and Kato [3]. It contains a basic theory on functional differential equations with infinite delay in finite dimensional spaces. Wu [6] developed a general theory of existence, comparison, invariance and monotonicity and provide some applications to reaction diffusion systems with general distributed delays when A is densely defined and generates a strongly contiuous semigroup. The mild solutions of equation (1.1) are given by the following variation of constants formula

$$x(t) = T(t)\varphi(0) + \int_0^t T(t-s)f(s, x_s)ds \text{ for } t \geq 0$$

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where $(T(t))_{t\geq 0}$ is the semigroup generated by A. In [1], the authors developed fundamental results on the existence, regularity and stability of solutions when A is non-densely defined and satisfies the Hille-Yosida condition. In this case, the integral solutions of equation (1.1) are given by this following variation of constant formula

$$x(t) = S'(t)\varphi(0) + \frac{d}{dt}\int_0^t S(t-s)f(s,x_s)ds \text{ for } t \geq 0$$

where $(S(t))_{t\geq 0}$ is the integrated semigroup generated by A. The aim of this work is to prove that the set consisting of integral solutions of equation (1.1) is connected. This property is known in the literature as Kneser's property. We refer to [4] for the original result in the frame of differential equations and for a similar result for functional equations. Another abstract version of the Kneser property, which is known as the Krasnoselskii-Perov theorem (see [7]), is obtained using degree theory. The main tool in the approach followed in this work is the theory of integrated semigroup. A brief reminder of this theory is provided in Section 2. In section 3, we establish that the sets of integral solutions of equation (1.1) is connected. Last section is devoted to an application.

2. Preliminary results

The purpose of this section is to collect some background materials required throughout this paper. These materials include integrated semigroup theory and differential operators with non-dense domain. We will only state results and for the details the reader may refer references. The following definitions are due to [2].

Definition 2.1. Let X be a Banach space. A family $(S(t))_{t\geq 0} \subset \mathcal{L}(X)$ is called an integrated semigroup if the following conditions are satisfied:

(i) S(0) = 0;

(ii) for any $x \in X$, $S(t)x$ is a continuous function of $t \geq 0$ with values in X;

(iii) for any $t, s \geq 0$, $S(s)S(t) = \int_0^s (S(t+\mu) - S(\mu))d\mu$.

Definition 2.2. An integrated semigroup $(S(t))_{t\geq 0}$ is called exponentially bounded if there exist $N \in \mathbb{R}^+$ and $\omega \in \mathbb{R}$ such that

$$|S(t)| \leq Ne^{\omega t} \text{ for } t \geq 0.$$

Moreover, $(S(t))_{t\geq 0}$ is called non-degenerate if $S(t)x = 0$ for all $t \geq 0$, implies that $x = 0$.

If $(S(t))_{t\geq 0}$ is an integrated semigroup, exponentially bounded, then the Laplace transform $R(\lambda) = \lambda \int_0^{+\infty} e^{-\lambda t}S(t)$ exists for all $\lambda \geq \omega$. If $(S(t))_{t\geq 0}$ is non-degenerate, there exists a unique operator A satisfying $]\omega, +\infty[\subset \rho(A)$ (the resolvent set of A) such that

$$R(\lambda) = R(\lambda, A) = (\lambda I - A)^{-1} \text{ for all } \mathcal{R}e\lambda > \omega.$$

This operator A is called the generator of $(S(t))_{t\geq 0}$. We have the following general definition.

Definition 2.3. An operator A is called a generator of an integrated semigroup, if there exists $\omega \in \mathbb{R}$ such that $]\omega, +\infty[\subset \rho(A)$ and there exists a strongly continuous exponentially bounded family $(S(t))_{t\geq 0}$ of linear bounded operators such that $S(0) = 0$ and

$$(\lambda I - A)^{-1} = \lambda \int_0^{+\infty} e^{-\lambda t} S(t) \text{ for all } \lambda > \omega.$$

Definition 2.4. An integrated semigroup $((S(t))_{t\geq 0}$ is called locally Lipschitz continuous, if for all $\tau \geq 0$ there exists a constant $k(\tau) \geq 0$ such that

$$|S(t) - S(s)| \leq k(\tau)|t - s| \text{ for all } t, s \in [0, \tau].$$

Definition 2.5. We say that a linear operator A satisfies the Hille-Yosida condition if there exist $N \geq 1$ and $\omega \in \mathbb{R}$ such that $]\omega, +\infty[\subset \rho(A)$ and

$$\sup\{(\lambda - \omega)^n |R(\lambda, A)|^n,\ n \in \mathbb{N}, \lambda > \omega\} < N.$$

The following theorem shows that the Hille-Yosida condition characterizes generators of locally Lipschitz continuous integrated semigroups.

Theorem 2.6. *The following assertions are equivalent.*

(i) *A is the generator of a locally Lipschitz continuous integrated semigroup;*

(ii) *A satisfies the Hille-Yosida condition.*

Remark. If A is the generator of a locally Lipschitz-continuous integrated semigroup $(S(t))_{t\geq 0}$ on X, then $(S'(t))_{t\geq 0} : \overline{D(A)} \to \overline{D(A)}$ is a c_0-semigroup ie there exist $\overline{N} \in \mathbb{R}^+$ and $\omega \in \mathbb{R}$ such that

$$|S'(t)y| \leq \overline{N} e^{\omega t}|y| \text{ for all } t \geq 0 \text{ and } y \in \overline{D(A)}.$$

Proposition 2.7. *Let $A : D(A) \subset X \to X$ be a linear operator which satisfies the Hille-Yosida condition, $(S(t))_{t\geq 0}$ be the integrated semigroup generated by A and $G : [0, a] \to X$ be a Bochner-integrable function. Then, the function $\mathcal{H} : [0, a] \to X$ defined by*

$$\mathcal{H}(t) = \int_0^t S(t - s)G(s)ds \tag{2.1}$$

is continuously differentiable on $[0, a]$ and satisfies for $\lambda > \omega$ and $t \in [0, a]$

$$\lambda R(\lambda, A)\mathcal{H}'(t) = \int_0^t S'(t - s)\lambda R(\lambda, A)G(s)ds. \tag{2.2}$$

From this theorem, we have

$$\begin{aligned}
\left|\lambda R(\lambda, A)\frac{d}{dt}\int_0^t S(t - s)G(s)ds\right| &= \left|\int_0^t S'(t - s)\lambda R(\lambda, A)G(s)ds\right| \\
\lim_{\lambda\to+\infty} |\lambda R(\lambda, A)| \left|\frac{d}{dt}\int_0^t S(t - s)G(s)ds\right| &= \left|\lim_{\lambda\to+\infty}\int_0^t S'(t - s)\lambda R(\lambda, A)G(s)ds\right| \\
\frac{d}{dt}\int_0^t S(t - s)G(s)ds &= \lim_{\lambda\to+\infty}\int_0^t S'(t - s)\lambda R(\lambda, A)G(s)ds.
\end{aligned}$$

Theorem 2.8. *[4] Let $\mathcal{F} : C([0,\tau];X) \to C([0,\tau];X)$, $\tau > 0$ be continuous and $\mathcal{S}$ be the set of fixed points of $\mathcal{F}$. Assume that there is a compact set $K \subset C([0,\tau];X)$ such that for each $\varepsilon > 0$, there is a set $K_\varepsilon \subset K$ with the following properties:*

(i) *the sets K_ε are connected;*

(ii) *$d(x, K_\varepsilon) < \varepsilon$ for all $x \in \mathcal{S}$;*

(iii) *$|y - \mathcal{F}y|_\infty < \delta(\varepsilon)$ for all $y \in K_\varepsilon$, where $\delta(\varepsilon) \to 0$, as $\varepsilon \to 0$.*

Then $\mathcal{S}$ is connected.

3. Kneser Property

In all this work, we assume that the state space $(\mathcal{B}, |.|_\mathcal{B})$ is a normed linear space of functions mapping $]-\infty, 0]$ into X and satisfying the following fundamental axioms.

($\mathbf{A}_1$) There exist a positive constant H and functions $K(.)$, $P(.) : \mathbb{R}^+ \to \mathbb{R}^+$, with K continuous and $\mathcal{M}$ locally bounded, such that for any $\sigma \in \mathbb{R}$ and $a > 0$, if $u :]-\infty, a] \to X$, $u_\sigma \in \mathcal{B}$, and $u(.)$ is continuous on $[\sigma, \sigma + a]$, then for every $t \in [\sigma, \sigma + a]$ the following conditions hold

(**i**) $u_t \in \mathcal{B}$.

(**ii**) $|u(t)| \leq H|u_t|_\mathcal{B}$, which is equivalent to $|\varphi(0)| \leq H|\varphi|_\mathcal{B}$ for every $\varphi \in \mathcal{B}$.

(**iii**) $|u_t|_\mathcal{B} \leq K(t-\sigma) \sup_{\sigma \leq s \leq t} |u(s)| + \mathcal{M}(t-\sigma)|u_\sigma|_\mathcal{B}$.

($\mathbf{A}_2$) For the function $u(.)$ in ($\mathbf{A}_1$), $t \mapsto u_t$ is a $\mathcal{B}$-valued continuous function for $t \in [\sigma, \sigma + a]$.

(**B**) The space $\mathcal{B}$ is complete.

We give the following definition

Definition 3.1. We say that a continuous function $x : [-r, +\infty[\to X$ is a integral solution of equation (1.1) if x satisfies the following equation

$$\begin{cases} x(t) = S'(t)\varphi(0) + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)f(s, x_s)ds \text{ for } t \geq 0 \\ \\ x_0 = \varphi. \end{cases}$$

Let Ω be a nonempty open subset of $\mathcal{B}$. For the existence of integral solutions, we make the following assumptions:

($\mathbf{H}_1$) A satisfies a Hille-Yosida condition;

($\mathbf{H}_2$) $(S'(t))_{t\geq 0}$ is a compact on $\overline{D(A)}$;

($\mathbf{H}_3$) $f : [0, +\infty[\times\Omega \to X$ is continuous

Theorem 3.2. *[1] Assume that* ($\boldsymbol{H}_1$), ($\boldsymbol{H}_2$) *and* ($\boldsymbol{H}_3$) *hold. Let* $\varphi \in \Omega$ *be such that* $\varphi(0) \in \overline{D(A)}$. *Then there exists* b *such that equation* (1.1) *has an integral solution* $x :]-\infty, b] \to X$.

Since $(S(t))_{t\geq 0}$ is an integrated semigroup and $(S'(t))_{t\geq 0}$ is a strongly continuous semigroup, then $(S(t))_{t\geq 0}$ and $(S'(t))_{t\geq 0}$ are uniformly bounded on bounded intervals. We can find a constant M such that $|S(t)|$ and $|S'(t)|$ are less or equal than M for all $t \in [0, b]$. Moreover let us pose $K_b = \sup\limits_{0\leq t\leq b} K(t)$ and $M_b = \sup\limits_{0\leq t\leq b} \mathcal{M}(t)$.

In what follows, we make the following assumptions.

($\mathbf{H}_4$) For each $R > 0$, there is a positive integrable function $\gamma_R \in L^1([0, b])$ such that for almost $t \in I = [0, b]$, $\sup\{|f(t, \psi)| : \ |\psi|_{\mathcal{B}} \leq R\} \leq \gamma_R(t)$.

($\mathbf{H}_5$) $\lim\limits_{R\to+\infty} \dfrac{MNK_b}{R} \displaystyle\int_0^b \gamma_R(s)ds < 1$.

Theorem 3.3. *Assume that* ($\boldsymbol{H}_1$), ($\boldsymbol{H}_2$), ($\boldsymbol{H}_3$), ($\boldsymbol{H}_4$) *and* ($\boldsymbol{H}_5$) *hold. Then the set* $\mathcal{S}$ *of integral solutions of equation* (1.1) *is compact in* $C([0, b]; X)$.

Proof. Consider the set C_φ be defined by

$$C_\varphi = \{x \in C([0, b]; X) \text{ such that } x(0) = \varphi(0)\}.$$

Consider the mapping $\mathcal{T}$ defined on C_φ by

$$(\mathcal{T}x)(t) = S'(t)\varphi(0) + \frac{d}{dt}\int_0^t S(t-s)f(s, \widetilde{x}_s)ds \ \text{ for } \ t \in [0, b].$$

We claim that there exists $n_0 \in \mathbb{N}$ such that $\mathcal{T} : B_{n_0} \to B_{n_0}$ where B_{n_0} is a closed ball with center at 0 and radius n_0 in $\mathcal{B}$. In fact, if we suppose that $\mathcal{T}B_{n_0} \not\subseteq B_{n_0}$, then we can select an increasing sequence R_j such that

$$\lim_{j\to+\infty} \frac{1}{R_j}\int_0^b \gamma_{R_j}(s)ds = \lim_{R\to+\infty} \frac{1}{R}\int_0^b \gamma_R(s)ds = \alpha < 1$$

as well a sequence $(n_j)_{j\geq 0}$ in $\mathbb{N}$ and a sequence $x^j \in B_{nj}$ such that $q_j = K_b n_j + M_b|\varphi| \leq R_j \leq K_b(n_j + 1) + M_b|\varphi|$ and $|\mathcal{T}(x^j)| > n_j$. Therefore, for each $t \in [0, b]$, we have that $|x^j_t| \leq q_j$. Using ($\mathbf{H}_1$) and by Definition 2.5 , it follows that

$$\begin{aligned}
n_j &< |(\mathcal{T}x^j)(t)| \leq |S'(t)\varphi(0)| + \left|\frac{d}{dt}\int_0^t S(t-s)f(s, x^j_s)ds\right| \\
&< M|\varphi(0)| + \left|\lim_{\lambda\to+\infty}\int_0^t S'(t-s)\lambda R(\lambda, A)f(s, x^j_s)ds\right| \\
&< M|\varphi(0)| + \lim_{\lambda\to+\infty}\frac{\lambda}{\lambda-\omega}MN\int_0^b \gamma_{R_j}(s)ds.
\end{aligned}$$

Consequently

$$\begin{aligned} 1 &< \frac{M|\varphi(0)|}{n_j} + \frac{MN}{n_j}\int_0^b \gamma_{R_j}(s)ds \\ &< \frac{M|\varphi(0)|}{n_j} + MN\frac{R_j}{n_j}\frac{1}{R_j}\int_0^b \gamma_{R_j}(s)ds \to MNK_b\alpha \text{ as } n_j \to +\infty \end{aligned}$$

which contradicts ($\mathbf{H}_5$).

To prove that the set $\mathcal{S}$ of integral solutions of equation (1.1) is compact in $C([0,b];X)$, it is sufficient to show that the set

$$\{(\mathcal{T}x)(t) - S'(t)\varphi(0) : \ |x|_\infty \leq R\}$$

is relatively compact for each $t \in]0,b]$ and Range($\mathcal{T}$) is equicontinuous on $[0,b]$. From Proposition 2.7, we have

$$\begin{aligned} \lambda R(\lambda, A)\frac{d}{dt}\int_0^t S(t-s)f(s,x_s)ds &= \int_0^t S'(t-s)\lambda R(\lambda,A)f(s,x_s)ds \\ &= S'(\varepsilon)\int_0^{t-\varepsilon} S'(t-\varepsilon-s)\lambda R(\lambda,A)f(s,x_s)ds \\ &\quad + \int_{t-\varepsilon}^t S'(t-s)\lambda R(\lambda,A)f(s,x_s)ds \end{aligned} \tag{3.1}$$

Since $S'(\varepsilon)$ is compact, there exists a compact set Ω_ε such that

$$\left\{S'(\varepsilon)\int_0^{t-\varepsilon} S'(t-\varepsilon-s)\lambda R(\lambda,A)f(s,x_s)ds\right\} \subset \Omega_\varepsilon.$$

For the second term on the right-hand side of equation (3.1), we obtain the following estimation

$$\Big|\int_{t-\varepsilon}^t S'(t-s)\lambda R(\lambda,A)f(s,x_s)ds\Big| \leq MN\int_{t-\varepsilon}^t \gamma_R(s)ds,$$

which shows that this term converges toward zero as $\varepsilon \to 0$ since γ_R is integrable. This implies that

$$\{(\mathcal{T}x)(t) - S'(t)\varphi(0) : \ |x|_\infty \leq R\}$$

is totally bounded and Range($\mathcal{T}$)(t) is relatively compact for $0 \leq t \leq b$.

On the other hand for every $0 \leq t_0 < t \leq b$ and $\lambda > \omega$, one has

$$\begin{aligned} (\mathcal{T}x)(t) - (\mathcal{T}x)(t_0) &= (S'(t) - S'(t_0))\varphi(0) + \lim_{\lambda\to+\infty}\Big[\int_{t_0}^t S'(t-s)\lambda R(\lambda,A)f(s,x_s)ds \\ &\quad + \int_0^{t_0}(S'(t-s) - S'(t_0-s))\lambda R(\lambda,A)f(s,x_s)ds\Big]. \end{aligned}$$

This implies that

$$\Big|(\mathcal{T}x)(t)-(\mathcal{T}x)(t_0)\Big| \leq |(S'(t)-S'(t_0))\varphi(0)| + \lim_{\lambda\to+\infty}\frac{\lambda}{\lambda-\omega}MN\int_{t_0}^{t}\gamma_R(s)ds$$

$$+\lim_{\lambda\to+\infty}\Big|(S'(t-t_0)-I)\int_0^{t_0}S'(t_0-s)\lambda R(\lambda,A)f(s,x_s)ds\Big|$$

$$\leq \ |(S'(t)-S'(t_0))\varphi(0)| + MN\int_{t_0}^{t}\gamma_R(s)ds + MN\int_{t_0}^{t}\gamma_R(s)ds|(S'(t-t_0)-I)|.$$

Using the continuity of $(S'(.)x)$ and the fact that γ_R is integrable, we obtain that

$$\lim_{t\to t_0}|(\mathcal{T}x)(t)-(\mathcal{T}x)(t_0)| = 0.$$

Using similar argument for $0 \leq t < t_0 \leq b$, we can conclude that Range$\mathcal{T}(t)$ is equicontinuous. By Arzela-Ascoli theorem, it follows that Range$\{\mathcal{T}x : \ |x|_\infty \leq R\}$ is relatively compact in $C([0,b];X)$. □

Theorem 3.4. *Assume that* $(\boldsymbol{H}_1)$, $(\boldsymbol{H}_2)$, $(\boldsymbol{H}_3)$, $(\boldsymbol{H}_4)$ *and* $(\boldsymbol{H}_5)$ *hold. Then* $\mathcal{S}$ *is connected in* $C([0,b];X)$.

Proof. Let us pose $h(t) = S'(t)\varphi(0)$. On the other hand, applying $(\mathbf{H}_4)$ and $(\mathbf{H}_5)$, we can choose a constant $\alpha > 0$ large enough such that $|x|_\infty \leq \alpha$ for all $x \in \mathcal{S}$ and

$$|h|_\infty + 3MN\int_0^b \gamma_R(s)ds \leq \alpha. \tag{3.2}$$

This implies that

$$K_b|h|_\infty + M_b|\varphi| + 3MNK_b\int_0^b\gamma_Q(s)ds \leq Q, \tag{3.3}$$

where $Q = K_b\alpha + M_b|\varphi|$. Let V be a set defined by

$$V = \Big\{\frac{d}{dt}\int_0^t S(t-s)f(s,x_s)ds : \ 0\leq t\leq b, \ x\in C([0,b];X), \ |x|_\infty \leq Q\Big\}$$

Proceeding as in the proof of Theorem 3.3, we can show that V is relatively compact in X. Without lost of generality, we assume that V is absolutely convex. We put $U = 2V$ and $U_1 = 3V$. We denote $N_1 = 2MN\int_0^b\gamma_Q(s)ds$. We divide the proof in several steps.

Step 1. Let $z \in C_\varphi$ be defined by

$$\begin{cases} z(t) = h(t) + \displaystyle\sum_{k=1}^{i-1}\Big(\frac{d}{dt}\int_{t_{k-1}}^{t_k}S(t-s)f(s,z_{t_{k-1}})ds + (t_k-t_{k-1})u_k\Big) \\ \qquad +\displaystyle\frac{d}{dt}\int_{t_{i-1}}^{t}S(t-s)f(s,z_{t_{i-1}})ds + (t-t_{i-1})u_i \ \text{ for } t_{i-1}<t<t_i \\ z(0) = \varphi(0), \end{cases} \tag{3.4}$$

where we choose u_k so that $\sum_{k=1}^{i}(t_k - t_{k-1})u_k \in U$ and $\left|\sum_{k=1}^{i}(t_k - t_{k-1})u_k\right| \leq N_1$ for all $i = 1, ..., n$ and $0 = t_0 < t_1 < .... < t_{n-1} < t_n = b$ is a division d of $[0, b]$. We show that $|z(t)| \leq \alpha$ for $0 \leq t \leq b$ independent of the division d of $[0, b]$ and the choice of points u_i.

Let z given by equation (3.4), then z is a continuous function and denote by $\psi(.)$ and $u(.)$ the steps functions defined by $\psi(0) = \varphi$, $u(0) = u_1$, $\psi(t) = z_{t_{k-1}}$ and $u(t) = u_k$ for $t_{k-1} < t < t_k$ and $k = 1, ..., n$. Thus we can rewrite $z(t)$ as follows

$$z(t) = h(t) + \frac{d}{dt}\int_0^t S(t-s)f(s, \psi(s))ds + \int_0^t u(s)ds. \tag{3.5}$$

From equation (3.2), we obtain that $|z(t)| \leq \alpha$ for $0 \leq t \leq t_1$. Assuming now that this property holds on $[0, t_{i-1}]$, we want to show that it is also verified for $t_{i-1} < t < t_i$.
In fact, from the axioms of phase and equation (3.3), we can see that $|z|_{\mathcal{B}} \leq Q$ for $0 < t < t_{i-1}$ and since $\sum_{k=1}^{i-1}(t_k - t_{k-1})u_k + (t - t_{i-1})u_i$ is a convex combination of $\sum_{k=1}^{i-1}(t_k - t_{k-1})u_k$ and $\sum_{k=1}^{i}(t_k - t_{k-1})u_k$, then from equation (3.5) we obtain

$$\begin{aligned}
|z(t)| &\leq |h(t)| + \left|\frac{d}{dt}\int_0^t S(t-s)f(s, \psi(s))ds\right| + \left|\sum_{k=1}^{i-1}(t_k - t_{k-1})u_k + (t - t_{i-1})u_i\right| \\
&\leq |h(t)| + \lim_{\lambda \to +\infty}\left|\int_0^t S'(t-s)\lambda R(\lambda, A)f(s, \psi(s))ds\right| + \left|\sum_{k=1}^{i-1}(t_k - t_{k-1})u_k + (t - t_{i-1})u_i\right| \\
&\leq |h|_\infty + MN\int_0^b \gamma_Q(s)ds + 2MN\int_0^b \gamma_Q(s)ds,
\end{aligned}$$

which establishes our assertion.

Step 2. Let W be the set formed by the functions z defined by equation (3.4). We show that W is relatively compact in $C([0, b]; X)$.

To simplify this construction, we consider the points t_k equally spaced with $\delta = t_k - t_{k-1}$. In addition to the conditions considered in **Step 1**, we suppose that

$$\left|\delta\sum_{k=i+1}^{j} u_k - \Big[S'((j-i)\delta) - I\Big]\delta\sum_{k=1}^{i} u_k\right| \leq 2MN\int_{i\delta}^{j\delta}\gamma_Q(s)ds \tag{3.6}$$

for $1 \leq i + 1 \leq j \leq n$.

Since h is a fixed function if we denote $\widetilde{z} = z - h$, we must prove that $W_0 = \{\widetilde{z} : \ z \in W\}$ is relatively compact.
First, by equation (3.5), we have

$$\widetilde{z}(t) = \frac{d}{dt}\int_0^t S(t-s)f(s, \psi(s))ds + \delta\sum_{k=1}^{i-1} u_k + (t - t_{i-1})u_i \in V + U \subseteq U_1$$

for every $z \in W$ and $t \in [0,b]$.
Now, we prove that W_0 is equicontinuous. Let $0 \leq t' \leq t \leq b$. From equation (3.5), we can write

$$\begin{aligned}
\widetilde{z}(t)-\widetilde{z}(t') &= \lim_{\lambda\to+\infty}\Big(\int_0^t S'(t-s)\lambda R(\lambda,A)f(s,\psi(s))ds-\int_0^{t'} S'(t-s)\lambda R(\lambda,A)f(s,\psi(s))ds\Big)\\
&\quad+\int_{t'}^t u(s)ds\\
&= \Big[S'(t-t')-I\Big]\lim_{\lambda\to+\infty}\int_0^{t'} S'(t'-s)\lambda R(\lambda,A)f(s,\psi(s))ds+\int_{t'}^t u(s)ds\\
&\quad+\lim_{\lambda\to+\infty}\Big(\int_{t'}^t S'(t-s)\lambda R(\lambda,A)f(s,\psi(s))ds\\
&= \Big[S'(t-t')-I\Big]\Big(\widetilde{z}(t')-\int_0^{t'} u(s)ds\Big)+\int_{t'}^t u(s)ds\\
&\quad+\lim_{\lambda\to+\infty}\int_{t'}^t S'(t-s)R(\lambda,A)f(s,\psi(s))ds.
\end{aligned}$$

In view of that V is a compact set and $\Big(\widetilde{z}(t')-\int_0^{t'} u(s)ds\Big)\in V$, in order to prove the assertion, it is sufficient to show that $\int_{t'}^t u(s)ds$ converges toward zero as $t-t'\to 0$ independent of the construction of z.
Since the set U is compact, then for $\varepsilon>0$ there is $\eta_0>0$ such that $|(S'(s)-I)u|\leq \frac{\varepsilon}{2}$ for all $s\in[0,\eta_0]$, $u\in U$ and $2MN\int_{t'}^t |f(s,\psi(s))|ds\leq\frac{\varepsilon}{2}$ for all $t,t'\in[0,b]$ such that $|t-t'|\leq\eta_0$. Let c_1 be a constant defined by

$$c_1=\sup\Big\{\frac{1}{s}|(S'(s)-I)u|:\ \eta_0\leq s\leq b,\ u\in U\Big\}$$

and we take $0<\eta\leq\min\Big\{\eta_0,\ \frac{\varepsilon\eta_0}{2N_1},\ \frac{\varepsilon}{2c_1}\Big\}$.
Initially, we assume that t and t' coincide with some points of the division. Thus, we suppose $t'=t_i$ and $t=t_j$. In this case, we have

$$\begin{aligned}
\int_{t'}^t u(s)ds &= \delta\sum_{k=i+1}^{j} u_k\\
&= \delta\sum_{k=i+1}^{j} u_k-\Big[S'((j-i)\delta)-I\Big]\delta\sum_{k=1}^{i} u_k+\Big[S'((j-i)\delta)-I\Big]\delta\sum_{k=1}^{i} u_k
\end{aligned}$$

and applying equation (3.6), we obtain the following estimation

$$\Big|\int_{t'}^t u(s)ds\Big|\leq 2MN\int_{i\delta}^{j\delta}\gamma_Q(s)ds+\Big|\Big[S'((j-i)\delta)-I\Big]\delta\sum_{k=1}^{i} u_k\Big|.$$

Consequently, since $\delta\sum_{k=1}^{i} u_k\in U$, if $t_j-t_i\leq\eta_0$, it follows that $\Big|\int_{t'}^t u(s)ds\Big|\leq\varepsilon$.
Let $t-t'\leq\eta$. Looking the relative location of points t_k, we analyze three possible situations.

In first case, we assume that there is no point t_k between t' and t. Hence, there is an index i such that $t_i < t' < t \leq t_{i+1}$ and $\displaystyle\int_{t'}^{t} u(s)ds = (t-t')u_{i+1}$. From equation (3.6), we have

$$|u_{i+1}| \leq \frac{2MN}{\delta}\int_{t_i}^{t_{i+1}} \gamma_Q(s)ds + \Big|\Big[S'(\delta)-I\Big]\sum_{k=1}^{i} u_k\Big|.$$

Hence, if $\delta \leq \eta_0$, using that $t-t' \leq \delta$, it follows that

$$\begin{aligned}(t-t')|u_{i+1}| &\leq \frac{\varepsilon}{2} + \frac{t-t'}{\delta}\Big|\Big[S'(\delta)-I\Big]\delta\sum_{k=1}^{i} u_k\Big| \\ &\leq \varepsilon,\end{aligned}$$

while if $\delta \geq \eta_0$, we obtain that

$$\begin{aligned}(t-t')|u_{i+1}| &\leq (t-t')\frac{N_1}{\eta_0} + \frac{t-t'}{\delta}\Big|\Big[S'(\delta)-I\Big]\delta\sum_{k=1}^{i} u_k\Big| \\ &\leq \varepsilon.\end{aligned}$$

Which establishes the assertion in this case.
Now, we assume that there exists an index i such that $t_{i-1} < t' < t_i < t < t_{i+1}$. Therefore, from our definitions, it follows that

$$\int_{t'}^{t} u(s)ds = (t_i - t')u_i + (t-t_i)u_{i+1}.$$

Since $t_i - t' < t - t'$ and $t - t_i < t - t'$, we can argue as in the preceding case.
Finally, we consider that there are index $i < j$ such that $t_{i-1} < t' < t_i < t_j < t < t_{j+1}$. Let us pose $v(t) = \displaystyle\int_0^t u(s)ds$. Then

$$|v(t) - v(t')| \leq |v(t) - v(t_j)| + |v(t_j) - v(t_i)| + |v(t_i) - v(t')|.$$

In view of that $t_j - t_i \leq t - t'$, from our initial remark, we obtain that $|v(t_j) - v(t_i)| \leq \varepsilon$. Thus, this case is reduced to estimate the first and third term on the right-hand side of the above expression. For the first term, we observe that $v(t)$ is a convex combination of $v(t_j)$ and $v(t_{j+1})$. Therefore, $|v(t) - v(t_j)| \leq |v(t_{j+1}) - v(t_j)|$ and since $\delta = t_{j+1} - t_j \leq t - t'$, we can repeat the previous argument. The third term is estimated similarly. From the Ascoli-Arzela theorem, it follows that W_0 is relatively compact and so is $W = h + W_0$.

Step 3. Now let $\varepsilon > 0$ be fixed. Without loss of generality, we also assume that $\varepsilon \leq \min\left\{\frac{\eta_0}{2},\ 2N_1\right\}$ and we take $\varepsilon_1 = \dfrac{\varepsilon}{2MNb}$. Using the compactness of sets $\mathcal{S}$ and W as well as the continuity of f , then there exists $0 \leq \delta_1 \leq 2K_b\varepsilon$ such that

$$|f(s,\psi_1) - f(s,\psi_2)| \leq \varepsilon_1 \tag{3.7}$$

for all $s \in [0,b]$ and for every $\psi_1,\ \psi_2 \in H(W \cup \mathcal{S})$ such that $|\psi_1 -\ \psi_2|_{\mathcal{B}} \leq \delta_1$. Similarly, there is $\delta_2 > 0$ such that

$$|x(s) - x(t)| \leq \frac{\delta_1}{4K_b}, \tag{3.8}$$

for all $x \in W \cup \mathcal{S}$ and $s, t \in [0, b]$ such that $|t - s| \leq \delta_2$. Now, we assume that $\delta = \dfrac{b}{n} \leq \min\left\{\dfrac{\delta_1 b}{2K_b\varepsilon},\ \delta_2\right\}$. In the sequel, we consider the division d defined by $t_i = i\delta$, $i = 0, ..., n$. Let W_ε be the set formed by the functions defined by equation (3.4) with $(u_1, ..., u_n) \in Z_\varepsilon$ where Z_ε is the set consisting of points $(u_1, ..., u_n) \in \left(\frac{2}{\delta}U\right)^n$ that satisfy the following conditions:

(i) $\delta \displaystyle\sum_{k=1}^{i} u_k \in U$;

(ii) $\left|\delta \displaystyle\sum_{k=1}^{i} u_k\right| \leq MNt_i\varepsilon_1$;

(iii) $\left|\delta \displaystyle\sum_{k=i+1}^{j} u_k - \Big[S'((j-i)\delta) - I\Big]\delta \sum_{k=1}^{i} u_k\right| \leq 2MN \int_{i\delta}^{j\delta} \gamma_Q(s)ds.$

for all $i = 1, ..., n$ and $j \geq 1$.

(ii) implies that $\left|\delta \displaystyle\sum_{k=1}^{i} u_k\right| \leq N_1$. Next, we establish some properties of W_ε.

Step 4. The set W_ε is connected.

This assertion is a consequence of the fact that the functions $z \in W_\varepsilon$ depend continuously on the choice of $(u_1, ..., u_n) \in Z_\varepsilon$ and Z_ε is convex by our construction.

Step 5. In this step, we show that the solutions of equation (1.1) can be approximated by the elements in W_ε.

Let $x \in \mathcal{S}$ be fixed. We proceed to define $z \in W_\varepsilon$ so that $|x - z|_\infty \leq \varepsilon$. We define $z(.)$ inductively on the intervals $[t_{i-1}, t_i]$. Let $i = 1$. In this case, $t_1 = \delta$ and we take

$$u_1 = \frac{1}{\delta}\frac{d}{dt}\int_0^{t_1} S(t_1 - s)[f(s, x_s) - f(s, \varphi)]ds.$$

From our construction, it follows that $u_1 \in \frac{1}{\delta}U$. Moreover, from equation (3.8), it follows that $|x_s - \varphi| \leq \dfrac{\delta_1}{4}$ so that equation (3.7) implies that $|f(s, x_s) - f(s, \varphi)| \leq \varepsilon_1$ for $s \in [0, \delta]$ and this yields that $|u_1| \leq M\varepsilon_1$. We define

$$z(t) = h(t) + \frac{d}{dt}\int_0^t S(t - s)f(s, \varphi)ds + tu_1,$$

for $0 \leq t \leq t_1$. It follows that

$$\begin{aligned} z(t_1) &= h(t_1) + \frac{d}{dt}\int_0^{t_1} S(t - s)f(s, \varphi)ds + t_1u_1 \\ &= h(t_1) + \frac{d}{dt}\int_0^{t_1} S(t - s)f(s, x_s)ds + t_1u_1 \\ &= x(t_1). \end{aligned}$$

Moreover, for $0 \leq t \leq t_1$ we have

$$\begin{aligned} |x(t)-z(t)| &\leq \left|\frac{d}{dt}\int_0^t S(t-s)[f(s,x_s)-f(s,\varphi)]ds\right| + |tu_1| \\ &\leq \lim_{\lambda\to+\infty}\left|\int_0^t S'(t-s)\lambda R(\lambda,A)[f(s,x_s)-f(s,\varphi)]ds\right| + |tu_1| \\ &\leq 2MN\varepsilon_1 t \\ &\leq \frac{\delta_1}{2K_b}. \end{aligned}$$

Proceeding by induction, we assume now that we have selected the elements u_k, $k = 1, ..., i-1$ such that $(u_1, ..., u_{i-1}, 0, ..., 0) \in Z_\varepsilon$ and the function $z(t)$ given by equation (3.4) for $t \in [0, t_{i-1}]$ satisfies $z(t_k) = x(t_k)$ and the following estimation

$$|x(t)-z(t)| \leq \frac{\delta_1}{2K_b} \text{ for } 0 \leq t \leq t_{i-1}.$$

We define now the function z on $[t_{i-1}, t_i]$. We begin by selecting

$$\begin{aligned} u_i &= \frac{1}{\delta}\sum_{k=1}^{i-1}\Big(\frac{d}{dt}\int_{t_{k-1}}^{t_k}[S(t_i-s)-S(t_{i-1}-s)][f(s,x_s)-f(s,z_{t_{k-1}})]ds\Big) \\ &+\frac{1}{\delta}\frac{d}{dt}\int_{t_{i-1}}^{t_i} S(t_i-s)[f(s,x_s)-f(s,z_{t_{i-1}})]ds. \end{aligned}$$

Using the function $\psi(.)$ defined previously, we can rewrite

$$u_i = \frac{1}{\delta}\frac{d}{dt}\int_0^{t_i} S(t_i-s)[f(s,x_s)-f(s,\psi)]ds - \frac{1}{\delta}\frac{d}{dt}\int_0^{t_{i-1}} S(t_{i-1}-s)[f(s,x_s)-f(s,\psi)]ds.$$

Initially, we establish that $(u_1, ..., u_i, 0, ..., 0) \in Z_\varepsilon$. From the above expression, it follows that $\delta u_i \in 2U$ and

$$\delta\sum_{k=1}^{i} u_k = \frac{d}{dt}\int_0^{t_i} S(t_i-s)[f(s,x_s)-f(s,\psi)]ds,$$

which implies that $\delta\sum_{k=1}^{i} u_k \in 2U$ and $\left|\delta\sum_{k=1}^{i} u_k\right| \leq N_1$. Moreover for $m+1 \leq j \leq i$, we have

$$\begin{aligned} \delta\sum_{k=m+1}^{j} u_k &= \delta\sum_{k=1}^{j} u_k - \delta\sum_{k=1}^{m} u_k \\ &= \frac{d}{dt}\int_0^{t_j} S(t_j-s)[f(s,x_s)-f(s,\psi)]ds - \frac{d}{dt}\int_0^{t_m} S(t_m-s)[f(s,x_s)-f(s,\psi)]ds. \end{aligned}$$

Hence

$$\begin{aligned}\delta\sum_{k=m+1}^{j}u_k &= \lim_{\lambda\to+\infty}\Big(\int_0^{t_j}S'(t_j-s)\lambda R(\lambda,A)[f(s,x_s)-f(s,\psi)]ds\\ &\quad -\int_0^{t_m}S'(t_m-s)\lambda R(\lambda,A)[f(s,x_s)-f(s,\psi)]ds\Big)\\ &= \Big[S'(t_j-t_m)-I\Big]\lim_{\lambda\to+\infty}\int_0^{t_m}S'(t_m-s)\lambda R(\lambda,A)[f(s,x_s)-f(s,\psi)]ds\\ &\quad +\lim_{\lambda\to+\infty}\int_{t_m}^{t_j}S'(t_j-s)\lambda R(\lambda,A)[f(s,x_s)-f(s,\psi)]ds,\end{aligned}$$

which yields

$$\Big(\delta\sum_{k=m+1}^{j}u_k-\Big[S'(t_j-t_m)-I\Big]\frac{d}{dt}\int_0^{t_m}S(t_m-s)[f(s,x_s)-f(s,\psi)]ds\Big)$$

$$= \lim_{\lambda\to+\infty}\int_{t_m}^{t_j}S'(t_j-s)\lambda R(\lambda,A)[f(s,x_s)-f(s,\psi)]ds.$$

This implies that

$$\Big|\delta\sum_{k=m+1}^{j}u_k-\Big[S'((j-m)\delta)-I\Big]\delta\sum_{k=1}^{m}u_k ds\Big|\leq 2MN\int_{t_m}^{t_j}\gamma_Q(s)ds.$$

In addition for $t_{i-1}<s\leq t_i$, we can write $x_s-\psi(s)=x_s-x_{t_{i-1}}+x_{t_{i-1}}-z_{t_{i-1}}$. From equation (3.8), we have $|x_s-x_{t_{i-1}}|_{\mathcal{B}}\leq\dfrac{\delta_1}{4}$ and by induction, we have $|x_{t_{i-1}}-z_{t_{i-1}}|_{\mathcal{B}}\leq\dfrac{\delta_1}{2}$. Combining these estimations with equation (3.7), we infer that $\delta\Big|\sum_{k=1}^{i}u_k\Big|\leq MNt_i\varepsilon_1$. This implies that $(u_1,...,u_i,0,...,0)\in Z_\varepsilon$.

Now, we define $z(t)$ for $t_{i-1}<t\leq t_i$ by means of formula equation (3.4). Using this expression as well as the choice of u_k, $k=1,...,i$, we infer that

$$\begin{aligned}x(t_i)-z(t_i) &= \frac{d}{dt}\int_0^{t_i}S(t_i-s)[f(s,x_s)-f(s,\psi)]ds-\delta\sum_{k=1}^{i}u_k\\ &= \frac{d}{dt}\int_0^{t_i}S(t_i-s)[f(s,x_s)-f(s,\psi)]ds-\frac{d}{dt}\int_0^{t_i}S(t_i-s)[f(s,x_s)-f(s,\psi)]ds\\ &= 0.\end{aligned}$$

Moreover, from equation (3.8) and the choice of δ, it follows that

$$\begin{aligned}|x(t)-z(t)| &\leq |x(t)-x(t_{i-1})|+|x(t_{i-1})-z(t)|\\ &\leq |x(t)-x(t_{i-1})|+|z(t)-z(t_{i-1})|\\ &\leq \frac{\delta_1}{2K_b},\end{aligned}$$

which establishes this assertion.

Step 6. In this step, we prove that the elements of W_ε are approximate solutions of equation (1.1). Specifically, we show that

$$\left|z(t) - h(t) - \frac{d}{dt}\int_0^t S(t-s)f(s,x_s)ds\right| \le \varepsilon,$$

for $t \in [0,b]$ and $z \in W_\varepsilon$.

In fact for $t_{i-1} < t \le t_i$, using equation (3.4), we have

$$z(t) - h(t) - \frac{d}{dt}\int_0^t S(t-s)f(s,x_s)ds \;=\; \frac{d}{dt}\int_0^t S(t-s)[f(s,\psi) - f(s,x_s)]ds + \sum_{k=1}^{i} u_k + (t-t_{i-1})u_i$$

and employing now equation (3.7), equation (3.8), and the choice of δ, we can establish the following estimation

$$\begin{aligned}
&\left|z(t) - h(t) - \frac{d}{dt}\int_0^t S(t-s)f(s,x_s)ds\right| \\
=\;& \left|\frac{d}{dt}\int_0^t S(t-s)[f(s,\psi) - f(s,x_s)]ds\right| + \left|\sum_{k=1}^{i} u_k + (t-t_{i-1})u_i\right| \\
=\;& \lim_{\lambda\to+\infty}\left|\int_0^t S'(t-s)\lambda R(\lambda,A)[f(s,\psi) - f(s,x_s)]ds\right| + \left|\sum_{k=1}^{i} u_k + (t-t_{i-1})u_i\right| \\
\le\;& 2MNb\varepsilon_1,
\end{aligned}$$

which show our assertion.
Since $\mathcal{S}$ is the set of fixed points of the map $\mathcal{T}$ defined previously, then using **Step 1** and **Step 6** and applying Theorem 2.8, we obtain that $\mathcal{S}$ is connected in the space $C([0,b];X)$. □

4. Application

For illustration, we propose to study in this work, the following model

$$(4.1)\qquad \begin{cases} \dfrac{\partial}{\partial t}z(t,\xi) = \dfrac{\partial^2}{\partial \xi^2}z(t,\xi) + h\Big(\displaystyle\int_{-\infty}^{0} g(\theta, z(t+\theta,\xi))d\theta\Big) \text{ for } t \ge 0 \text{ and } \xi \in [0,\pi] \\ z(t,0) = z(t,\pi) = 0 \text{ for } t \ge 0 \\ z(\theta,\xi) = \varphi(\theta)(\xi) \text{ for } \theta \in]-\infty, 0] \text{ and } \xi \in [0,\pi], \end{cases}$$

where $h : \mathbb{R} \to X$ is a continuous function and $g : \mathbb{R}^- \times \mathbb{R} \to \mathbb{R}$ is lipschitzian with respect to the second argument. To rewrite equation (4.1) in the abstract form, we introduce the space $X = C[0,\pi];\mathbb{R})$, the space of continuous function from $[0,\pi]$ to $\mathbb{R}$ equipped with the uniform norm topology. Let $A : X \to X$ be defined by

$$\begin{cases} D(A) = \{y \in X \cap C^2([0,\pi],\mathbb{R}) :\; y'' \in X\} \\ Ay = y''. \end{cases}$$

It is well known that $]0,+\infty[\subset D(A)$ and $|(\lambda I-A)^{-1}|\leq \dfrac{1}{\lambda}$, this implies that $(\mathbf{H}_1)$ is satisfied. Moreover A generates a semigroup which satisfies $(\mathbf{H}_2)$. By the continuity of h, we obtain $(\mathbf{H}_3)$. We define the phase space

$$\mathcal{B}=BUC(\mathbb{R}^-;X)$$

where $BUC(\mathbb{R}^-;X)$ is the space of bounded uniformly continuous functions from $\mathbb{R}^-$ into X with the norm $|\varphi|=\sup_{\theta\leq 0}|\varphi(\theta)|$. This space satisfies axioms $(\mathbf{A}_1)$, $(\mathbf{A}_2)$ and $(\mathbf{B})$. Let $F:\mathcal{B}\to X$ be defined by

$$F(\varphi)(\xi)=\int_{-\infty}^{0} g(\theta,\varphi(\theta)(\xi))d\theta \text{ for } \xi\in[0,\pi] \text{ and } t\geq 0.$$

We define the function f by

$$f(t,\varphi)(\xi)=h(F(\varphi)(\xi)) \text{ for } \xi\in[0,\pi] \text{ and } t\geq 0.$$

Let us pose $v(t)=z(t,\xi)$.Then equation (4.1) takes the following abstract form

$$\begin{cases} v'(t)=Av(t)+f(t,v_t),\ t\geq 0 \\ \\ v_0=\varphi\in\mathcal{B} \end{cases}$$

We suppose that there exists a positive function $k(.)\in L^1(]-\infty,0];\mathbb{R})$ such that

$$|g(\theta,\varphi)|\leq k(\theta)|\varphi|_{\mathcal{B}} \text{ for } \theta\leq 0.$$

We assume in addition that there exist constant c_1 and $0\leq\beta<1$ such that

$$|h(t)|\leq c_1|t|^{\beta} \text{ for } t\in\mathbb{R}.$$

For every $\varphi\in\mathcal{B}$, we have $|f(t,\varphi)|\leq c_1c^{\beta}|\varphi|_{\mathcal{B}}^{\beta}$. Let us pose $\gamma_R(t)=c_1c^{\beta}R^{\beta}$, then $(\mathbf{H}_4)$ is satified. Moreover, we can see that $(\mathbf{H}_5)$ is satified for all $b>0$ when $\beta<1$, while $(\mathbf{H}_5)$ is satified for $b>0$ small enough when $\beta=1$.

Proposition 4.1. *Under the above assumptions, for $\varphi\in\Omega$, there exists $b>0$ such that equation (4.1) has an integral solution $v:]-\infty,b]\to X$. Moreover, the set $\mathcal{S}$ of integral solutions is connected in $C([0,b];X)$.*

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