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ON THE FRACTIONAL HYBRID STURM-LIOUVILLE AND LANGEVIN DIFFERENTIAL EQUATIONS AND INCLUSIONS IN THE FRAME OF ψ-HILFER OPERATOR

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Abstract. In this investigation, we explore the existence and uniqueness of solutions for fractional hybrid differential equations and inclusions of Langevin and Sturm-Liouville within the sense of the ψ–Hilfer fractional derivatives. We characterize an unused operator based on the integral solution of the given boundary value inclusion problem, and after that we utilize the presumptions of Dhage's fixed point for the operator within the hybrid case. The theorem is connected for the boundary value problem in the single-valued case uniqueness of solution which is decided by utilizing Banach's contraction mapping rule. Moreover, the stability analysis within the Ulam-Hyers sense of a given system is considered. At last, illustrations are advertised to guarantee the legitimacy of our obtained results.

1. Introduction

The field of fractional calculus is concerned with the generalization of the integer-order differentiation and integration to an arbitrary real or complex arrangement. This field has played a critical part in different branches of science such as material science, chemistry, chemical material science, electrical systems, control of energetic frameworks, science, designing, natural science, optics, and flag handling [1–5]. A later advancement on fractional differential and integral equations ($\mathbb{FDE}$s, $\mathbb{FIE}$s) are considered in a few later books [6–9]. There exist valuable researches on the $\mathbb{FDE}$s and inclusions [10–16].

In recent years, hybrid fractional differential equations ($\mathbb{HFDE}$s) have achieved a great deal of interest and attention of several researchers. For some developments on the existence results for $\mathbb{HFDE}$s, refer to [17–21]. The Langevin equation (first formulated by Langevin in 1908 to give an elaborated description of Brownian motion) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [22]. In general, the nonlinear Langevin equation has achieved a great deal of interest and attention from several researchers.

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For some developments on the existence results of the nonlinear Langevin equation, we refer to [23–26]. Dhage *et al.* in 2013 considered the initial value problems for HDE,

$$\frac{\mathrm{d}}{\mathrm{d}z}\Big[\varrho(z) - \mathfrak{f}\left(z, \varrho(z)\right)\Big] = \theta\left(z, \varrho(z)\right), \quad z \in [\epsilon_0, \epsilon_0 + \tau), \varrho(\epsilon_0) = \varrho_0 \in \mathbb{R}, \tag{1}$$

where $\mathfrak{f}$, $\theta : ([\epsilon_0, \epsilon_0 + \tau) \times \mathbb{R})$ [27]. The authors improved the concept of HFDEs with complex order and considered the HFDE,

$$^{C}\mathcal{D}^{\zeta}_{0+}\left[\frac{\varrho(z)}{\mathfrak{g}(z,\varrho(z))}\right] = \theta\left(z, \varrho(z)\right), \tag{2}$$

for $z \in [0, \top]$, via initial condition $\varrho(0) = \varrho_0$, where $^{C}\mathcal{D}^{\zeta}_{0+}$ is the Caputo fractional derivative of order $\zeta \in \mathbb{C}$, $\zeta = p+\mathrm{i}q$, $0 < p \leq 1$, $q \in \mathbb{R}$, $\mathfrak{g} \in C([0, \top]\times\mathbb{C}, \mathbb{C}\setminus\{0\})$ and $\theta \in C([0, \top]\times\mathbb{C}, \mathbb{C})$ [28]. Samei *et al.* in [29] investigated the existence of solutions for the hybrid Caputo-Hadamard fractional differential inclusion (FDI),

$$^{C}_{\mathrm{H}}\mathcal{D}^{\zeta}\left[\frac{\varrho(z)-\mathfrak{f}\big(z,\varrho(z),\mathcal{I}^{\beta_1}h_1(z,\varrho(z)),\mathcal{I}^{\beta_2}h_2(z,\varrho(z)),\ldots,\mathcal{I}^{\beta_n}h_n(z,\varrho(z))\big)}{\mathfrak{g}(z,\varrho(z),\mathcal{I}^{\alpha_1}\varrho(z),\mathcal{I}^{\alpha_2}\varrho(z),\ldots,\mathcal{I}^{\alpha_m}\varrho(z))}\right] \in \Theta\left(z, \varrho(z)\right), \tag{3}$$

under conditions $\varrho(1) = \mu_1(\varrho(z))$, $\varrho(e) = \mu_2(\varrho(z))$ for $z \in [1, e]$, where $^{C}_{\mathrm{H}}\mathcal{D}^{\gamma}$ and $\mathcal{I}^{\gamma}$ denote the Caputo-Hadamard fractional derivative and Hadamard integral of order γ, respectively, $n, m \in \mathbb{N}$, $\zeta \in (1, 2]$, $\beta_i > 0$, $i = 1, 2, \ldots, n$, $\alpha_i > 0$, $i = 1, 2, \ldots, m$, the function $\mathfrak{f} : [1, e] \times \mathbb{R}^{n+1} \to \mathbb{R}$, $\mathfrak{g} : [1, e] \times \mathbb{R}^{m+1} \to \mathbb{R} \setminus \{0\}$, $h_i : [1, e] \times \mathbb{R} \to \mathbb{R}$ for $i = 1, 2, \ldots, m$, functions μ_i, $(i = 1, 2)$ map $C[1, e]$ into $\mathbb{R}$, and the multifunction $\Theta : [1, e] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ satisfies certain conditions. In 2020, Almeida studied by the following fractional functional DE by using ψ-Caputo fractional derivative [12]. Also, Etemad *et al.* investigated the fractional hybrid multi-term Caputo integro-differential inclusion

$$^{C}\mathcal{D}^{\zeta}_{0}\left[\frac{\varrho(z)}{\mathfrak{g}(z,\varrho(z),\hbar_1(\varrho(z)),\hbar_2(\varrho(z)),\ldots,\hbar_n(\varrho(z)))}\right] \in \Theta\Big(z, \varrho(z), \upmu_1(\varrho(z)), \upmu_2(\varrho(z)), \ldots, \upmu_m(\varrho(z))\Big), \tag{4}$$

under three-point integral hybrid boundary value conditions for $z \in [0, 1]$, $1 < \zeta \leq 2$, $\mathfrak{g} \in C([0, 1] \times \mathbb{R}^{n+1})$ and $\Theta : [0, 1] \times \mathbb{R}^{m+1} \to P(\mathbb{R})$ is a set-valued map via some certain properties [30]. In 2018, Mert *et al.* used a nabla fractional difference integration to formulate a nabla discrete fractional Sturm-Liouville equation through left and right Caputo and Riemann fractional differences and continued to study the new formulated Sturm Liouville problem by making use of a discrete fractional isoperimetric variational problem [31]. The authors investigated the following fractional neutral inclusion

$$^{C}\mathcal{D}^{\zeta}\left[\varrho(z)\theta\left(z, \varrho(z)\right)\right] \in \uptheta_1\left(\varrho(z)\right) + \uptheta_2\left(\varrho(z)\right), \tag{5}$$

a.e. $z \in [0, \mathfrak{s}]$, $\mathfrak{s}$ is positive in nature, $0 < \zeta \leq 1$, $\uptheta_1$ represents the infinitesimal generator of a C_0-strongly continuous semigroup $T(z)$, $z \geq 0$, defined from $\uptheta_1 : D(\uptheta_1) \to Y$, $\varrho(\cdot)$ assumes values in the Banach space Y, ϱ_0 is for an element of Y, $\uptheta_2 : [0, \mathfrak{s}]\times Y \to Y$ denotes a multivalued map, $\theta : [0, \mathfrak{s}] \times Y \to Y$ is equicontinuous and bounded [32]. In [33] Nazir *et al.* investigated the system with nonlinear boundary value problems for sequential HFDEs as

$$\begin{cases} ^{C}\mathcal{D}^{\zeta_1}\left[\frac{^{C}\mathcal{D}^{\zeta_2}\varrho(z)\sum_{i=1}^{m}\mathcal{I}^{\eta_i}\vartheta_i(z,\varrho(z))}{\hbar_1(z,\varrho(z))}\right] = \theta_1\left(z, \grave{\varrho}(z), \mathcal{I}^{\gamma}\grave{\varrho}(z)\right), \\ ^{C}\mathcal{D}^{\zeta_1}\left[\frac{^{C}\mathcal{D}^{\zeta_2}\acute{\varrho}(z)\sum_{i=1}^{m}\mathcal{I}^{\eta_i}\vartheta_i(z,\acute{\varrho}(z))}{\hbar_2(z,\acute{\varrho}(z))}\right] = \theta_2\left(z, \varrho(z), \mathcal{I}^{\gamma}\varrho(z)\right), \end{cases} \tag{6}$$

via conditions $^{C}\mathcal{D}^{\zeta_1}\varrho(0) = {}^{C}\mathcal{D}^{\zeta_1}\acute{\varrho}(0) = 0$ and

$$\delta_1\varrho(1) + \grave{\delta}_1\grave{\varrho}(\upchi_{11}) = \upsigma_1(\varrho(\upchi_{12})), \qquad \delta_2\grave{\varrho}(1) + \grave{\delta}_2\varrho(\upchi_{21}) = \upsigma_2(\grave{\varrho}(\upchi_{22})), \tag{7}$$

where $0 < \zeta_1, \zeta_2 \leq 1$, $1 < \zeta_1 + \zeta_2 \leq 2$, $0 < \chi_{ij} < 1$ and $\mathcal{I}^{\gamma}$ is the Riemann-Liouville (RL) fractional integral order γ, $\delta_1, \grave{\delta}_1$, $\delta_2 \cdot \grave{\delta}_2 > 0$, $\hbar_i \in C(\mathrm{J} \times \mathbb{R})$, $\theta_i \in C(\mathrm{J} \times \mathbb{R}^2)$, $i, j = 1, 2$. In [34], Devi *et al.* deal with the existence and uniqueness of solutions for nonlinear Langevin $\mathbb{FDE}$ having fractional derivative of different orders with nonlocal integral

$$(8) \qquad {}^{C}\mathcal{D}^{\zeta_1}\left[{}^{C}\mathcal{D}^{\zeta_2} + \varkappa\right]\varrho(z) = \theta\left(z, \varrho(z), \mathcal{D}^{\zeta_3}\varrho(z)\right), \qquad z \in [0, 1],$$

and anti-periodic-type boundary conditions $\varrho(0) = 0$, $\mathcal{D}^{\zeta_2}\varrho(0) = 0$,

$$(9) \qquad \mathcal{D}^{\zeta_2}\varrho(\eta_1) + \mathcal{D}^{\zeta_2}\varrho(\eta_2) = \alpha\mathcal{I}^{\zeta_3}\varrho(\vartheta),$$

and investigated the Ulam-Hyres stability (UH) of solutions where and $\mathcal{D}^{\zeta_3}$ is the RL derivative of fractional order ζ_3, $\theta \in C\left([0, 1] \times \mathbb{R}^2\right)$, $\varkappa > 0$, $1 < \zeta_1 + \zeta_2 \leq 2$, $0 < \eta_1, \eta_2, \vartheta < 1$, $\alpha \in \mathbb{R}$. The existence result is derived by applying Krasnoselskii's fixed point theorem (FPT) and the uniqueness of result is established by applying Banach contraction mapping principle. An example is offered to ensure the validity of our obtained results. The authors in [35] discussed about two nonlinear $\mathbb{FD}$ hybrid systems subjected to periodic boundary conditions are given by

$$(10) \qquad {}^{C}\mathcal{D}^{\zeta,\psi}_{\mathfrak{s}_1^+}\left[\varrho(z)\theta_1\left(z, \varrho(z)\right)\right] = \theta_2\left(z, \varrho(z)\right), \qquad \zeta \in (0, 1),$$

with $\varrho(\mathfrak{s}_1) = \varrho(\mathfrak{s}_2)$ and

$$(11) \qquad {}^{C}\mathcal{D}^{\zeta,\psi}_{\mathfrak{s}_1^+}\left[\varrho(z)\theta_1\left(z, \varrho(z)\right)\right] = \theta_2\left(z, \varrho(z)\right), \qquad \zeta \in (1, 2),$$

with $\varrho(\mathfrak{s}_1) = \varrho(\mathfrak{s}_2)$ and $\varrho'(\mathfrak{s}_1) = \varrho'(\mathfrak{s}_2)$ where $z \in \mathrm{J} := [\mathfrak{s}_1, \mathfrak{s}_2]$, ${}^{C}\mathcal{D}^{\zeta,\psi}_{\mathfrak{s}_1^+}$ is the ψ-Caputo fractional derivative $\theta_1 : \mathrm{J} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ and $\theta_2 \in (\mathrm{J} \times \mathbb{R})$ with θ_1 and θ_2 are identically zero at the origin and $\theta_2(z, 0) = 0$. Lachouri *et al.* proved the existence of solutions for the following nonlinear $\mathbb{HFDE}$ in the frame of ψ-Hilfer fractional derivative

$$(12) \qquad {}^{H}\mathcal{D}^{\zeta_1,\zeta_2,\psi}_{\mathfrak{s}_1^+}\varrho(z) \in \theta\left(\varrho(z), \varrho(z)\right), \qquad z \in (\mathfrak{s}_1, \mathfrak{s}_2), \mathfrak{s}_1 > 0,$$

with nonlocal integral boundary conditions $\varrho(\mathfrak{s}_1) = 0$, and

$$(13) \qquad \varrho(\mathfrak{s}_2) = \sum_{i=1}^{m} \delta_i\, \mathcal{I}^{\eta_i,\psi}_{\mathfrak{s}_1^+}\varrho(\chi_i),$$

where derivative order $1 < \zeta_1 < 2$ and type $0 \leq \zeta_2 \leq 1$, $\mathcal{I}^{\eta_i,\psi}_{\mathfrak{s}_1^+}\varrho(\chi_i)$ is the ψ-RL fractional integral of order $\eta_i > 0$, θ is a set-valued map from $[\mathfrak{s}_1, \mathfrak{s}_2] \times \mathbb{R}$ to the collection of $\mathcal{P}(\mathbb{R}) \subset \mathbb{R}$, $-\infty < \mathfrak{s}_1 < \mathfrak{s}_2 < \infty$, $\delta_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$, $0 \leq \mathfrak{s}_1 \leq \chi_1 < \chi_2 < \cdots < \chi_m \leq \mathfrak{s}_2$ [36].

This paper deals with the existence of solutions for the following boundary value problem to the nonlinear hybrid Sturm-Liouville and Langevin $\mathbb{FDI}$,

$$(14) \qquad {}^{\psi}_{\mathrm{H}}\mathcal{D}^{\epsilon_1,\zeta}_{\mathfrak{s}_1}\left[\mathrm{p}(z)\, {}^{\psi}_{\mathrm{H}}\mathcal{D}^{\epsilon_2,\zeta}_{\mathfrak{s}_1}\left[\frac{\varrho(z)}{\omega(z,\varrho(z))}\right] - \mathrm{q}(z)\varrho(z)\right] \in \Theta\left(z, \varrho(z)\right),$$

$\forall z \in \mathrm{J} := [\mathfrak{s}_1, \mathfrak{s}_2]$, with hybrid boundary conditions

$$(15) \qquad \begin{cases} \varrho(\mathfrak{s}_1) = 0, \\ \mathrm{p}(\mathfrak{s}_2)\, {}^{\psi}_{\mathrm{H}}\mathcal{D}^{\epsilon_2,\zeta}_{\mathfrak{s}_1}\left[\frac{\varrho(z)}{\omega(z,\varrho(z))}\right]_{z=\mathfrak{s}_2} - \mathrm{q}(\mathfrak{s}_2)\varrho(\mathfrak{s}_2) = 0, \end{cases}$$

where ${}^{\psi}_{\mathrm{H}}\mathcal{D}^{\sigma,\zeta}_{\mathfrak{s}_1}$ is the ψ-Hilfer fractional derivative of order $\sigma = \{\epsilon_1, \epsilon_2\} \in \Delta := (0, 1)$, type $\zeta \in \overline{\Delta} := [0, 1]$ and $\psi \in \mathcal{C}(\mathrm{J})$ s.t ψ is increasing and $\psi'(z) \neq 0$ for all $z \in \mathrm{J}$. Also, ω :

$\mathrm{J}\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$, $\mathrm{p},\mathrm{q}\in C(\mathrm{J})$ and $\Theta:\mathrm{J}\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ stands for a multi-valued map via some specific properties. In the sequel, by applying some notions of functional analysis, we study the solutions' existence for the nonlinear hybrid Langevin and Sturm-Liouville $\mathbb{FDE}$,

$$ {}^{\psi}_{\mathrm{H}}\mathcal{D}^{\epsilon_1,\zeta}_{\mathfrak{s}_1}\left[\mathrm{p}(z){}^{\psi}_{\mathrm{H}}\mathcal{D}^{\epsilon_2,\zeta}_{\mathfrak{s}_1}\left[\frac{\varrho(z)}{\omega(z,\varrho(z))}\right]-\mathrm{q}(z)\varrho(z)\right]=\Theta\left(z,\varrho(z)\right), \tag{16}$$

$\forall z\in\mathrm{J}$, with the conditions (15). In fact, the single-valued problem (16) is a special case of the first multi-valued fractional boundary value problem (14) by assuming $\Theta(z)=\{\Theta_{\varrho}(z)\}$, where $\Theta_{\varrho}(z)=\Theta(z,\varrho(z))$ is continuous.

Remark 1.1. *Our results for the problem (16) remain true for the following cases:*

- RL *type problem:* $\psi(z)=z$ *and* $\zeta=0$.
- *Caputo type problem:* $\psi(z)=z$ *and* $\zeta=1$.
- ψ-RL *type problem:* $\zeta=0$.
- ψ-*Caputo type problem:* $\zeta=1$.
- *Hilfer type problem:* $\psi(z)=z$.
- *Hilfer-Hadamard type problem:* $\psi(z)=\log z$.
- ψ-*Hilfer type problem:* $\psi(z)=z^{\rho}$, $\rho>0$.

A brief outline of the paper is as follows: Section 2 provides the definitions and preliminary facts that we need to our analysis. In Section 3, we prove the existence for the problem (14), and the existence, uniqueness and UH stability results for the problem (16) are also investigated. Three examples are given in Section 3.3 which show our results better. This work is completed with a conclusion.

2. Preliminaries

We consider the Banach space $\widehat{\mathcal{C}}:=C(\mathrm{J})$ with the norm $\|\varrho\|_{\infty}=\sup\{|\varrho(z)\ :\ z\in\mathrm{J}\}$, and $\psi,\varrho\in\widehat{\mathcal{C}}^{n}$ s.t ψ is increasing and $\psi'(z)\neq0$ for all $z\in\mathrm{J}$. The ψ-Hilfer fractional derivative of a function ϱ of order $\epsilon\in\Delta$ and type $\zeta\in\overline{\Delta}$ is expressed by [7],

$$ {}^{\psi}\mathcal{D}^{\epsilon,\zeta}_{\mathfrak{s}_1^+}\varrho(z)={}^{\psi}\mathcal{I}^{\zeta(1-\epsilon)}_{\mathfrak{s}_1^+}\ D_{\psi}\ {}^{\psi}\mathcal{I}^{(1-\zeta)(1-\epsilon)}_{\mathfrak{s}_1^+}\varrho(z),\quad D_{\psi}=\left(\frac{1}{\psi'(z)}\frac{\mathrm{d}}{\mathrm{d}z}\right). \tag{17}$$

The relation (17) can be written as

$$ {}^{\psi}\mathcal{D}^{\epsilon,\zeta}_{\mathfrak{s}_1}\varrho(z)={}^{\psi}\mathcal{I}^{\gamma-\epsilon}_{\mathfrak{s}_1^+}\ {}^{\psi}_{\mathrm{RL}}\mathcal{D}^{\gamma}_{\mathfrak{s}_1^+}\varrho(z), \tag{18}$$

with $\gamma=\epsilon+\zeta\left(1-\epsilon\right)$, ${}^{\psi}\mathcal{I}^{\gamma-\epsilon}_{\mathfrak{s}_1^+}$ and ${}^{\psi}_{\mathrm{RL}}\mathcal{D}^{\gamma}_{\mathfrak{s}_1}$ are called the ψ-RL fractional integral and derivative defined by [1],

$$ {}^{\psi}\mathcal{I}^{\epsilon}_{\mathfrak{s}_1^+}\varrho(z)=\int_{\mathfrak{s}_1}^{z}\frac{\psi'(\eta)(\psi(z)-\psi(\eta))^{\epsilon-1}}{\Gamma(\epsilon)}\varrho\left(\eta\right)\,\mathrm{d}\eta, \tag{19}$$

and ${}^{\psi}_{\mathrm{RL}}\mathcal{D}^{\epsilon}_{\mathfrak{s}_1}\varrho(z)=D_{\psi}\ {}^{\psi}\mathcal{I}^{1-\epsilon}_{\mathfrak{s}_1^+}\varrho(z)$, respectively. The ψ-Caputo fractional derivative is given by [6],

$$ {}^{\psi}_{\mathrm{C}}\mathcal{D}^{\epsilon}_{\mathfrak{s}_1}\varrho(z)={}^{\psi}\mathcal{I}^{1-\epsilon}_{\mathfrak{s}_1^+}\ D^{n}_{\psi}\varrho(z). \tag{20}$$

The classical fractional operators were introduced in [1, 5, 8], will obtain whenever we put $\psi(z)=z$ in relations (17), (19), and (42),

Lemma 2.1 ([1,7]). *Let* $\epsilon > 0$, $\zeta > 0$ *and* $\varrho \in \widehat{\mathcal{C}}^n$. *Then* ${}^{\psi}\mathcal{D}^{\epsilon}_{\mathfrak{s}_1}\, {}^{\psi}\mathcal{I}^{\epsilon}_{\mathfrak{s}_1^+}\, \varrho(z) = \varrho(z)$ *and*

$$ {}^{\psi}\mathcal{I}^{\epsilon}_{\mathfrak{s}_1^+}\, {}^{\psi}\mathcal{I}^{\zeta}_{\mathfrak{s}_1^+}\, \varrho(z) = {}^{\psi}\mathcal{I}^{\epsilon+\zeta}_{\mathfrak{s}_1^+}\varrho(z). \tag{21} $$

Lemma 2.2 ([1,7]). *Let* $\epsilon, \zeta > 0$. *If* $h_{\mathfrak{s}_1}(z) := (\psi(z) - \psi(\mathfrak{s}_1))^{\sigma-1}$, $(\sigma > 0)$ *then*

$$ {}^{\psi}\mathcal{I}^{\epsilon}_{\mathfrak{s}_1^+} h_{\mathfrak{s}_1}(z) = \tfrac{\Gamma(\sigma)}{\Gamma(\epsilon+\sigma)} (\psi(z) - \psi(\mathfrak{s}_1))^{\epsilon+\sigma-1}, \tag{22} $$

$$ {}^{\psi}\mathcal{D}^{\epsilon,\zeta}_{\mathfrak{s}_1} h_{\mathfrak{s}_1}(z) = \tfrac{\Gamma(\sigma)}{\Gamma(\sigma-\epsilon)} (\psi(z) - \psi(\mathfrak{s}_1))^{\epsilon-\sigma-1}, \tag{23} $$

whenever $\sigma > \gamma = \epsilon + \zeta(1-\epsilon)$, ${}^{\psi}\mathcal{D}^{\epsilon,\zeta}_{\mathfrak{s}_1} h_{\mathfrak{s}_1}(z) = 0$, *whenever* $\sigma = \gamma$.

Lemma 2.3 ([7]). *Let* $0 < \epsilon < 1$ *and* $\omega \in \widehat{\mathcal{C}}^1$, *then for* $z \in \mathrm{J}$,

$$ \left({}^{\psi}\mathcal{I}^{\epsilon}_{\mathfrak{s}_1^+}\, {}^{\psi}_{\mathrm{H}}\mathcal{D}^{\epsilon,\zeta}_{\mathfrak{s}_1}\, \omega\right)(z) = \omega(z) - \tfrac{1}{\Gamma(\epsilon)} \left({}^{\psi}\mathcal{I}^{1-\epsilon}_{\mathfrak{s}_1^+}\, \omega\right)(\mathfrak{s}_1)(\psi(z) - \psi(\mathfrak{s}_1))^{\epsilon-1}. \tag{24} $$

Lemma 2.4 ([1]). *Let* $\epsilon \in \Delta$ *and* $\varrho \in \widehat{\mathcal{C}}$. *Then*

$$ \left({}^{\psi}\mathcal{I}^{\epsilon}_{\mathfrak{s}_1^+}\, \varrho\right)(\mathfrak{s}_1) = \lim_{z \to \mathfrak{s}_1} \left({}^{\psi}\mathcal{I}^{\epsilon}_{\mathfrak{s}_1^+}\, \varrho\right)(z) = 0. \tag{25} $$

Assume that $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is a normed space. An element $\varrho \in \mathcal{Y}$ is a fixed point of the set-valued map $\Theta : \mathcal{Y} \to \mathcal{P}(\mathcal{Y})$ if $\varrho \in \Theta(\varrho)$ [37]. The Pompeiu-Hausdorff metric $\mathrm{H}_{\mathrm{d}_{\mathcal{Y}}}$ is defined by [37],

$$ \begin{cases} \mathrm{H}_{\mathrm{d}_{\mathcal{Y}}} : \mathcal{P}(\mathcal{Y}) \times \mathcal{P}(\mathcal{Y}) \to \mathbb{R} \cup \{\infty\}, \\ \mathrm{H}_{\mathrm{d}_{\mathcal{Y}}}(U_1, U_2) = \max\Big\{ \sup\limits_{\mathfrak{u}_1 \in U_1} \mathrm{d}_{\mathcal{Y}}(\mathfrak{u}_1, U_2), \sup\limits_{\mathfrak{u}_2 \in U_2} \mathrm{d}_{\mathcal{Y}}(U_1, \mathfrak{u}_2) \Big\}, \end{cases} \tag{26} $$

where $\mathrm{d}_{\mathcal{Y}}(U_1, \mathfrak{u}_2) = \inf_{\mathfrak{u}_1 \in U_1} \mathrm{d}_{\mathcal{Y}}(\mathfrak{u}_1, \mathfrak{u}_2)$ and $\mathrm{d}_{\mathcal{Y}}(\mathfrak{u}_1, U_2) = \inf_{\mathfrak{u}_2 \in U_2} \mathrm{d}_{\mathcal{Y}}(\mathfrak{u}_1, \mathfrak{u}_2)$. A set of selections of Θ at an element $\varrho \in C(\Delta)$, is defined by

$$ S_{\Theta,\varrho} := \Big\{ \vartheta \in \mathcal{L}^1(\Delta, \mathbb{R}) \,:\, \vartheta(z) \in \Theta(z, \varrho(z)) \Big\}, \tag{27} $$

for almost all $z \in \Delta$ [37, 38]. If Θ is an arbitrary set-valued map, then for each function $\varrho \in C(\Delta, \mathcal{Y})$, we have $S_{\Theta,\varrho} \neq \emptyset$ whenever $\dim \mathcal{Y} < \infty$ [37].

Theorem 2.5 ([39]). *Let* $\mathcal{Y}$ *be a separable Banach space,* $\Theta : \Delta \times \mathcal{Y} \to \mathcal{P}_{cp,cv}(\mathcal{Y})$ *an* $\mathcal{L}^1$-*Carathéodory and* $\Psi : \mathcal{L}^1(\Delta, \mathcal{Y}) \to C(\Delta, \mathcal{Y})$ *a continuous linear mapping. Then*

$$ \Psi \circ S_{\Theta} : C(\Delta, \mathcal{Y}) \to \mathcal{P}_{cp,cv}(C(\Delta, \mathcal{Y})), \tag{28} $$

as an operator in $C(\Delta, \mathcal{Y}) \times C(\Delta, \mathcal{Y})$ *with action* $\varrho \mapsto (\Psi \circ S_{\Theta})(\varrho) = \Psi(S_{\Theta,\varrho})$, *admits a closed graph, where* $\mathcal{P}_{cp}(\mathcal{Y})$ *and* $\mathcal{P}_{cv}(\mathcal{Y})$ *are the set of all compact subsets and the set of all convex subsets of* $\mathcal{Y}$, *respectively.*

Theorem 2.6 ([40]). *Let* $\mathcal{Y}$ *be a Banach algebra and* $\theta : \mathcal{Y} \to \mathcal{Y}$ *and* $\widehat{\Theta} : \mathcal{Y} \to \mathcal{P}_{cp,cv}(\mathcal{Y})$ *be provided that* (i) θ *is Lipschitz with* $\mathring{\eta} > 0$; (ii) $\widehat{\Theta}$ *is upper semi-continuous and compact;* (iii) $2\mathring{\eta}\mathring{M} < 1$, *s.t.* $\mathring{M} = \|\widehat{\Theta}(\mathcal{Y})\|$. *Then either a solution exists for* $\varrho \in (\theta\varrho)(\widehat{\Theta}\varrho)$ *or*

$$ \mathcal{O} = \Big\{ \varrho^* \in \mathcal{Y} \,:\, a\varrho^* \in (\theta\varrho^*)\left(\widehat{\Theta}\varrho^*\right),\, a > 1 \Big\}, \tag{29} $$

is unbounded.

To conclude this section, we display the following FPTs. The following definition will be used in the sequel.

Definition 2.7 ([41, 42]). *A self-operator $\widehat{\omega}$ on a Banach space $\widehat{\mathcal{C}}$ is announced as Lipschitz if exists $\mathring{\eta}_{\widehat{\omega}} > 0$ satisfying*

$$\|\widehat{\omega}(\varrho) - \widehat{\omega}(\acute{\varrho})\| \leq \mathring{\eta}_{\widehat{\omega}} \|\varrho - \acute{\varrho}\|, \qquad \varrho, \acute{\varrho} \in \widehat{\mathcal{C}}. \tag{30}$$

We should make utilize of the hybrid fixed point result ascribed to Dhage [41, 43] and contraction vital credited to Banach as a principal device for illustrating the existence-uniqueness result of the coupled solutions of the proposed system given in this paper.

Theorem 2.8 ([41, 43]). *Let $C \subset \widehat{\mathcal{C}}$ be a convex, bounded and closed set and the operators $\Theta_1, \Theta_2 : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$, $\Theta_3 : C \to \widehat{\mathcal{C}}$ be so that*

(s1) *Θ_1, Θ_2 are Lipschitz maps with Lipschitz constants $\mathring{\eta}_{\Theta_1}$, $\mathring{\eta}_{\Theta_2}$, respectively;*
(s2) *Θ_3 is continuous and compact;*
(s3) *$\varrho = \Theta_1 \varrho\, \Theta_3 \acute{\varrho} + \Theta_2 \varrho$, for all $\acute{\varrho} \in C$ yields $\varrho \in C$;*
(s4) *$\mathring{\eta}_{\Theta_1} \mathring{M}_{\Theta_3} + \mathring{\eta}_{\Theta_2} < 1$, where $\mathring{M}_{\Theta_3} = \|\Theta_3(C)\| = \sup\{\|\Theta_3 \varrho\| : \varrho \in C\}$.*

Then the operator equation $\varrho = \Theta_1 \varrho\, \Theta_3 \varrho + \Theta_2 \varrho$ possesses a solution in C.

Theorem 2.9 ([44]). *A contraction mapping $\Theta : C \to C$ possesses a unique fixed point whenever the nonempty set $C \subset \widehat{\mathcal{C}}$ be closed.*

3. Results for the single-valued $\mathbb{FDE}$ (16)

In this section, we are concerned with the existence, uniqueness and UH stability of solutions of problem (16). First, we state and prove the key lemma 3.1.

Lemma 3.1. *Let $\wp \in \widehat{\mathcal{C}}$ be given function. Then, the following generalization of hybrid Langevin and Sturm-Liouville $\mathbb{FDE}$*

$$ {}^{\psi}_{\mathrm{H}}\mathcal{D}^{\epsilon_1,\zeta}_{\mathfrak{s}_1}\left[\mathrm{p}(z)\, {}^{\psi}_{\mathrm{H}}\mathcal{D}^{\epsilon_2,\zeta}_{\mathfrak{s}_1}\left[\frac{\varrho(z)}{\omega(z,\varrho(z))}\right] - \mathrm{q}(z)\varrho(z)\right] = \wp\left(z\right), \qquad z \in \mathrm{J}, \tag{31}$$

with hybrid boundary conditions $\varrho(\mathfrak{s}_1) = 0$ and

$$\mathrm{p}(\mathfrak{s}_2)\, {}^{\psi}_{\mathrm{H}}\mathcal{D}^{\epsilon_2,\zeta}_{\mathfrak{s}_1}\left[\frac{\varrho(z)}{\omega(z,\varrho(z))}\right]_{z=\mathfrak{s}_2} - \mathrm{q}(\mathfrak{s}_2)\varrho(\mathfrak{s}_2) = 0, \tag{32}$$

is equivalent to the integral equation

$$\begin{aligned}\varrho(z) =\, &\omega(z,\varrho(z))\left\{ {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\frac{1}{\mathrm{p}}\, {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\, \Theta\left(z,\varrho(z)\right)\right) - {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\frac{\mathrm{q}}{\mathrm{p}}\varrho\right)(z) \right.\\ &\left. - \frac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)} \frac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)}\, {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\Theta\left(\mathfrak{s}_2,\varrho(\mathfrak{s}_2)\right)\left(\psi(z)-\psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1}\right\},\end{aligned} \tag{33}$$

where Θ is defined in (14) and

$$\gamma = \left(\epsilon_1 + \epsilon_2\right)\left(1-\zeta\right) + \zeta. \tag{34}$$

Proof. Taking the ψ-Riemman-Liouville fractional integral of orders ϵ_1 into the equation (31), we have

$$\mathrm{p}(z)\, {}^{\psi}_{\mathrm{H}}\mathcal{D}^{\epsilon_2,\zeta}_{\mathfrak{s}_1}\left[\frac{\varrho(z)}{\omega(z,\varrho(z))}\right] - \mathrm{q}(z)\varrho(z) = {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\wp(z) + \frac{k_1}{\Gamma(\gamma)}\left(\psi(z)-\psi(\mathfrak{s}_1)\right)^{\gamma-1}, \tag{35}$$

where $k_1 \in \mathbb{R}$. The boundary conditions (31) implies that

$$k_1 = -\Gamma(\gamma)\left(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1)\right)^{1-\gamma}\, {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\wp(\mathfrak{s}_2). \tag{36}$$

Taking the ψ-Riemman-Liouville fractional integral of orders ϵ_2 to (35), we get

$$\begin{aligned}\left[\frac{\varrho(z)}{\omega(z,\varrho(z))}\right] =& \Bigg\{ {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\frac{1}{\mathrm{p}}\,{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\wp\right)(z) - {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\frac{\mathrm{q}}{\mathrm{p}}\varrho\right)(z) \\ & - \frac{(\Gamma(\gamma))^2(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\Gamma(\epsilon_2+\gamma)\mathrm{p}(z)}\,{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\wp(\mathfrak{s}_2)\Big(\psi(z)-\psi(\mathfrak{s}_1)\Big)^{\gamma+\epsilon_2-1}\Bigg\} \\ & + \frac{k_2}{\Gamma(\gamma)}\left(\psi(z)-\psi(\mathfrak{s}_1)\right)^{\gamma-1}, \end{aligned} \tag{37}$$

where $k_2 \in \mathbb{R}$. Thanks to boundary conditions (31), $k_2 = 0$. On the other hand, if we take the ψ-Hilfer derivative of (33) and use Lemma 2.3, then it is easy to obtain problem (31). ☐

Before introducing the main results, we investigate Equation (33) under the following assumptions:

H1) There exists $\mathring{M}_\omega, \mathring{M}_\Theta > 0$ s.t $|\omega(z,\varrho)| \le \mathring{M}_\omega$ and $|\Theta(z,\varrho)| \le \mathring{M}_\Theta$, for each $z \in \mathrm{J}$, $\varrho \in \mathbb{R}$;

H2) There exists $\mathring{\eta}_\omega > 0$, $\mathring{\eta}_\Theta > 0$ s.t

$$|\omega(z,\varrho)-\omega(z,\acute{\varrho})| \le \mathring{\eta}_\omega|\varrho-\acute{\varrho}|, \qquad |\Theta(z,\varrho)-\Theta(z,\acute{\varrho})| \le \mathring{\eta}_\Theta|\varrho-\acute{\varrho}|, \tag{38}$$

for $(z,\{\varrho,\acute{\varrho}\}) \in \mathrm{J}\times\mathbb{R}$;

H3) There exists $\mathring{\hbar} \in L^\infty(\mathrm{J},\mathbb{R}^+)$ and a continuous nondecreasing function $\mathring{\varphi} : [0,\infty) \to (0,\infty)$ s.t

$$|\Theta(z,\varrho)| \le \mathring{\hbar}(z)\mathring{\varphi}\left(|\varrho|\right), \qquad z \in \mathrm{J},\ \varrho \in \mathbb{R}; \tag{39}$$

H4) The constants in the hypotheses (H1)–(H3) obey the following assertions

$$\kappa \ge \left[\mathring{\eta}_\omega\kappa + \omega_\circ\right]\left(\mathcal{A}+\mathcal{B}\kappa\right), \tag{40}$$

where $\omega_\circ = \sup\left\{|\omega(z,0)| \,:\, z \in \mathrm{J}\right\}$ and

$$\begin{aligned}\mathcal{A} &= \frac{\|\mathring{\hbar}\|\mathring{\varphi}(\kappa)}{\mathrm{p}^*}\left\{\frac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_1+\epsilon_2}}{\Gamma(\epsilon_1+\epsilon_2+1)} + \frac{\Gamma(\gamma)(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_1+\epsilon_2}}{\Gamma(\gamma+\epsilon_2)\Gamma(\epsilon_1+1)}\right\}, \\ \mathcal{B} &= \frac{\mathrm{q}^*}{\mathrm{p}^*}\frac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_2}}{\Gamma(\epsilon_2+1)}, \qquad \mathrm{p}^* = \sup_{z\in\mathrm{J}}\mathrm{p}(z),\ \mathrm{q}^* = \sup_{z\in\mathrm{J}}\mathrm{q}(z). \end{aligned} \tag{41}$$

3.1. Existence results via hybrid FPT of Dhage.

Definition 3.2. *A solution $\varrho \in \widehat{\mathcal{C}}$ of the hybrid fractional integral equation (19) is called a solution of the $\mathbb{FDE}$ (16) defined on* J.

Theorem 3.3. *Let the hypotheses* (H1)-(H3) *hold. Furthermore, if $\mathring{\eta}_\omega\mathring{M}_{\widehat{\Theta}} < 1$, here $\mathring{M}_{\widehat{\Theta}} = \mathcal{A}+\mathcal{B}\kappa$, then the boundary value problem to hybrid Langevin and Sturm-Liouville $\mathbb{FDE}$ (16) has a least one solution defined on* J.

Proof. By Lemma 3.1, the solution of the boundary value problem to the hybrid Langevin and Sturm-Liouville $\mathbb{FDE}$ in (16) is the solution to the fractional integral equation,

$$\begin{aligned}\varrho(z) =& \omega(z,\varrho(z))\Bigg\{ {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\frac{1}{\mathrm{p}}{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\wp\right)(z) - {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\frac{\mathrm{q}}{\mathrm{p}}\varrho\right)(z) \\ & - \frac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)}\frac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)}\,{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\wp(\mathfrak{s}_2)\left(\psi(z)-\psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1}\Bigg\}, \end{aligned} \tag{42}$$

where $\wp \in \widehat{\mathcal{C}}$. Clearly, $\widehat{\mathrm{B}}_\kappa$ is a closed, convex and bounded subset of the Banach space $\widehat{\mathcal{C}}$. Define the operators $\widehat{\omega} : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$, and $\widehat{\Theta} : \widehat{\mathrm{B}}_\kappa \to \widehat{\mathcal{C}}$ by

$$(\widehat{\omega}\varrho)(z) = \omega(z,\varrho(z)),$$

$$(\widehat{\Theta}\varrho)(z) = \Big\{ {}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_2}\left(\tfrac{1}{\mathrm{p}}{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_1}\wp\right)(z) - {}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_2}\left(\tfrac{\mathrm{q}}{\mathrm{p}}\varrho\right)(z)$$

$$- \tfrac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)}\tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)}{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_1}\wp(\mathfrak{s}_2)\left(\psi(z)-\psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1}\Big\}, \tag{43}$$

where $\widehat{\mathrm{B}}_\kappa = \{\varrho \in \widehat{\mathcal{C}} : \|\varrho\| \leq \kappa\}$. Then the coupled system of hybrid fractional integral equation (42) can be written as the system of operator equations as $\widehat{\omega}\varrho(z)\,\widehat{\Theta}\varrho(z) = \varrho(z)$, for $z \in \mathrm{J}$. Now we continue the prove in several cases.

Step I: First we show that $\widehat{\omega}$ is lipschitzian on $\widehat{\mathcal{C}}$ with Lipschitz constants $\mathring{\eta}_\omega$. Let $\varrho, \acute{\varrho} \in \widehat{\mathcal{C}}$ be arbitrary. Then, using (H4), for $z \in \mathrm{J}$, we have

$$|(\widehat{\omega}\varrho)(z) - (\widehat{\omega}\acute{\varrho})(z)| = |\omega(z,\varrho(z)) - \omega(z,\acute{\varrho}(z))| \leq \mathring{\eta}_\omega|\varrho(z)-\acute{\varrho}(z)| \leq \mathring{\eta}_\omega\|\varrho-\acute{\varrho}\|. \tag{44}$$

Taking the supremum over z, we obtain $\|\widehat{\omega}\varrho - \widehat{\omega}\acute{\varrho}\| \leq \mathring{\eta}_\omega\|\varrho-\acute{\varrho}\|$, for all $\varrho, \acute{\varrho} \in \widehat{\mathcal{C}}$, that is, $\widehat{\omega}$ is a Lipschitzian with Lipschitz constant $\mathring{\eta}_\omega$.

Step II: Now we show that $\widehat{\Theta}$ is compact and continuous operator from $\widehat{\mathrm{B}}_\kappa$ into $\widehat{\mathcal{C}}$. For continuity of $\widehat{\mathrm{B}}_\kappa$, let a sequence $(\varrho_n) \subseteq \widehat{\mathrm{B}}_\kappa$ converging to a point $\varrho \in \widehat{\mathrm{B}}_\kappa$. Then, Lebesgue Dominated Convergence Theorem implies that

$$\begin{aligned}
\lim_{n\to\infty}\widehat{\Theta}(\varrho_n)(z) &= \lim_{n\to\infty}\Big\{{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_2}\left(\tfrac{1}{\mathrm{p}}{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_1}\Theta(z,\varrho_n(z))\right) - {}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_2}\left(\tfrac{\mathrm{q}}{\mathrm{p}}\varrho_n\right)(z)\\
&\quad - \tfrac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)}\tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)}{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_1}\Theta(\mathfrak{s}_2,\varrho_n(\mathfrak{s}_2))\left(\psi(z)-\psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1}\Big\}\\
&= {}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_2}\left(\tfrac{1}{\mathrm{p}}{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_1}\lim_{n\to\infty}\Theta(z,\varrho_n(z))\right) - {}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_2}\left(\tfrac{\mathrm{q}}{\mathrm{p}}\lim_{n\to\infty}\varrho_n\right)(z)\\
&\quad - \tfrac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)}\tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)}{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_1}\lim_{n\to\infty}\Theta(\mathfrak{s}_2,\varrho_n(\mathfrak{s}_2))\left(\psi(z)-\psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1}\\
&= {}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_2}\left(\tfrac{1}{\mathrm{p}}{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_1}\Theta(z,\varrho(z))\right) - {}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_2}\left(\tfrac{\mathrm{q}}{\mathrm{p}}\varrho\right)(z)\\
&\quad - \tfrac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)}\tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)}{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_1}\Theta(\mathfrak{s}_2,\varrho(\mathfrak{s}_2))\left(\psi(z)-\psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1}\\
&= \left(\widehat{\Theta}\varrho\right)(z), \qquad \forall z \in \mathrm{J}.
\end{aligned} \tag{45}$$

Hence $\widehat{\Theta}\varrho_n$ converges to $\widehat{\Theta}\varrho$ pointwise on $\overline{\Delta}$. Next, we show that $\widehat{\Theta}$ is a compact operator on $\widehat{\mathrm{B}}_\kappa$. Let $\varrho \in \widehat{\mathrm{B}}_\kappa$. Then, using (H3), we have

$$\begin{aligned}
\left|\left(\widehat{\Theta}\varrho_n\right)(z)\right| &= \Big|{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_2}\left(\tfrac{1}{\mathrm{p}}{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_1}\Theta(z,\varrho(z))\right) - {}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_2}\left(\tfrac{\mathrm{q}}{\mathrm{p}}\varrho\right)(z)\\
&\quad - \tfrac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)}\tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)}{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_1}\Theta(\mathfrak{s}_2,\varrho(\mathfrak{s}_2))\left(\psi(z)-\psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1}\Big|\\
&\leq {}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_2}\left(\tfrac{1}{\mathrm{p}}{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_1}|\Theta(z,\varrho(z))|\right) - {}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_2}\left(\tfrac{\mathrm{q}^*}{\mathrm{p}^*}|\varrho|\right)(z)\\
&\quad - \tfrac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)}\tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)}{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_1}|\Theta(\mathfrak{s}_2,\varrho_n(\mathfrak{s}_2))|\left(\psi(z)-\psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1}\\
&\leq \tfrac{\|\mathring{\hbar}\|}{\mathrm{p}^*}\tfrac{\mathring{\varphi}(\kappa)}{\Gamma(\epsilon_1+\epsilon_2+1)}\left(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1)\right)^{\epsilon_1+\epsilon_2} + \tfrac{\mathrm{q}^*}{\mathrm{p}^*}\tfrac{\kappa}{\Gamma(\epsilon_2+1)}\left(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1)\right)^{\epsilon_2}\\
&\quad + \tfrac{\|\mathring{\hbar}\|}{\mathrm{p}^*}\tfrac{(\Gamma(\gamma))^2\mathring{\varphi}(\kappa)}{\Gamma(\gamma+\epsilon_2)\Gamma(\epsilon_1+1)}\left(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1)\right)^{\epsilon_1+\epsilon_2}.
\end{aligned} \tag{46}$$

Taking the supremum over z in the inequality (46), we obtain

$$\begin{aligned}
\left\|\left(\widehat{\Theta}\varrho\right)(z)\right\| &\leq \tfrac{\|\mathring{\hbar}\|\mathring{\varphi}(\kappa)}{\mathrm{p}^*}\left\{\tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_1+\epsilon_2}}{\Gamma(\epsilon_1+\epsilon_2+1)} + \tfrac{(\Gamma(\gamma))^2(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_1+\epsilon_2}}{\Gamma(\gamma+\epsilon_2)\Gamma(\epsilon_1+1)}\right\}\\
&\quad + \tfrac{\mathrm{q}^*}{\mathrm{p}^*}\tfrac{\kappa(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_2}}{\Gamma(\epsilon_2+1)} = \mathcal{A} + \mathcal{B}\kappa,
\end{aligned} \tag{47}$$

$\forall \varrho \in \widehat{\mathrm{B}}_{\kappa}$. Hence $\widehat{\Theta}$ is a uniformly bounded operator on $\widehat{\mathrm{B}}_{\kappa}$ with constant $\mathring{M}_{\widehat{\Theta}} = \mathcal{A} + \mathcal{B}\kappa$. We set the notation for the sake of computational convenience, $\forall \chi > 0$,

$$\mathcal{Q}_{\psi}^{\chi}(z, \xi) = \frac{\psi'(\xi)(\psi(z) - \psi(\xi))^{\chi-1}}{\Gamma(\chi)}. \tag{48}$$

Next, to confirm the equicontinuity of $\widehat{\Theta}$. Choose $z_1, z_2 \in \mathrm{J}$, with $z_1 < z_2$ and let $\varrho \in \widehat{\mathrm{B}}_{\kappa}$ be an arbitrary point, then

$$\begin{aligned}
\left|\widehat{\Theta}\varrho\,(t_1) - \widehat{\Theta}\varrho\,(z_2)\right| =& \frac{1}{\mathrm{p}^*}\bigg| \int_{\mathfrak{s}_1}^{z_1} \mathcal{Q}_{\psi}^{\epsilon_1+\epsilon_2}(z_1,\xi)\omega\,(\xi, z(\xi))\,\mathrm{d}\xi \\
& - \int_{\mathfrak{s}_1}^{z_2} \mathcal{Q}_{\psi}^{\epsilon_1+\epsilon_2}(z_2,\xi)\omega\,(\xi, \varrho(\xi))\,\mathrm{d}\xi \bigg| \\
& + \frac{\mathrm{q}^*}{\mathrm{p}^*}\bigg| \int_{\mathfrak{s}_1}^{z_1} \mathcal{Q}_{\psi}^{\epsilon_2}(z_1,\xi)\varrho(\xi)\,\mathrm{d}\xi - \int_{\mathfrak{s}_1}^{z_2} \mathcal{Q}_{\psi}^{\epsilon_1+\epsilon_2}(z_2,\xi)\varrho(\xi)\,\mathrm{d}\xi \bigg| \\
& + \frac{\Gamma(\gamma)}{\mathrm{p}^*}\frac{\Gamma(\gamma)}{\Gamma(\epsilon_2+\gamma)} \left(\psi(\mathfrak{s}_2) - \psi(\mathfrak{s}_1)\right)^{1-\gamma}\, {}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_1} \left|\Theta(\mathfrak{s}_2, \varrho(\mathfrak{s}_2))\right| \\
& \times \left((\psi(z_2) - \psi(\mathfrak{s}_1))^{\gamma+\epsilon_2-1} - (\psi(z_1) - \psi(\mathfrak{s}_1))^{\gamma+\epsilon_2-1} \right) \\
\leq & \frac{1}{\mathrm{p}^*}\bigg| \int_{\mathfrak{s}_1}^{z_2} \left[\mathcal{Q}_{\psi}^{\epsilon_1+\epsilon_2}(z_1,\xi) - \mathcal{Q}_{\psi}^{\epsilon_1+\epsilon_2}(z_2,\xi)\right] \omega\,(\xi, \varrho(\xi))\,\mathrm{d}\xi \bigg| \\
& + \frac{1}{\mathrm{p}^*}\bigg| \int_{z_2}^{z_1} \mathcal{Q}_{\psi}^{\epsilon_1+\epsilon_2}(z_1,\xi)\omega\,(\xi, \varrho(\xi))\,\mathrm{d}\xi \bigg| \\
& + \frac{\mathrm{q}^*}{\mathrm{p}^*}\bigg| \int_{\mathfrak{s}_1}^{z_2} \left[\mathcal{Q}_{\psi}^{\epsilon_2}(z_1,\xi) - \mathcal{Q}_{\psi}^{\epsilon_2}(z_2,\xi)\right] \omega\,(\xi, \varrho(\xi))\,\mathrm{d}\xi \bigg| \\
& + \frac{\mathrm{q}^*}{\mathrm{p}^*}\bigg| \int_{z_2}^{z_1} \mathcal{Q}_{\psi}^{\epsilon_2}(z_1,\xi)\,\omega\,(\xi, \varrho(\xi))\,\mathrm{d}\xi \bigg| \\
& + \frac{\Gamma(\gamma)}{\mathrm{p}^*}\frac{\Gamma(\gamma)}{\Gamma(\epsilon_2+\gamma)} \left(\psi(\mathfrak{s}_2) - \psi(\mathfrak{s}_1)\right)^{1-\gamma}\, {}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_1} \left|\Theta(\mathfrak{s}_2, \varrho(\mathfrak{s}_2))\right| \\
& \times \left((\psi(z_2) - \psi(\mathfrak{s}_1))^{\gamma+\epsilon_2-1} - (\psi(z_1) - \psi(\mathfrak{s}_1))^{\gamma+\epsilon_2-1} \right) \\
\leq & \frac{1}{\mathrm{p}^*}\frac{\varphi(\|\varrho\|)}{\Gamma(\epsilon_1+\epsilon_2+1)} \bigg[\bigg| \left(\psi\,(z_1) - \psi(\mathfrak{s}_1)\right)^{\epsilon_1+\epsilon_2} \\
& - \left(\psi\,(z_2) - \psi(\mathfrak{s}_1)\right)^{\epsilon_1+\epsilon_2} - \left(\psi\,(z_1) - \psi\,(z_2)\right)^{\epsilon_1+\epsilon_2} \bigg| \\
& + \left(\psi\,(z_1) - \psi\,(z_2)\right)^{\epsilon_1+\epsilon_2} \bigg] \\
& + \frac{\mathrm{q}^*}{\mathrm{p}^*}\frac{\|\varrho\|}{\Gamma(\epsilon_2+1)} \bigg[\bigg| \left(\psi\,(Z_1) - \psi(\mathfrak{s}_1)\right)^{\epsilon_2} - \left(\psi\,(z_2) - \psi(\epsilon_1)\right)^{\epsilon_2} \\
& - \left(\psi\,(z_1) - \psi\,(z_2)\right)^{\epsilon_2} \bigg| + \left(\psi\,(Z_1) - \psi\,(z_2)\right)^{\epsilon_2} \bigg] \\
& + \frac{\mathring{\varphi}(\|\varrho\|)}{\mathrm{p}^*}\frac{(\Gamma(\gamma))^2(\psi(\mathfrak{s}_2) - \psi(\mathfrak{s}_1))^{\epsilon_1-\gamma+1}}{\Gamma(\epsilon_1+1)\Gamma(\epsilon_2+\gamma)} \\
& \times \left((\psi(z_2) - \psi(\mathfrak{s}_1))^{\gamma+\epsilon_2-1} - (\psi(z_1) - \psi(\mathfrak{s}_1))^{\gamma+\epsilon_2-1} \right).
\end{aligned}$$

This implies

$$\left|\left(\widehat{\Theta}\varrho\right)(z_1) - \left(\widehat{\Theta}\varrho\right)(z_2)\right| \to 0, \tag{49}$$

as $z_2 \to z_1$ uniformly $\forall\, \varrho \in \widehat{\mathrm{B}}_{\kappa}$. Now, $\widehat{\Theta}\,(\widehat{\mathrm{B}}_{\kappa}) \subset \widehat{\mathcal{C}}$ is uniformly bounded and equicontinuous subset, it is compact subset of $\widehat{\mathcal{C}}$ in view of Arzelà–Ascoli theorem. Consequently, $\widehat{\mathrm{B}}_{\kappa}$ is compact and continuous operator on $\widehat{\mathcal{C}}$.

Step III: Now we prove the third condition (s3) of Theorem 2.8 holds. Let for $\varrho, \acute{\varrho}$ be two elements in $\widehat{\mathcal{C}}$ s.t $\varrho = \widehat{\omega}\varrho\,\widehat{\Theta}\acute{\varrho}$. Then, we have

$$
\begin{aligned}
|\varrho(z)| = \left|\widehat{\omega}\varrho(z)\,\widehat{\Theta}\acute{\varrho}(z)\right| &\leq \Big[|\omega(z,\varrho) - \omega(z,0)| + |\omega(z,0)|\Big] \\
&\times \Bigg\{ {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\tfrac{1}{\mathrm{p}^*}{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}|\Theta(z,\varrho(z))|\right) - {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\tfrac{\mathrm{q}^*}{\mathrm{p}^*}|\varrho(z)|\right) \\
&- \tfrac{\Gamma(\gamma)}{\mathrm{p}^*}\tfrac{\Gamma(\gamma)}{\Gamma(\gamma+\epsilon_2)}{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}|\Theta(\mathfrak{s}_2,\varrho(\mathfrak{s}_2))|\,(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_2}\Bigg\}
\end{aligned}
$$

(50)
$$
\leq \Big[\kappa\mathring{\eta} + \omega_\circ\Big]\left(\mathcal{A} + \mathcal{B}\|\varrho\|\right).
$$

Therefore $\|\varrho\| \leq \left[\mathring{\eta}\kappa + \omega_\circ\right]\left(\mathcal{A} + \mathcal{B}\|\varrho\|\right)$, and so

(51)
$$
\|\varrho\| \leq \left[\mathring{\eta}\kappa + \omega_\circ\right]\left(\mathcal{A} + \mathcal{B}\kappa\right) \leq \kappa.
$$

We have that $\|\varrho\| \leq \kappa$ and so the hypothesis (H3) of Theorem 2.8 is satisfied.

Step IV: Finally, we have

(52)
$$
\mathring{M}_{\widehat{\Theta}} = \left\|\widehat{\Theta}\left(\widehat{\mathrm{B}}_\kappa\right)\right\| = \sup\left\{\left\|\widehat{\Theta}\varrho\right\| : \varrho \in \widehat{\mathrm{B}}_\kappa\right\} \leq \mathcal{A} + \mathcal{B}\kappa.
$$

From above estimate, we obtain $\mathring{\eta}_\omega\mathring{M}_{\widehat{\Theta}} \leq \|\varrho\|\left(\mathcal{A} + \mathcal{B}\kappa\right) < 1$, and so the assumption (s4) of Theorem 2.8 is fulfilled. Thus, the operators $\widehat{\omega}$ and $\widehat{\Theta}$ satisfy all the conditions of Theorem 2.8 and so, the operator equation $\widehat{\omega}\varrho\,\widehat{\Theta}\varrho = \varrho$ has a solution in $\widehat{\mathrm{B}}_\kappa$. Consequently, the hybrid Langevin and Sturm-Liouville $\mathbb{FDE}$ (16) has a solution defined on J. This completes the proof. □

3.2. **Existence and uniqueness via Banach contraction principle.** In this subsection, we give the equivalence of the boundary value problem (16) and prove the existence, uniqueness and estimate of solution of (16). Now, via Theorem 2.9, we obtain the existence and uniqueness results of a system for ψ–Hilfer boundary value problems (16).

Theorem 3.4. *Assume that the continuous functions* $\Theta : \mathrm{J}\times\mathbb{R} \to \mathbb{R}$ *and* $\omega : \mathrm{J}\times\mathbb{R} \to \mathbb{R}\backslash\{0\}$ *is so that the assumptions* (H1)-(H2) *are satisfied. Then, the hybrid Langevin and Sturm-Liouville* $\mathbb{FDE}$ (16) *possesses one and only one solution if* $\Lambda < 1$, *subject to constant*

$$
\begin{aligned}
\Lambda := &\left\{\tfrac{\mathring{M}_\Theta}{\mathrm{p}^*}\tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_1+\epsilon_2}}{\Gamma(\epsilon_1+\epsilon_2+1)} + \tfrac{\mathrm{q}^*}{\mathrm{p}^*}\tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_1}}{\Gamma(\epsilon_2+1)}\right. \\
&\left.+ \tfrac{\mathring{M}_\Theta}{\mathrm{p}^*}\tfrac{(\Gamma(\gamma))^2(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_1+\epsilon_2}}{\Gamma(\gamma+\epsilon_2)\Gamma(\epsilon_1+1)}\right\}\mathring{\eta}_\omega + \left\{\tfrac{\mathring{\eta}_\Theta}{\mathrm{p}^*}\tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_1+\epsilon_2}}{\Gamma(\epsilon_1+\epsilon_2+1)}\right. \\
&\left.+ \tfrac{\mathrm{q}^*}{\mathrm{p}^*}\tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_2}}{\Gamma(\epsilon_2+1)} + \tfrac{\mathring{\eta}_\Theta}{\mathrm{p}^*}\tfrac{(\Gamma(\gamma))^2(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_1+\epsilon_2}}{\Gamma(\gamma+\epsilon_2)\Gamma(\epsilon_1+1)}\right\}\mathring{M}_\omega. \qquad (53)
\end{aligned}
$$

Then, the problem (16) *has a unique solution on the interval* J.

Proof. According to Lemma 3.1, we consider the operators $\mathbb{N} : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$ defined by $(\mathbb{N}\varrho)(z) = \varrho(z)$, that is,

$$
\begin{aligned}
(\mathbb{N}\varrho)(z) := \omega(z,\varrho(z))&\left\{{}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\tfrac{1}{\mathrm{p}}{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\wp\right) - {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\tfrac{\mathrm{q}}{\mathrm{p}}\varrho\right)(z)\right. \\
&\left.-\tfrac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)}\tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)}{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\wp(\mathfrak{s}_2)\,(\psi(z)-\psi(\mathfrak{s}_1))^{\gamma+\epsilon_2-1}\right\}. \qquad (54)
\end{aligned}
$$

On the other side, for $z \in \mathrm{J}$, and using (H1) and (H2), we have

$$
\begin{aligned}
|(\mathbb{N}\varrho)(z) - (\mathbb{N}\acute{\varrho})(z)| = {}& (|\omega(z,\varrho(z)) - \omega(z,\acute{\varrho}(z))|) \Big\{ {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+} \Big(\tfrac{1}{\mathrm{p}} {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+} |\Theta(z,\varrho(z))| \Big) \\
& - {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+} \Big(\tfrac{\mathrm{q}}{\mathrm{p}} \varrho \Big)(z) - \tfrac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)} \tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)} {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+} \big|\Theta(\mathfrak{s}_2,\varrho(\mathfrak{s}_2))\big| (\psi(z) \\
& - \psi(\mathfrak{s}_1))^{\gamma+\epsilon_2-1} \Big\} + \big|\omega(z,\varrho(z))\big| \Big\{ {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+} \Big(\tfrac{1}{\mathrm{p}} {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+} \big|\Theta(z,\varrho(z)) \\
& - \Theta(z,\acute{\varrho}(z))\big| \Big) - {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+} \Big(\tfrac{\mathrm{q}}{\mathrm{p}} |\varrho - \acute{\varrho}| \Big)(z) \\
& - \tfrac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)} \tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)} {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+} \big|\Theta(\mathfrak{s}_2,\varrho(\mathfrak{s}_2)) \\
& - \Theta(\mathfrak{s}_2,\acute{\varrho}(\mathfrak{s}_2))\big| (\psi(z) - \psi(\mathfrak{s}_1))^{\gamma+\epsilon_2-1} \Big\} \\
\leq {}& \mathring{\eta}_\omega |\varrho - \acute{\varrho}| \Big\{ {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+} \Big(\tfrac{1}{\mathrm{p}^*} {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+} \mathring{M}_\Theta \Big) - {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+} (\tfrac{\mathrm{q}^*}{\mathrm{p}^*} \varrho)(z) \\
& - \tfrac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)} \tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_2}}{\mathrm{p}^*} {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+} \mathring{M}_\Theta \Big\} \\
& + \mathring{M}_\omega \Big\{ {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+} \Big(\tfrac{1}{\mathrm{p}^*} {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+} \mathring{\eta}_\Theta |\varrho(z) - \acute{\varrho}(z)| \Big) - {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+} \Big(\tfrac{\mathrm{q}^*}{\mathrm{p}^*} |\varrho - \acute{\varrho}| \Big)(z) \\
& - \tfrac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)} \tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_2}}{\mathrm{p}^*} {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+} \mathring{\eta}_\Theta |\varrho(z) - \acute{\varrho}(z)| \Big\} \\
\leq {}& \Big\{ \tfrac{\mathring{M}_\Theta}{\mathrm{p}^*} \tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_1+\epsilon_2}}{\Gamma(\epsilon_1+\epsilon_2+1)} + \tfrac{\mathrm{q}^* \mathring{\hbar}}{\mathrm{p}^*} \tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_2}}{\Gamma(\epsilon_2+1)} \\
& + \tfrac{\mathring{M}_\Theta}{\mathrm{p}^*} \tfrac{(\Gamma(\gamma))^2 (\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_1+\epsilon_2}}{\Gamma(\gamma+\epsilon_2)\Gamma(\epsilon_1+1)} \Big\} \mathring{\eta}_\omega |\varrho - \acute{\varrho}| \\
& + \Big\{ \tfrac{\mathring{\eta}_\Theta}{\mathrm{p}^*} \tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_1+\epsilon_2}}{\Gamma(\epsilon_1+\epsilon_2+1)} + \tfrac{\mathrm{q}^*}{\mathrm{p}^*} \tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_2}}{\Gamma(\epsilon_2+1)} \\
& + \tfrac{\mathring{\eta}_\Theta}{\mathrm{p}^*} \tfrac{(\Gamma(\gamma))^2 (\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_1+\epsilon_2}}{\Gamma(\gamma+\epsilon_2)\Gamma(\epsilon_1+1)} \Big\} \mathring{M}_\omega |\varrho - \acute{\varrho}|,
\end{aligned}
$$

which implies $\|\mathbb{N}\varrho - \mathbb{N}\acute{\varrho}\| \leq \Lambda \|\varrho - \acute{\varrho}\|$. Thus $\mathbb{N}$ is a contraction mapping and Theorem 2.9 implies that $\mathbb{N}$ has a unique fixed point. Hence (16) has a unique solution. □

3.3. **Stability results.** In the recent section, we interested to studied UH and generalized UH stability of the hybrid Langevin and Sturm-Liouville $\mathbb{FDE}$ (16).

Definition 3.5. *The hybrid Langevin and Sturm-Liouville* $\mathbb{FDE}$ (16) *is*

- UH *stable if there exists a real number* $c_\omega > 0$, *s.t for each* $\vartheta \in \mathbb{R}^+$ *and for each* $\varrho \in \widehat{\mathcal{C}}$ *satisfying*

(55) $$\left| {}^{\psi}_{\mathrm{H}}\mathcal{D}^{\epsilon_1,\zeta}_{\mathfrak{s}_1} \left[\mathrm{p}(z) {}^{\psi}_{\mathrm{H}}\mathcal{D}^{\epsilon_2,\zeta}_{\mathfrak{s}_1} \left[\frac{\varrho(z)}{\omega(z,\varrho(z))} \right] - \mathrm{q}(z)\varrho(z) \right] - \Theta\left(z,\varrho(z)\right) \right| \leq \vartheta, \quad z \in \mathrm{J},$$

with hybrid boundary conditions

(56) $$\begin{cases} \varrho(\mathfrak{s}_1) = 0, \\ \mathrm{p}(\mathfrak{s}_2) \, {}^{\psi}_{\mathrm{H}}\mathcal{D}^{\epsilon_2,\zeta}_{\mathfrak{s}_1} \left[\frac{\varrho(z)}{\omega(z,\varrho(z))} \right]_{z=\mathfrak{s}_2} - \mathrm{q}(\mathfrak{s}_2)\varrho(\mathfrak{s}_2) = 0, \end{cases}$$

there exists a unique solution $\breve{\varrho} \in \widehat{\mathcal{C}}$ *of* (16) *with* $\|\varrho - \breve{\varrho}\| \leq c_\omega \vartheta$;

- *generalized* UH *stable if there exists a real number* $C_\omega \in C(\mathbb{R}^+, \mathbb{R}^+)$, *with* $C_\omega(0) = 0$ *s.t for each* $\vartheta \in \mathbb{R}$ *and for each* $\varrho \in \widehat{\mathcal{C}}$ *satisfying* (55) *and* (56), *there exists a unique solution* $\breve{\varrho} \in \widehat{\mathcal{C}}$ *of* (16) *with* $\|\varrho - \breve{\varrho}\| \leq C_\omega(\vartheta)$.

Remark 3.6. *A function $\breve{\varrho} \in \widehat{\mathcal{C}}$ is a solution of inequality* (3.5) *if and only if there exists a $\sigma \in \widehat{\mathcal{C}}$ (which depends on solution $\breve{\varrho}$) s.t for all $z \in \mathrm{J}$, $|\sigma(z)| \leq \vartheta$, and*

$$
{}^{\psi}_{\mathrm{H}}\mathcal{D}^{\epsilon_1,\zeta}_{\mathfrak{s}_1}\left[\mathrm{p}(z)\,{}^{\psi}_{\mathrm{H}}\mathcal{D}^{\epsilon_2,\zeta}_{\mathfrak{s}_1}\left[\frac{\varrho(z)}{\omega(z,\varrho(z))}\right]-\mathrm{q}(z)\varrho(z)\right]=\Theta\left(z,\varrho(z)\right)+\sigma(z). \tag{57}
$$

Now, we discuss the UH stability of solution to the problem (16).

Theorem 3.7. *Suppose that the conditions* (H1), (H2) *and* $\Lambda < 1$ *are fulfilled. Then, the solution of* (16) *is* UH *and generalized* UH *stable.*

Proof. Let $\vartheta > 0$, $\breve{\varrho} \in \widehat{\mathcal{C}}$ be a function which satisfies the inequality (55) and let $\varrho \in \widehat{\mathcal{C}}$ the unique solution of the following problem

$$
{}^{\psi}_{\mathrm{H}}\mathcal{D}^{\epsilon_1,\zeta}_{\mathfrak{s}_1}\left[\mathrm{p}(z){}^{\psi}_{\mathrm{H}}\mathcal{D}^{\epsilon_2,\zeta}_{\mathfrak{s}_1}\left[\frac{\varrho(z)}{\omega(z,\varrho(z))}\right]-\mathrm{q}(z)\varrho(z)\right]=\Theta\left(z,\varrho(z)\right), \tag{58}
$$

for each $z \in \mathrm{J}$, with hybrid boundary conditions (15). Now, Lemma 3.1 implies that

$$
\begin{aligned}
\varrho(z) =&\omega(z,\varrho(z))\left\{{}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\frac{1}{\mathrm{p}}{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\wp(z)\right)-{}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\frac{\mathrm{q}}{\mathrm{p}}\varrho\right)(z)\right.\\
&\left.-\frac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)}\frac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)}{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\wp(\mathfrak{s}_2)\left(\psi(z)-\psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1}\right\}.
\end{aligned} \tag{59}
$$

Since we have assumed that $\breve{\varrho}$ is a solution of (3.5), hence we have by Remark 3.6,

$$
{}^{\psi}_{\mathrm{H}}\mathcal{D}^{\epsilon_1,\zeta}_{\mathfrak{s}_1}\left[\mathrm{p}(z)\,{}^{\psi}_{\mathrm{H}}\mathcal{D}^{\epsilon_2,\zeta}_{\mathfrak{s}_1}\left[\frac{\breve{\varrho}(z)}{\omega(z,\breve{\varrho}(z))}\right]-\mathrm{q}(z)\breve{\varrho}(z)\right]=\Theta\left(z,\breve{\varrho}(z)\right)+\sigma(z), \tag{60}
$$

with hybrid boundary conditions

$$
\begin{cases}
\varrho(\mathfrak{s}_1)=0,\\
\mathrm{p}(\mathfrak{s}_2)\,{}^{\psi}_{\mathrm{H}}\mathcal{D}^{\epsilon_2,\zeta}_{\mathfrak{s}_1}\left[\frac{\breve{\varrho}(z)}{\omega(z,\breve{\varrho}(z))}\right]_{z=\mathfrak{s}_2}-\mathrm{q}(\mathfrak{s}_2)\breve{\varrho}(\mathfrak{s}_2)=0.
\end{cases} \tag{61}
$$

Again, Lemma 3.1 implies that

$$
\begin{aligned}
\breve{\varrho}(z) =&\omega(z,\breve{\varrho}(z))\left\{{}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\frac{1}{\mathrm{p}}{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\wp(z)\right)-{}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\frac{\mathrm{q}}{\mathrm{p}}\breve{\varrho}\right)(z)\right.\\
&\left.-\frac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)}\frac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)}{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\wp(\mathfrak{s}_2)\left(\psi(z)-\psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1}\right\}\\
&+\omega(z,\breve{\varrho}(z))\left\{{}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\frac{1}{\mathrm{p}}{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\sigma(z)\right)-{}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\frac{\mathrm{q}}{\mathrm{p}}\breve{\varrho}\right)(z)\right.\\
&\left.-\frac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)}\frac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)}{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\sigma(\mathfrak{s}_2)\left(\psi(z)-\psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1}\right\}.
\end{aligned} \tag{62}
$$

On the other hand, we have,

$$
\begin{aligned}
|\Theta\varrho(z)-\Theta\breve{\varrho}(z)| \leq &\left|\breve{\varrho}(z)-\omega(z,\varrho(z))\left\{{}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\frac{1}{\mathrm{p}}{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\wp(z)\right)-{}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\frac{\mathrm{q}}{\mathrm{p}}\varrho\right)(z)\right.\right.\\
&\left.\left.-\frac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)}\frac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)}{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\wp(\mathfrak{s}_2)\left(\psi(z)-\psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1}\right\}\right|,
\end{aligned} \tag{63}
$$

for $z \in \mathrm{J}$. Hence using part 1 of Remark 3.6 and (H2), we obtain $|\breve{\varrho}-\varrho| \leq \mathring{M}_{\omega}\mu\vartheta+\Lambda\|\varrho-\breve{\varrho}\|$, where Λ is defined in (53) and μ given by

$$
\mu := \frac{1}{\mathrm{p}^*}\left\{\frac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_1+\epsilon_2}}{\Gamma(\epsilon_1+\epsilon_2+1)}+\frac{(\Gamma(\gamma))^2(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_1+\epsilon_2}}{\Gamma(\gamma+\epsilon_2)\Gamma(\epsilon_1+1)}\right\}. \tag{64}
$$

In consequence, it follows that $\|\breve{\varrho}-\varrho\| \leq \frac{1}{1-\Lambda}\mathring{M}_{\omega}\mu\vartheta$. Therefor the UH stability condition is satisfied whenever $c_{\omega} = \frac{1}{1-\Lambda}\mathring{M}_{\omega}\mu$. More generally, for $C_{\omega}(\vartheta) = \frac{1}{1-\Lambda}\mathring{M}_{\omega}\mu\vartheta$, $C_{\omega}(0) = 0$, the generalized UH stability condition is also satisfied. This completes the proof. □

4. Results for multi-valued $\mathbb{FDI}$ (14)

Now, in the first place, we aim to investigate the solution existence for given multi-valued hybrid Sturm-Liouville and Langevin $\mathbb{FDI}$ (14). To reach this purpose, we provide an auxiliary lemma. Before starting it, we introduce the Banach algebra $\mathcal{Y} = \left\{\varrho(z) : \varrho(z) \in \widehat{\mathcal{C}}\right\}$, with the norm $\|\varrho\|_{\mathcal{Y}} = \sup_{z\in \mathrm{J}} |\varrho(z)|$ and with the multiplication $(\varrho \cdot \varrho')(z) = \varrho(z)\, \varrho'(z)$, for all $\varrho, \varrho' \in \mathcal{Y}$. Note the following equivalent solution in the framework of an integral equation.

Lemma 4.1. *Let $\wp \in \widehat{\mathcal{C}}$. Then a function ϱ^* is a solution for the given linear multi-valued hybrid Langevin and Sturm-Liouville $\mathbb{FDE}$*

$$\tag{65} {}^{\psi}_{\mathrm{H}}\mathcal{D}^{\epsilon_1,\zeta}_{\mathfrak{s}_1}\left[\mathrm{p}(z)\,{}^{\psi}_{\mathrm{H}}\mathcal{D}^{\epsilon_2,\zeta}_{\mathfrak{s}_1}\left[\frac{\varrho(z)}{\omega(z,\varrho(z))}\right] - \mathrm{q}(z)\varrho(z)\right] = \wp\,(z)\,,$$

with hybrid boundary conditions (15) which is given as

$$\tag{66} \begin{aligned}\varrho^*(z) =&\,\omega(z,\varrho(z))\left\{{}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\frac{1}{\mathrm{p}}{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\wp(z)\right) - {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\frac{\mathrm{q}}{\mathrm{p}}\varrho\right)(z)\right.\\ &\left. -\frac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)}\frac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)}\,{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\wp(\mathfrak{s}_2)\left(\psi(z)-\psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1}\right\},\end{aligned}$$

where γ is defined in Eq. (34).

Based on lemma 4.1, we are going to define the solution of supposed multi-valued hybrid Sturm-Liouville and Langevin $\mathbb{FDI}$ (14).

Definition 4.2. *The absolutely continuous function $\varrho : \mathrm{J} \to \mathbb{R}$ is called a solution to the multi-valued Sturm-Liouville and Langevin* H$\mathbb{FDI}$ (14) *if an integrable mapping $\mathfrak{m} \in \mathcal{L}^1(\mathrm{J}, \mathbb{R})$ with $\mathfrak{m}(z) \in \Theta(z, \varrho(z))$, for almost all $z \in \mathrm{J}$ satisfying hybrid boundary conditions (15) and*

$$\tag{67} \begin{aligned}\varrho(z) =&\,\omega(z,\varrho(z))\left\{{}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\frac{1}{\mathrm{p}}{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\mathfrak{m}(z)\right) - {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\frac{\mathrm{q}}{\mathrm{p}}\varrho\right)(z)\right.\\ &\left. -\frac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)}\frac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)}\,{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\mathfrak{m}(\mathfrak{s}_2)\left(\psi(z)-\psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1}\right\}.\end{aligned}$$

Now, we prove our first result about the inclusion problem (14).

Theorem 4.3. *Consider set-valued map Θ and continuous function ω according to the Sturm-Liouville and Langevin* H$\mathbb{FDI}$ (14). *Assume that*

C1) *the set-valued map $\Theta : \mathrm{J} \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ has $\mathcal{L}^1$-Carathéodory property;*

C2) *there exists a bounded mapping $\breve{\mathrm{M}} : \mathrm{J} \to \mathbb{R}^+$ s.t for $\varrho \in \mathbb{R}$ and each $z \in \mathrm{J}$,*

$$\tag{68} |\omega\,(z, \varrho(z))| \le \breve{\mathrm{M}}(z)\,|\varrho(z)|\,;$$

C3) *there is a positive mapping $\Upsilon(z) \in \mathcal{L}^1(\mathrm{J}, \mathbb{R}^+)$ s.t*

$$\|\Theta(z,\varrho)\| = \sup\left\{|\bar{\varrho}| : \bar{\varrho} \in \Theta(z,\varrho(z))\right\} \le \Upsilon(z), \qquad \forall\, \varrho \in \mathbb{R}, \forall\, z \in \mathrm{J};$$

C4) *there exists $\tilde{\kappa} \in (0,\infty)$ s.t*

$$\tag{69} \tilde{\kappa} > \left[\breve{\mathrm{M}}^*\tilde{\kappa} + \omega^*\right]\left[\mu\|\Upsilon\|_{L^1} + \mathcal{B}\tilde{\kappa}\right],$$

where $\mathcal{B}$, μ are defined in Eqs. (41), Eq. (64) respectively and

$$\tag{70} \|\Upsilon\|_{\mathcal{L}^1} = \int_0^1 |\Upsilon(\xi)|\,\mathrm{d}\xi, \quad \omega^* = \sup_{z\in\mathrm{J}}|\omega(z,0)|, \quad \breve{\mathrm{M}}^* = \sup_{z\in\mathrm{J}}|\breve{\mathrm{M}}(z)|.$$

If $\breve{\mathrm{M}}^*\,(\mu\|\Upsilon\|_{\mathcal{L}^1} + \mathcal{B}\tilde{\kappa}) < 0.5$, then the hybrid inclusion problem (14) has a solution.

Proof. For every $\varrho \in \mathcal{Y}$, define the set of selections of the operator Θ, for almost all $z \in \mathrm{J}$, by

$$(\mathcal{SEL})_{\Theta,\varrho} = \left\{ \bar{\bar{\varrho}} \in \mathcal{L}^1(\mathrm{J}) \,:\, \bar{\bar{\varrho}}(z) \in \Theta\left(z, \varrho(z)\right) \right\}, \tag{71}$$

and define the set-valued map $\widehat{\Theta}_1 : \mathcal{Y} \to \mathcal{P}(\mathcal{Y})$, $\forall z \in \mathrm{J}$, by

$$\widehat{\Theta}_1(\varrho) = \left\{ \begin{array}{l} g \in \mathcal{Y} : \\ g(z) = \omega(z, \varrho(z)) \left\{ {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+} \left(\frac{1}{\mathrm{p}} {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+} \bar{\bar{\varrho}}(z) \right) - \right. \\ \quad {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+} \left(\frac{\mathrm{q}}{\mathrm{p}} \varrho \right)(z) - \frac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)} \frac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)} \\ \quad \left. \times {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+} \bar{\bar{\varrho}}(\mathfrak{s}_2) \left(\psi(z) - \psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1} \right\}, \qquad \bar{\bar{\varrho}} \in (\mathcal{SEL})_{\Theta,\varrho} \end{array} \right\}. \tag{72}$$

It is obvious that the function g_0 is a solution for the Sturm-Liouville and Langevin HFDI (14) iff g_0 is a fixed point of the operator $\widehat{\Theta}_1$. Now, define the single-valued mappings $\widehat{\omega} : \mathcal{Y} \to \mathcal{Y}$ and $\widehat{\Theta}_2 : \mathcal{Y} \to \mathcal{P}(\mathcal{Y})$ for all $z \in \mathrm{J}$, by $(\widehat{\omega}\varrho)(z) = \omega\left(z, \varrho(z)\right)$, and

$$\widehat{\Theta}_2(\varrho) = \left\{ \begin{array}{l} \acute{g} \in \mathcal{Y} : \\ \acute{g}(z) = \left\{ {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+} \left(\frac{1}{\mathrm{p}} {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+} \bar{\bar{\varrho}}(z) \right) - {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+} \left(\frac{\mathrm{q}}{\mathrm{p}} \varrho \right)(z) \right. \\ \quad - \frac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)} \frac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)} \\ \quad \left. \times {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+} \bar{\bar{\varrho}}(\mathfrak{s}_2) \left(\psi(z) - \psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1} \right\}, \qquad \bar{\bar{\varrho}} \in (\mathcal{SEL})_{\Theta,\varrho} \end{array} \right\}. \tag{73}$$

Then, we obtain $\widehat{\Theta}_1(\varrho) = (\widehat{\omega}\varrho)\widehat{\Theta}\varrho$. First, we prove that the operator $\widehat{\omega}$ is Lipschitzian. Let $\varrho_1, \varrho_2 \in \mathcal{Y}$. Thus the assumption (C2) implies that

$$\left| \left(\widehat{\omega}\varrho_1\right)(z) - \left(\widehat{\omega}\varrho_2\right)(z) \right| = \left| \omega\left(z, \varrho_1(z)\right) - \omega\left(z, \varrho_2(z)\right) \right| \leq \breve{\mathrm{M}}(z) \left| \varrho_1(z) - \varrho_2(z) \right|, \tag{74}$$

for all $z \in \overline{\Delta}$. Hence, we get

$$\left\| \widehat{\omega}\varrho_1 - \widehat{\omega}\varrho_2 \right\|_{\mathcal{Y}} \leq \breve{\mathrm{M}} \left\| \varrho_1 - \varrho_2 \right\|_{\mathcal{Y}}, \qquad \forall\, \varrho_1, \varrho_2 \in \mathcal{Y}. \tag{75}$$

This shows that the operator $\widehat{\omega}$ is Lipschitzian with a Lipschitz constant $\mathring{M} > 0$. In this step, we prove that the set-valued map $\widehat{\Theta}_2$ has convex values. Let $\varrho_1, \varrho_2 \in \widehat{\Theta}_2\varrho$. Choose $\bar{\bar{\varrho}}_1, \bar{\bar{\varrho}}_2 \in (\mathcal{SEL})_{\Theta,\varrho}$ s.t

$$\begin{aligned} \varrho_i(z) = \Big\{ & {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+} \left(\frac{1}{\mathrm{p}} {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+} \bar{\bar{\varrho}}_i(z) \right) - {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+} \left(\frac{\mathrm{q}}{\mathrm{p}} \varrho \right)(z) \\ & - \frac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)} \frac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)} {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+} \bar{\bar{\varrho}}_i(\mathfrak{s}_2) \left(\psi(z) - \psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1} \Big\}, \end{aligned} \tag{76}$$

where $i = 1, 2$, for almost all $z \in \mathrm{J}$. Then, we have

$$\begin{aligned} \lambda\varrho_1(z) + (1-\lambda)\varrho_2(z) = & \Big\{ {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+} \left(\frac{1}{\mathrm{p}} {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+} \left[\lambda\bar{\bar{\varrho}}_1(z) + (1-\lambda)\bar{\bar{\varrho}}_2(z)\right] \right) \\ & - {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+} \left(\frac{\mathrm{q}}{\mathrm{p}} \varrho \right)(z) - \frac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)} \frac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)} {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+} \big[\lambda\bar{\bar{\varrho}}_1(z) \\ & + (1-\lambda)\bar{\bar{\varrho}}_2(z)\big](\mathfrak{s}_2) \left(\psi(z) - \psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1} \Big\}, \end{aligned} \tag{77}$$

for almost all $z \in \mathrm{J}$, where $\lambda \in (0, 1)$. Since $\widehat{\Theta}$ has convex values, $(\mathcal{SEL})_{\widehat{\Theta},\varrho}$ is convex-valued. This follows that $\lambda\bar{\bar{\varrho}}_1(z) + (1-\lambda)\bar{\bar{\varrho}}_2(z) \in (\mathcal{SEL})_{\Theta,\varrho}$, for all $z \in \mathrm{J}$, and so $\widehat{\Theta}_2\varrho$ is a convex set $\forall\, \varrho \in \mathcal{Y}$. Now, we prove that the operator $\widehat{\Theta}_2$ is completely continuous. In order to do this, we have to prove two equi-continuity and uniform boundedness properties for the set $\widehat{\Theta}_2(\mathcal{Y})$. First, we show that $\widehat{\Theta}_2$ maps all bounded sets into bounded subsets of $\mathcal{Y}$. For a positive number

$\kappa^\circ \in \mathbb{R}$, consider the bounded ball $\widehat{\mathrm{B}}_{\kappa^\circ} = \{\varrho \in \mathcal{Y} : \|\varrho\|_{\mathcal{Y}} \leq \kappa^\circ\}$. For every $\varrho \in \widehat{\mathrm{B}}_{\kappa^\circ}$ and $\acute{g} \in \widehat{\Theta}_2\varrho$, there exists a function $\bar{\bar{\varrho}} \in (\mathcal{SEL})_{\Theta,\varrho}$ s.t

$$\begin{aligned}\acute{g}(z) = \Big\{ & {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\tfrac{1}{\mathrm{p}}{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\bar{\bar{\varrho}}(z)\right) - {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\tfrac{\mathrm{q}}{\mathrm{p}}\varrho\right)(z) \\ & -\tfrac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)}\tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)}{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\bar{\bar{\varrho}}(\mathfrak{s}_2)\left(\psi(z)-\psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1}\Big\},\end{aligned} \tag{78}$$

$\forall\, z \in \mathrm{J}$. Then, we have

$$\begin{aligned}|\acute{g}(z)| &= \Big|\Big\{{}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\tfrac{1}{\mathrm{p}}{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\bar{\bar{\varrho}}(z)\right) - {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\tfrac{\mathrm{q}}{\mathrm{p}}\varrho\right)(z) \\ &\quad -\tfrac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)}\tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)}{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\bar{\bar{\varrho}}(\mathfrak{s}_2)\left(\psi(z)-\psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1}\Big\}\Big| \\ &\leq {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\tfrac{1}{\mathrm{p}}{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}|\bar{\bar{\varrho}}(z)|\right) - {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\tfrac{\mathrm{q}}{\mathrm{p}}|\varrho|\right)(z) \\ &\quad - \tfrac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)}\tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)}{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}|\bar{\bar{\varrho}}(\mathfrak{s}_2)|\left(\psi(z)-\psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1} \\ &\leq {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\tfrac{1}{\mathrm{p}^*}{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}|\Upsilon(z)|\right) - {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\tfrac{\mathrm{q}^*}{\mathrm{p}^*}\|\varrho\|\right)(z) \\ &\quad - \tfrac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)}\tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_2}}{\mathrm{p}^*(z)}{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}|\Upsilon(\mathfrak{s}_2)| \\ &\leq \tfrac{\|v\|_{L^1}}{\mathrm{p}^*}\tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_1+\epsilon_2}}{\Gamma(\epsilon_1+\epsilon_2+1)} + \tfrac{\mathrm{q}^*}{\mathrm{p}^*}\tfrac{\|\varrho\|(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_2}}{\Gamma(\epsilon_2+1)} \\ &\quad + \tfrac{\|\Upsilon\|_{L^1}}{\mathrm{p}^*}\tfrac{(\Gamma(\gamma))^2(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_1+\epsilon_2}}{\Gamma(\gamma+\epsilon_2)\Gamma(\epsilon_1+1)}.\end{aligned} \tag{79}$$

Hence, we have

$$\begin{aligned}\|\acute{g}\| \leq{}& \tfrac{\|\Upsilon\|_{L^1}}{\mathrm{p}^*}\left\{\tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_1+\epsilon_2}}{\Gamma(\epsilon_1+\epsilon_2+1)} + \tfrac{(\Gamma(\gamma))^2(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_1+\epsilon_2}}{\Gamma(\gamma+\epsilon_2)\Gamma(\epsilon_1+1)}\right\} \\ & + \tfrac{\mathrm{q}^*}{\mathrm{p}^*}\tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_2}}{\Gamma(\epsilon_2+1)}\kappa^\circ = \mu\|\Upsilon\|_{L^1} + \mathcal{B}\kappa^\circ,\end{aligned} \tag{80}$$

where $\mathcal{B}$ and μ are given in (41) and (64), respectively. Thus, $\|\acute{g}\| \leq \mu\|\Upsilon\|_{L^1} + \mathcal{B}\kappa^\circ$. Indeed the set $\widehat{\Theta}_2(\mathcal{Y})$ is uniformly bounded. Let $\varrho \in \widehat{\mathrm{B}}_{\kappa^\circ}$ and $\acute{g} \in \widehat{\Theta}_2\varrho$. Choose $\bar{\bar{\varrho}} \in (\mathcal{SEL})_{\Theta,\varrho}$ s.t

$$\begin{aligned}\acute{g}(z) = \Big\{ & {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\tfrac{1}{\mathrm{p}}{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\bar{\bar{\varrho}}(z)\right) - {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+}\left(\tfrac{\mathrm{q}}{\mathrm{p}}\varrho\right)(z) \\ & -\tfrac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)}\tfrac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)}{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}\bar{\bar{\varrho}}(\mathfrak{s}_2)\left(\psi(z)-\psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1}\Big\},\end{aligned} \tag{81}$$

for all $z \in \mathrm{J}$. Assume that $z_1, z_2 \in \mathrm{J}$ with $z_1 < z_2$. Then, we have

$$\begin{aligned}|\acute{g}(z_2) - \acute{g}(z_1)| ={}& \tfrac{1}{p^*}\Big|\int_a^{t_1} \mathcal{Q}^{\epsilon_1+\epsilon_2}_{\psi}(z_1,\xi)\omega\left(\xi,\varrho(\xi)\right)\,\mathrm{d}\xi \\ & - \int_{\mathfrak{s}_1}^{z_2} \mathcal{Q}^{\epsilon_1+\epsilon_2}_{\psi}(z_2,\xi)\omega\left(\xi,\varrho(\xi)\right)\mathrm{d}\xi\Big| \\ & + \tfrac{\mathrm{q}^*}{\mathrm{p}^*}\Big|\int_{\mathfrak{s}_1}^{z_1} \mathcal{Q}^{\epsilon_2}_{\psi}(z_1,\xi)\varrho(\xi)\mathrm{d}\xi - \int_{\mathfrak{s}_1}^{z_2} \mathcal{Q}^{\epsilon_1+\epsilon_2}_{\psi}(z_2,\xi)\varrho(\xi)\mathrm{d}\xi\Big| \\ & + \tfrac{(\Gamma(\gamma))^2}{\mathrm{p}^*\Gamma(\epsilon_2+\gamma)}\left(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1)\right)^{1-\gamma}{}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+}|\Theta(\mathfrak{s}_2,\varrho(\mathfrak{s}_2))| \\ & \cdot\left[(\psi(z_2)-\psi(\mathfrak{s}_1))^{\gamma+\epsilon_2-1} - (\psi(z_1)-\psi(\mathfrak{s}_1))^{\gamma+\epsilon_2-1}\right] \\ \leq{}& \tfrac{1}{\mathrm{p}^*}\Big|\int_{\mathfrak{s}_1}^{z_2}\left[\mathcal{Q}^{\epsilon_1+\epsilon_2}_{\psi}(z_1,\xi) - \mathcal{Q}^{\epsilon_1+\epsilon_2}_{\psi}(z_2,\xi)\right]\omega\left(\xi,u(\xi)\right)\mathrm{d}\xi\Big| \\ & + \tfrac{1}{\mathrm{p}^*}\Big|\int_{z_2}^{z_1} \mathcal{Q}^{\epsilon_1+\epsilon_2}_{\psi}(z_1,\xi)\omega\left(\xi,\varrho(\xi)\right)\,\mathrm{d}\xi\Big|\end{aligned}$$

$$
\begin{aligned}
&+\frac{q^*}{p^*}\left|\int_{\mathfrak{s}_1}^{z_2}\left[\mathcal{Q}_{\psi}^{\epsilon_2}(z_1,\xi)-\mathcal{Q}_{\psi}^{\epsilon_2}(z_2,\xi)\right]\omega\left(\xi,\varrho(\xi)\right)\,\mathrm{d}\xi\right| \\
&+\frac{q^*}{p^*}\left|\int_{z_2}^{z_1}\mathcal{Q}_{\psi}^{\epsilon_2}(z_1,\xi)\omega\left(\xi,\varrho(\xi)\right)\,\mathrm{d}\xi\right| \\
&+\frac{(\Gamma(\gamma))^2}{p^*\Gamma(\epsilon_2+\gamma)}\left(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1)\right)^{1-\gamma}\,{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_1}\left|\Theta(\mathfrak{s}_2,\varrho(\mathfrak{s}_2))\right| \\
&\cdot\left[(\psi(z_2)-\psi(\mathfrak{s}_1))^{\gamma+\epsilon_2-1}-(\psi(z_1)-\psi(\mathfrak{s}_1))^{\gamma+\epsilon_2-1}\right] \\
\le\ &\frac{\|\Upsilon\|_{L^1}}{p^*\Gamma(\epsilon_1+\epsilon_2+1)}\left[\left|(\psi(z_1)-\psi(\mathfrak{s}_1))^{\epsilon_1+\epsilon_2}-(\psi(z_2)-\psi(\mathfrak{s}_1))^{\epsilon_1+\epsilon_2}\right.\right. \\
&\left.\left.-(\psi(z_1)-\psi(z_2))^{\epsilon_1+\epsilon_2}\right|+(\psi(z_1)-\psi(z_2))^{\epsilon_1+\epsilon_2}\right] \\
&+\frac{q^*\|\varrho\|}{p^*\Gamma(\epsilon_2+1)}\left[\left|(\psi(z_1)-\psi(\mathfrak{s}_1))^{\epsilon_2}-(\psi(z_2)-\psi(\mathfrak{s}_1))^{\epsilon_2}\right.\right. \\
&\left.\left.-(\psi(z_1)-\psi(z_2))^{\epsilon_2}\right|+(\psi(z_1)-\psi(z_2))^{\epsilon_2}\right] \\
&+\frac{\|\Upsilon\|_{L^1}}{p^*}\frac{(\Gamma(\gamma))^2(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_1-\gamma+1}}{\Gamma(\epsilon_1+1)\Gamma(\epsilon_2+\gamma)} \\
&\cdot\left[(\psi(z_2)-\psi(\mathfrak{s}_1))^{\gamma+\epsilon_2-1}-(\psi(z_1)-\psi(\mathfrak{s}_1))^{\gamma+\epsilon_2-1}\right]\to 0.
\end{aligned}\tag{82}
$$

Notice that the right-hand side tends to zero independently of $\varrho\in\widehat{\mathrm{B}}_{\kappa^\circ}$ as $z_2\to z_1$. By using the Arzelà-Ascoli theorem, the complete continuity of $\widehat{\Theta}_2:\widehat{\mathcal{C}}\to\mathcal{P}(\widehat{\mathcal{C}})$ is deduced. In the following, we prove that $\widehat{\Theta}_2$ has a closed graph and this implies the upper semi-continuity of the operator $\widehat{\Theta}_2$. Assume that $\varrho_n\in\widehat{\mathrm{B}}_{\kappa^\circ}$ and $\acute{g}_n\in(\widehat{\Theta}_2\varrho_n)$ with $\varrho_n\to\varrho^*$ and $\acute{g}_n\to\acute{g}^*$. We claim that $\acute{g}^*\in(\widehat{\Theta}_2\varrho^*)$. For every $n\ge 1$ and $\acute{g}_n\in(\widehat{\Theta}_2\varrho_n)$, choose $\bar{\bar{\varrho}}_n\in(\mathcal{SEL})_{\Theta,\varrho_n}$ s.t

$$
\begin{aligned}
\acute{g}_n(z)=\Big\{&{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_2}\left(\tfrac{1}{p}\,{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_1}\,\bar{\bar{\varrho}}_n(z)\right)-{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_2}\left(\tfrac{q}{p}\varrho_n\right)(z) \\
&-\frac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)}\frac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{p(z)}\,{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_1}\bar{\bar{\varrho}}_n(\mathfrak{s}_2)\left(\psi(z)-\psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1}\Big\},
\end{aligned}\tag{83}
$$

for all $z\in\mathrm{J}$. It is sufficient to show that there exists a function $\bar{\bar{\varrho}}^*\in(\mathcal{SEL})_{\Theta,\varrho^*}$ s.t

$$
\begin{aligned}
\acute{g}^*(z)=\Big\{&{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_2}\left(\tfrac{1}{p}\,{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_1}\,\bar{\bar{\varrho}}^*(z)\right)-{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_2}\left(\tfrac{q}{p}\varrho^*\right)(z) \\
&-\frac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)}\frac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{p(z)}\,{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_1}\bar{\bar{\varrho}}^*(\mathfrak{s}_2)\left(\psi(z)-\psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1}\Big\},
\end{aligned}\tag{84}
$$

for all $z\in\mathrm{J}$. Define the continuous linear operator $\mathfrak{O}:\mathcal{L}^1(\mathrm{J},\mathbb{R})\to\mathcal{Y}=\widehat{\mathcal{C}}$ by

$$
\begin{aligned}
\mathfrak{O}(\bar{\bar{\varrho}}(z))=\varrho(z)=\Big\{&{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_2}\left(\tfrac{1}{p}\,{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_1}\,\bar{\bar{\varrho}}(z)\right)-{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_2}\left(\tfrac{q}{p}\varrho\right)(z) \\
&-\frac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)}\frac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{p(z)}\,{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_1}\bar{\bar{\varrho}}(\mathfrak{s}_2)\left(\psi(z)-\psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1}\Big\},
\end{aligned}\tag{85}
$$

for each $z\in\mathrm{J}$. Hence,

$$
\begin{aligned}
\|\acute{g}_n(z)-\acute{g}^*(z)\|=\Big\|\Big\{&{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_2}\Big(\tfrac{1}{p}\,{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_1}\left(\bar{\bar{\varrho}}_n(z)-\bar{\bar{\varrho}}^*(z)\right)\Big)-{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_2}\Big(\tfrac{q}{p}\left(\varrho_n-\varrho^*\right)\Big)(z) \\
&-\frac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)}\frac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{p(z)}\,{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_1}\big(\bar{\bar{\varrho}}_n(\mathfrak{s}_2) \\
&-\bar{\bar{\varrho}}^*(\mathfrak{s}_2)\big)\left(\psi(z)-\psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1}\Big\}\Big\|\to 0,
\end{aligned}\tag{86}
$$

for all $z\in\mathrm{J}$. Thus, by using Theorem 2.5, it is deduced that the operator $\mathfrak{O}\circ(\mathcal{SEL})_{\Theta,\varrho}$ has a closed graph. Also, since $\acute{g}_n\in\mathfrak{O}\left((\mathcal{SEL})_{\Theta,\varrho_n}\right)$ and $\varrho_n\to\varrho^*$, so $\exists\,\bar{\bar{\varrho}}^*\in(\mathcal{SEL})_{\Theta,\varrho^*}$ s.t

$$
\begin{aligned}
\acute{g}^*(z)=\Big\{&{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_2}\left(\tfrac{1}{p}\,{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_1}\,\bar{\bar{\varrho}}^*(z)\right)-{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_2}\left(\tfrac{q}{p}\varrho^*\right)(z) \\
&-\frac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)}\frac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{p(z)}\,{}^{\psi}\mathcal{I}_{\mathfrak{s}_1^+}^{\epsilon_1}\bar{\bar{\varrho}}^*(\mathfrak{s}_2)\left(\psi(z)-\psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1}\Big\},
\end{aligned}\tag{87}
$$

for each $z \in \mathrm{J}$. Indeed, $\acute{g}^* \in (\widehat{\Theta}_2 \varrho^*)$ and so $\widehat{\Theta}_2$ has a closed graph. Indeed the operator $\widehat{\Theta}_2$ is upper semi-continuous. Since the operator $\widehat{\Theta}_2$ has compact values, so it is a compact and upper semi-continuous operator. By using the assumption (C3), we get

$$\begin{aligned} \widehat{\Delta} = \left\|\widehat{\Theta}_2(\mathcal{Y})\right\| &= \sup_{z\in \mathrm{J}} \left\{ \left|\widehat{\Theta}_2 \varrho\right| \ : \ \varrho \in \mathcal{Y} \right\} \\ &= \frac{\|\Upsilon\|_{L^1}}{\mathrm{p}^*} \left\{ \frac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_1+\epsilon_2}}{\Gamma(\epsilon_1+\epsilon_2+1)} + \frac{(\Gamma(\gamma))^2(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_1+\epsilon_2}}{\Gamma(\gamma+\epsilon_2)\Gamma(\epsilon_1+1)} \right\} \\ &\quad + \frac{\mathrm{q}^*}{\mathrm{p}^*} \frac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{\epsilon_2}}{\Gamma(\epsilon_2+1)} \vartheta^* = \mu \|\Upsilon\|_{L^1} + \mathcal{B}\kappa^\circ. \end{aligned} \tag{88}$$

Then, $\widehat{\Delta}\breve{\mathrm{M}}^* < 0.5$. Now by using Theorem 2.6 for $\widehat{\Theta}$, we get one of the conditions (i) or (ii) holds. We first investigate the condition (ii). By considering Theorem 2.6 and the assumption (C4), we consider and element ϱ in $\mathcal{O}$, Eq. (29), with $\|\varrho\| = \tilde{\kappa}$. Then,

$$a\varrho(z) \in (\widehat{\omega}\varrho)(z)(\widehat{\Theta}\varrho)(z), \qquad \forall\, a > 1. \tag{89}$$

Choose the related function $\bar{\bar{\varrho}} \in (\mathcal{SEL})_{\widehat{\Theta},\varrho}$. Then for each $a > 1$, we have

$$\begin{aligned} \varrho(z) = \frac{1}{a}\omega(z,\varrho(z)) &\left\{ {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+} \left(\frac{1}{\mathrm{p}} {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+} \bar{\bar{\varrho}}(z) \right) - {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+} \left(\frac{\mathrm{q}}{\mathrm{p}} \varrho \right)(z) \right. \\ &\left. - \frac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)} \frac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)} {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+} \bar{\bar{\varrho}}(\mathfrak{s}_2) \left(\psi(z) - \psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1} \right\}, \quad \forall z \in \mathrm{J}. \end{aligned} \tag{90}$$

Thus, one can write

$$\begin{aligned} |\varrho(z)| &= \frac{1}{a}|\omega(z,\varrho(z))| \left\{ {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+} \left(\frac{1}{\mathrm{p}} {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+} |\bar{\bar{\varrho}}(z)| \right) - {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+} \left(\frac{\mathrm{q}}{\mathrm{p}} |\varrho| \right)(z) \right. \\ &\quad \left. - \frac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)} \frac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)} {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+} |\bar{\bar{\varrho}}(\mathfrak{s}_2)| \left(\psi(z) - \psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1} \right\} \\ &= \frac{1}{a}\left[|\omega(z,\varrho(z)) - \omega(z,0)| + |\omega(z,0)| \right] \left\{ {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+} \left(\frac{1}{\mathrm{p}} {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+} |\bar{\bar{\varrho}}(z)| \right) - {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+} \left(\frac{\mathrm{q}}{\mathrm{p}} |\varrho| \right)(z) \right. \\ &\quad \left. - \frac{(\Gamma(\gamma))^2}{\Gamma(\epsilon_2+\gamma)} \frac{(\psi(\mathfrak{s}_2)-\psi(\mathfrak{s}_1))^{1-\gamma}}{\mathrm{p}(z)} {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+} |\bar{\bar{\varrho}}(\mathfrak{s}_2)| \left(\psi(z) - \psi(\mathfrak{s}_1)\right)^{\gamma+\epsilon_2-1} \right\} \\ &\le \frac{1}{a}\left[\breve{\mathrm{M}}^* \|\varrho\ | + \omega^* \right] \left\{ {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+} \left(\frac{1}{\mathrm{p}^*} {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+} \Upsilon(z) \right) \right. \\ &\quad \left. - {}^{\psi}\mathcal{I}^{\epsilon_2}_{\mathfrak{s}_1^+} \left(\frac{\mathrm{q}^*}{\mathrm{p}^*} \|\varrho\| \right) - \frac{(\Gamma(\gamma))^2}{\mathrm{p}^*\Gamma(\epsilon_2+\gamma)} {}^{\psi}\mathcal{I}^{\epsilon_1}_{\mathfrak{s}_1^+} \Upsilon(\mathfrak{s}_2) \left(\psi(z) - \psi(\mathfrak{s}_1)\right)^{\epsilon_2} \right\} \\ &\le \left[\breve{\mathrm{M}}^* \tilde{\kappa} + \omega^* \right] \left[\mu \|\Upsilon\|_{L^1} + \mathcal{B}\tilde{\kappa} \right]. \end{aligned} \tag{91}$$

for all $z \in \mathrm{J}$. Hence, we get

$$\tilde{k} \le \left[\breve{\mathrm{M}}^* \tilde{\kappa} + \omega^* \right] \left[\mu \|\Upsilon\|_{L^1} + \mathcal{B}\tilde{\kappa} \right]. \tag{92}$$

According to the condition (69), one can see that Theorem 2.6, condition (ii) is impossible. Thus, $\varrho \in (\widehat{\omega}\varrho)(\widehat{\Theta}_2\varrho)$. Indeed, the operator $\widehat{\Theta}_1$ has a fixed point and so the Sturm-Liouville and Langevin $\mathbb{HFDI}$ (14) has at least one solution. □

5. Illustrative examples with numerical applications

In this section, in order to illustrate our results, we consider some examples. First, we review the Theorem 3.3.

Example 5.1. *Let us consider the Langevin and Sturm-Liouville* $\mathbb{HFDE}$,

$${}^{\psi}_{\mathrm{H}}\mathcal{D}^{7/8,1/7}_{0.2} \left[\frac{\sqrt{z}}{\sqrt{z+1}} \, {}^{\psi}_{\mathrm{H}}\mathcal{D}^{10/11,1/7}_{0.2} \left[\frac{12}{z \sin|\varrho(z)|} \varrho(z) \right] - \frac{1}{z^2} \varrho(z) \right] = \frac{z\sqrt{|\varrho(z)|}}{23\sqrt{|\varrho(z)|+1}}, \tag{93}$$

$\forall z \in \mathrm{J} = [\mathfrak{s}_1, \mathfrak{s}_2] = [0.2, 0.95]$, *with hybrid boundary conditions*

$$\begin{cases} \varrho(0.2) = 0, \\ \mathrm{p}(0.95)\, {}^{\psi}_{\mathrm{H}}\mathcal{D}^{10/11, 1/7}_{0.1} \left[\frac{\varrho(z)}{\omega(z,\varrho(z))}\right]_{z=0.95} - \mathrm{q}(0.95)\varrho(0.95) = 0. \end{cases} \tag{94}$$

Clearly, $\epsilon_1 = \frac{7}{8} \in \Delta$, $\zeta = \frac{1}{7} \in \overline{\Delta}$, $\epsilon_2 = \frac{10}{11} \in \Delta$, $\mathrm{p}(z) = \frac{\sqrt{z}}{\sqrt{z+1}}$, $\mathrm{q}(z) = \frac{1}{z^2}$, *function* $\omega : \mathrm{J} \times \mathbb{R} \to \mathbb{R}$ *is defined by* $\omega(z, \varrho(z)) = \frac{z}{12} \sin |\varrho(z)|$ *and* Θ *is defined as*

$$\Theta\left(z, \varrho(z)\right) = \frac{z\sqrt{|\varrho(z)|}}{23\sqrt{|\varrho(z)|+1}}, \tag{95}$$

$\gamma = (\epsilon_1 + \epsilon_2)(1 - \zeta) + \zeta \simeq 1.672077$. *Observe that*

$$\begin{aligned} |\omega(z, \varrho(z))| &= \left|\tfrac{z}{12} \sin |\varrho(z)|\right| \leq \tfrac{z}{12} \leq \tfrac{0.95}{12} = \mathring{M}_{\omega}, \\ |\Theta\left(z, \varrho(z)\right)| &= \left|\frac{z\sqrt{|\varrho(z)|}}{23\sqrt{|\varrho(z)|+1}}\right| \leq \tfrac{z}{23} \leq \tfrac{0.95}{23} = \mathring{M}_{\Theta}. \end{aligned} \tag{96}$$

So, condition (H1) *holds. Also, we get*

$$\begin{aligned} |\omega(z, \varrho(z)) - \omega(z, \acute{\varrho}(z))| &\leq \tfrac{z}{12} \left|\varrho(z) - \acute{\varrho}(z)\right|, \\ |\Theta\left(z, \varrho(z)\right) - \Theta\left(z, \acute{\varrho}(z)\right)| &\leq \tfrac{z}{23} \left|\varrho(z) - \acute{\varrho}(z)\right|. \end{aligned} \tag{97}$$

By choosing $\mathring{\eta}_{\omega} = \frac{0.95}{12}$ *and* $\mathring{\eta}_{\Theta} = \frac{0.95}{23}$, *Condition* (H2) *holds. Furthermore, Condition* (H3) *holds, whenever we put* $\mathring{\hbar}(z) = \frac{z}{23}$, $\mathring{\varphi}(\varrho) = (\frac{|\varrho|}{|\varrho|+1})^{0.5}$. *Also, by using Eq.* (41), *we obtain* $\omega_{\circ} = \sup_{z \in \mathrm{J}} |\omega(z, 0)| = 0$,

$$\mathrm{p}^* = \sup_{z \in \mathrm{J}} \mathrm{p}(z) = \sup_{z \in \mathrm{J}} \tfrac{\sqrt{z}}{\sqrt{z+1}} \simeq 0.6980, \qquad \mathrm{q}^* = \sup_{z \in \mathrm{J}} \mathrm{q}(z) = \sup_{z \in \mathrm{J}} \tfrac{1}{z^2} = 25. \tag{98}$$

Now, we consider four cases for ψ *as*

$$\psi_1(z) = 2^z, \quad \psi_2(z) = z, \quad \psi_3(z) = \ln z, \quad \psi_4(z) = \sqrt{z}, \tag{99}$$

where we know that ψ_2, ψ_3 *and* ψ_4 *give the Caputo, Caputo–Hadamard and Katugampola derivatives in this example. By using Eq.* (41), *we have*

$$\begin{aligned} \mathcal{A}_i = \tfrac{1}{1.67207} \left(\tfrac{\mathfrak{s}_2}{23}\right) \mathring{\varphi}(\kappa) \Bigg\{ & \frac{(\psi_i(\mathfrak{s}_2) - \psi_i(0.2))^{7/8 + \frac{10}{11}}}{\Gamma\left(\frac{7}{8} + \frac{10}{11} + 1\right)} \\ & + \frac{\Gamma\left(\gamma\right) (\psi_i(\mathfrak{s}_2) - \psi_i(0.2))^{\frac{7}{8} + \frac{10}{11}}}{\Gamma\left(\gamma + \frac{10}{11}\right) \Gamma\left(\frac{7}{8} + 1\right)} \Bigg\} \simeq \begin{cases} 0.034100, & \psi_1(z) = 2^z, \\ 0.031566, & \psi_2(z) = z, \\ 0.116346, & \psi_3(z) = \ln z, \\ 0.009014, & \psi_4(z) = \sqrt{z}, \end{cases} \end{aligned} \tag{100}$$

$$\mathcal{B} = \frac{\mathrm{q}^*}{\mathrm{p}^*} \frac{(\psi(\mathfrak{s}_2) - \psi(0.2))^{\epsilon_2}}{\Gamma\left(\frac{10}{11} + 1\right)} \simeq \begin{cases} 29.973241, & \psi_1(z) = 2^z, \\ 28.859326, & \psi_2(z) = z, \\ 54.719154, & \psi_3(z) = \ln z, \\ 16.264823, & \psi_4(z) = \sqrt{z}, \end{cases} \tag{101}$$

One can see these results in Tables 1, 2, 3 and 4. We can see graphical representation of $\mathcal{A}$, $\mathcal{B}$ *and* $\mathring{\eta}_{\omega}$, $\mathring{\eta}_{\omega}\mathring{M}_{\widehat{\Theta}}$ *for* $\psi_i \in \{2^z, z, \ln z, \sqrt{z}\}$ *in Figurs 1 and 2. In tables 1, 2, 3 and 4 at the end values of the interval* $[0.2, 0.95]$, *are marked with a circle for* κ *and* $\mathring{M}_{\widehat{\Theta}}$. *Actually,* $\mathring{\eta}_{\omega}\mathring{M}_{\widehat{\Theta}} < 1$

for $z \in [0.2, 0.95]$, $[0.2, 0.95]$, $[0.2, 0.6]$, $[0.2, 0.95]$ *whenever* $\psi_1(z) = 2^z$, $\psi_1(z) = z$ *and* $\kappa = 0.4$ *and* $\kappa = 0.4$, $\psi_1(z) = \ln z$ *and* $\kappa = 0.3$, $\psi_1(z) = \sqrt{z}$ *and* $\kappa = 0.55$ *respectively.*

TABLE 1. Numerical results of $\mathcal{A}$, $\mathcal{B}$, and $\mathring{M}_{\widehat{\Theta}}$ for $z \in [0.2, 0.95]$ of Langevin and Sturm-Liouville $\mathbb{HFDE}$ (93) with $\psi_1(z) = 2^z$.

	$\boldsymbol{\psi_1(z) = 2^z}$, $\kappa = 0.4$				
z	$\mathcal{A}$	$\mathcal{B}$	$\kappa >$	$\mathring{M}_{\widehat{\Theta}}$	$\mathring{\eta}_\omega \mathring{M}_{\widehat{\Theta}} < 1$
0.20	0.00000	0.00000	0.00000	0.00000	0.00000
0.25	0.00011	2.24498	0.02844	0.89810	0.07110
0.30	0.00042	4.18077	0.05297	1.67273	0.13242
0.35	0.00095	6.05363	0.07671	2.42241	0.19177
0.40	0.00173	7.90780	0.10022	3.16485	0.25055
0.45	0.00276	9.76356	0.12376	3.90819	0.30940
0.50	0.00409	11.63284	0.14748	4.65723	0.36870
0.55	0.00574	13.52372	0.17148	5.41523	0.42871
0.60	0.00774	15.44220	0.19585	6.18462	0.48962
0.65	0.01011	17.39304	0.22063	6.96733	0.55158
0.70	0.01289	19.38024	0.24589	7.76498	0.61473
0.75	0.01610	21.40727	0.27167	8.57901	0.67917
0.80	0.01979	23.47727	0.29801	9.41070	0.74501
0.85	0.02400	25.59313	0.32494	10.26125	0.81235
0.90	0.02875	27.75758	0.35251	11.13178	0.88127
0.95	0.03410	29.97324	0.38074	12.02340	0.95185

According to the numericals results in Table 1, all conditions of Theorem 3.3 holds. So the Langevin and Sturm-Liouville $\mathbb{HFDE}$ (93) *has at least one solution on* $[0.2, 0.95]$.

TABLE 2. Numerical results of $\mathcal{A}$, $\mathcal{B}$, and $\mathring{M}_{\widehat{\Theta}}$ for $z \in [0.2, 0.95]$ of Langevin and Sturm-Liouville $\mathbb{HFDE}$ (93) with $\psi_2(z) = z$.

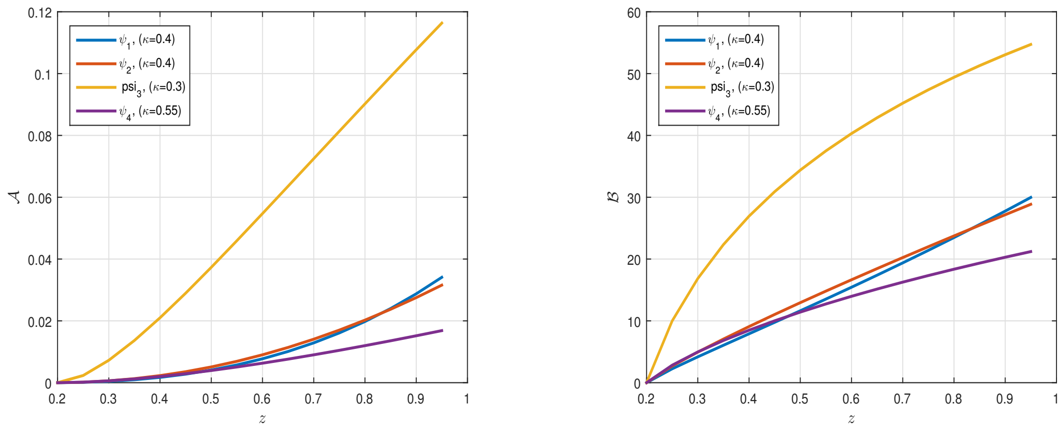
	$\boldsymbol{\psi_2(z) = z}$, $\kappa = 0.4$				
z	$\mathcal{A}$	$\mathcal{B}$	$\kappa >$	$\mathring{M}_{\widehat{\Theta}}$	$\mathring{\eta}_\omega \mathring{M}_{\widehat{\Theta}} < 1$
0.20	0.00000	0.00000	0.00000	0.00000	0.00000
0.25	0.00016	2.69902	0.03419	1.07977	0.08548
0.30	0.00060	4.95003	0.06272	1.98061	0.15680
0.35	0.00130	7.05809	0.08944	2.82454	0.22361
0.40	0.00229	9.07839	0.11507	3.63364	0.28766
0.45	0.00355	11.03583	0.13990	4.41788	0.34975
0.50	0.00509	12.94460	0.16413	5.18293	0.41032
0.55	0.00692	14.81382	0.18786	5.93245	0.46965
0.60	0.00902	16.64984	0.21118	6.66896	0.52796
0.65	0.01141	18.45732	0.23415	7.39434	0.58538
0.70	0.01408	20.23981	0.25682	8.11000	0.64204
0.75	0.01702	22.00011	0.27921	8.81707	0.69802
0.80	0.02025	23.74050	0.30135	9.51645	0.75339
0.85	0.02375	25.46284	0.32328	10.20888	0.80820
0.90	0.02752	27.16867	0.34501	10.89499	0.86252
0.95	0.03157	28.85933	0.36655	11.57530	0.91638

TABLE 3. Numerical results of $\mathcal{A}$, $\mathcal{B}$, and $\mathring{M}_{\widehat{\Theta}}$ for $z \in [0.2, 0.95]$ of Langevin and Sturm-Liouville HFDE (93) with $\psi_3(z) = \ln z$.

z	$\boldsymbol{\psi_3(z) = \ln z}$, $\kappa = 0.3$				
	$\mathcal{A}$	$\mathcal{B}$	$\kappa >$	$\mathring{M}_{\widehat{\Theta}}$	$\mathring{\eta}_\omega \mathring{M}_{\widehat{\Theta}} < 1$
0.20	0.00000	0.00000	0.00000	0.00000	0.00000
0.25	0.00233	9.99123	0.07124	2.99969	0.23748
0.30	0.00725	16.84872	0.12022	5.06187	0.40073
0.35	0.01366	22.33630	0.15947	6.71455	0.53157
0.40	0.02100	26.93580	0.19242	8.10174	0.64139
0.45	0.02896	30.90069	0.22086	9.29917	0.73618
0.50	0.03732	34.38640	0.24589	10.35324	0.81963
0.55	0.04594	37.49649	0.26825	11.29489	0.89418
0.60	0.05472	40.30385	0.28846	12.14588	0.96155
0.65	0.06359	42.86185	0.30690	12.92214	1.02300
0.70	0.07248	45.21087	0.32385	13.63574	1.07950
0.75	0.08136	47.38217	0.33953	14.29601	1.13177
0.80	0.09021	49.40047	0.35412	14.91036	1.18040
0.85	0.09900	51.28570	0.36776	15.48471	1.22587
0.90	0.10772	53.05411	0.38057	16.02395	1.26856
0.95	0.11635	54.71915	0.39264	16.53209	1.30879

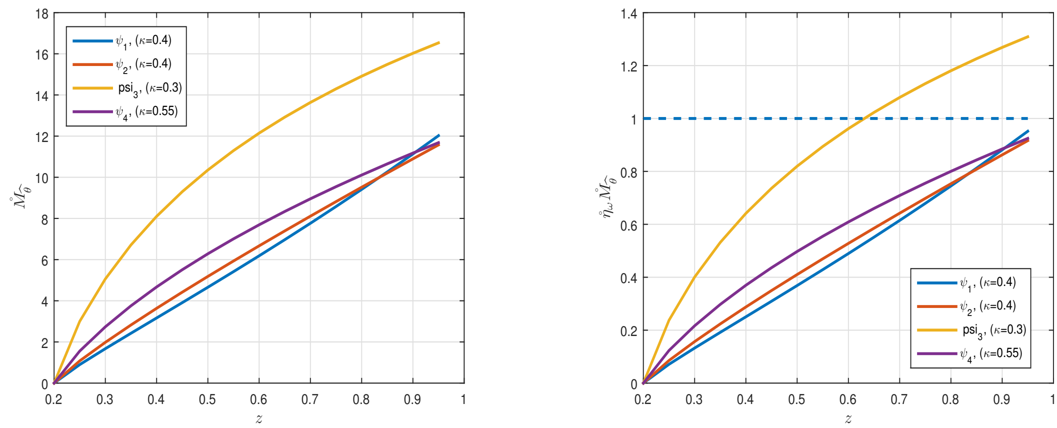
TABLE 4. Numerical results of $\mathcal{A}$, $\mathcal{B}$, and $\mathring{M}_{\widehat{\Theta}}$ for $z \in [0.2, 0.95]$ of Langevin and Sturm-Liouville HFDE (93) with $\psi_4(z) = \sqrt{z}$.

z	$\boldsymbol{\psi_4(z) = \sqrt{z}}$, $\kappa = 0.55$				
	$\mathcal{A}$	$\mathcal{B}$	$\kappa >$	$\mathring{M}_{\widehat{\Theta}}$	$\mathring{\eta}_\omega \mathring{M}_{\widehat{\Theta}} < 1$
0.20	0.00000	0.00000	0.00000	0.00000	0.00000
0.25	0.00018	2.83018	0.06778	1.55678	0.12324
0.30	0.00060	4.97206	0.11910	2.73524	0.21654
0.35	0.00122	6.82675	0.16354	3.75593	0.29734
0.40	0.00199	8.48945	0.20339	4.67119	0.36980
0.45	0.00291	10.00938	0.23983	5.50806	0.43606
0.50	0.00394	11.41703	0.27359	6.28331	0.49743
0.55	0.00508	12.73316	0.30515	7.00832	0.55483
0.60	0.00631	13.97275	0.33489	7.69132	0.60890
0.65	0.00763	15.14705	0.36307	8.33850	0.66013
0.70	0.00901	16.26482	0.38990	8.95467	0.70891
0.75	0.01047	17.33302	0.41555	9.54363	0.75554
0.80	0.01199	18.35727	0.44014	10.10849	0.80026
0.85	0.01356	19.34226	0.46380	10.65180	0.84327
0.90	0.01518	20.29185	0.48661	11.17570	0.88474
0.95	0.01685	21.20936	0.50865	11.68200	0.92482



(A) $\mathcal{A}$ (B) $\mathcal{B}$

FIGURE 1. Graphical representation of $\mathcal{A}$, $\mathcal{B}$ for ψ_i and $z \in [0.2, 0.95]$ in Example 5.1.



(A) $\mathring{M}_{\widehat{\Theta}}$ (B) $\mathring{\eta}_{\omega}\mathring{M}_{\widehat{\Theta}}$

FIGURE 2. Graphical representation of $\mathring{M}_{\widehat{\Theta}}$ and $\mathring{\eta}_{\omega}\mathring{M}_{\widehat{\Theta}}$ for ψ_i and $z \in [0.2, 0.95]$ in Example 5.1.

To check UH *stability, we consider the conditions of Theorem 3.7, In this case, by using relations* (64) *and* (53) *we have,*

$$\mu_i \simeq \begin{cases} 0.04303, & \psi_1(z) = 2^z, \\ 0.05885, & \psi_2(z) = z, \\ 0.02881, & \psi_4(z) = \sqrt{z}, \end{cases} \qquad \Lambda_i \simeq \begin{cases} 0.92604, & \psi_1(z) = 2^z, \\ 0.75433, & \psi_2(z) = z, \\ 0.75775, & \psi_4(z) = \sqrt{z}, \end{cases} \tag{102}$$

and

$$c_\omega = \frac{\mu}{1-\Lambda} \simeq \begin{cases} 0.58188, & \psi_1(z) = 2^z, \\ 0.11621, & \psi_2(z) = z, \\ 0.11892, & \psi_4(z) = \sqrt{z}. \end{cases} \tag{103}$$

Table 5 shows the numerical results of UH *stability of Problem* (93)*. As can be seen,* $c_\omega = 0.58188$, $c_\omega = 0.11621$, $c_\omega = 0.11892$ *whenever* $\psi_1(z) = 2^z$, $\psi_2(z) = z$, $\psi_4(z) = \sqrt{z}$ *respectively and stable for* $z \in [0.20, 0.35]$, $[0.2, 0.3]$, $[0.2, 0.3]$ *respectively.*

TABLE 5. Numerical results of μ_i, Λ_i and $c_\omega = \frac{\mu}{1-\Lambda}$ for $z \in [0.2, 0.95]$ of Langevin and Sturm-Liouville $\mathbb{HFDE}$ (93).

	$\psi_1(z) = 2^z$			$\psi_2(z) = z$			$\psi_4(z) = \sqrt{z}$		
z	μ	Λ	c_ω	μ	Λ	c_ω	μ	Λ	c_ω
0.20	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.25	0.00569	0.33709	0.00859	0.00829	0.40665	0.01397	0.00913	0.42679	0.01593
0.30	0.02023	0.63508	0.05544	0.02855	0.75433	<u>0.11621</u>	0.02881	0.75775	<u>0.11892</u>
0.35	0.04303	0.92604	<u>0.58188</u>	0.05885	1.08287	−0.71012	0.05498	1.04671	−1.17710
0.40	0.07420	1.21589	−0.34369	0.09832	1.39960	−0.24605	0.08575	1.30711	−0.27923
0.45	0.11404	1.50736	−0.22478	0.14640	1.70784	−0.20682	0.11998	1.54605	−0.21971
0.50	0.16300	1.80209	−0.20322	0.20268	2.00951	−0.20077	0.15690	1.76801	−0.20429
0.55	0.22160	2.10122	−0.20123	0.26684	2.30581	−0.20435	0.19599	1.97604	−0.20080
0.60	0.29042	2.40560	−0.20662	0.33862	2.59762	−0.21195	0.23686	2.17238	−0.20203
0.65	0.37015	2.71593	−0.21571	0.41780	2.88555	−0.22158	0.27922	2.35872	−0.20550
0.70	0.46150	3.03280	−0.22703	0.50420	3.17009	−0.23234	0.32284	2.53636	−0.21013
0.75	0.56528	3.35675	−0.23986	0.59766	3.45162	−0.24378	0.36755	2.70637	−0.21540
0.80	0.68234	3.68824	−0.25382	0.69803	3.73044	−0.25565	0.41320	2.86959	−0.22101
0.85	0.81360	4.02774	−0.26871	0.80518	4.00681	−0.26778	0.45966	3.02674	−0.22680
0.90	0.96006	4.37567	−0.28441	0.91899	4.28094	−0.28010	0.50685	3.17840	−0.23267
0.95	1.12280	4.73247	−0.30082	1.03936	4.55301	−0.29253	0.55468	3.32509	−0.23856

In the next Example 5.2, we review the Theorem 3.4.

Example 5.2. *We consider the Langevin and Sturm-Liouville* $\mathbb{HFDE}$

$$ {}^{\psi}_{\mathrm{H}}\mathcal{D}^{3/7,5/7}_{1.1}\left[\frac{1}{z^2+1}\,{}^{\psi}_{\mathrm{H}}\mathcal{D}^{4/5,5/7}_{1.1}\left[\frac{3\pi(1+\arctan|\varrho(z)|)\varrho(z)}{(|z|+1)\arctan|\varrho(z)|}\right] - \frac{\varrho(z)\exp(|z|)}{\exp(|z|)+1}\right] = \frac{z\sin|\varrho(z)|}{z+15}, \tag{104}$$

$\forall z \in \mathrm{J} = [\mathfrak{s}_1, \mathfrak{s}_2] = [1.1, 1.8]$, *with hybrid boundary conditions*

$$\begin{cases} \varrho(1.1) = 0, \\ \mathrm{p}(1.8)\,{}^{\psi}_{\mathrm{H}}\mathcal{D}^{\epsilon_2,\zeta}_{1.1}\left[\frac{\varrho(z)}{\omega(z,\varrho(z))}\right]_{z=1.8} - \mathrm{q}(1.8)\varrho(1.8) = 0. \end{cases} \tag{105}$$

Clearly, $\epsilon_1 = \frac{4}{5} \in \Delta$, $\zeta = \frac{1}{7} \in \overline{\Delta}$, $\epsilon_2 = \frac{9}{11} \in \Delta$, $\mathrm{p}(z) = \frac{1}{z^2+1}$, $\mathrm{q}(z) = \frac{\exp(|z|)}{\exp(|z|)+1}$, *and*

$$\omega(z,\varrho(z)) = \frac{(|z|+1)\arctan|\varrho(z)|}{3\pi(1+\arctan|\varrho(z)|)}, \qquad \Theta(z,\varrho(z)) = \frac{z\sin|\varrho(z)|}{z+15}. \tag{106}$$

So, condition (H1) *holds. Observe that*

$$\begin{aligned} |\omega(z,\varrho(z))| &= \left|\frac{(|z|+1)\arctan|\varrho(z)|}{3\pi(1+\arctan|\varrho(z)|)}\right| \le \frac{|z|+1}{3\pi} \le 0.297089 =: \mathring{M}_\omega, \\ |\Theta(z,\varrho(z))| &= \left|\frac{z\sin|\varrho(z)|}{z+15}\right| \le \frac{z}{z+15} \le 0.107142 =: \mathring{M}_\Theta, \end{aligned} \tag{107}$$

and

$$\begin{aligned} |\omega(z,\varrho(z)) - \omega(z,\acute{\varrho}(z))| &\le \frac{|z|+1}{3\pi}\left|\varrho(z) - \acute{\varrho}(z)\right|, \\ |\Theta(z,\varrho(z)) - \Theta(z,\acute{\varrho}(z))| &\le \left|\frac{z}{z+15}\right|\left|\varrho(z) - \acute{\varrho}(z)\right|. \end{aligned} \tag{108}$$

By choosing $\mathring{\eta}_\omega = \frac{2.8}{3\pi}$ *and* $\mathring{\eta}_\Theta = \frac{1.8}{16.8}$*, Condition* (H2) *holds. Now we choose three values for* $\epsilon_1 \in \left\{\frac{1}{7}, \frac{1}{2}, \frac{7}{8}\right\}$*, with* $\psi(z) = z$*. From Eqs.* (34) *and* (53)*, we obtain*

$$\gamma_i = (\epsilon_1 + \epsilon_2)(1 - \zeta) + \zeta = \left(\epsilon_1 + \tfrac{4}{5}\right)\left(1 - \tfrac{5}{7}\right) + \tfrac{5}{7} \simeq \begin{cases} 0.983673, & \epsilon_1 = \frac{1}{7}, \\ 1.085714, & \epsilon_1 = \frac{1}{2}, \\ 1.192857, & \epsilon_1 = \frac{7}{8}, \end{cases} \tag{109}$$

and

$$\begin{aligned} \Lambda_i = {} & \left\{ \frac{\mathring{M}_\Theta}{\mathrm{p}^*} \frac{(\mathfrak{s}_2 - 1.1)^{\epsilon_1 + 4/5}}{\Gamma\left(\epsilon_1 + \frac{4}{5} + 1\right)} + \frac{\mathrm{q}^*}{\mathrm{p}^*} \frac{(\mathfrak{s}_2 - 1.1)^{\epsilon_1}}{\Gamma\left(\frac{4}{5} + 1\right)} + \frac{\mathring{M}_\Theta}{\mathrm{p}^*} \frac{(\Gamma(\gamma_i))^2 (\mathfrak{s}_2 - 1.1)^{\epsilon_1 + 4/5}}{\Gamma\left(\gamma_i + \frac{4}{5}\right)\Gamma(\epsilon_1 + 1)} \right\} \mathring{\eta}_\omega \\ & + \left\{ \frac{\mathring{\eta}_\Theta}{\mathrm{p}^*} \frac{(\mathfrak{s}_2 - 1.1)^{\epsilon_1 + 4/5}}{\Gamma\left(\epsilon_1 + \frac{4}{5} + 1\right)} + \frac{\mathrm{q}^*}{\mathrm{p}^*} \frac{(\mathfrak{s}_2 - 1.1)^{4/5}}{\Gamma\left(\frac{4}{5} + 1\right)} + \frac{\mathring{\eta}_\Theta}{\mathrm{p}^*} \frac{(\Gamma(\gamma_i))^2 (\mathfrak{s}_2 - 1.1)^{\epsilon_1 + 4/5}}{\Gamma\left(\gamma_i + \frac{4}{5}\right)\Gamma(\epsilon_1 + 1)} \right\} \mathring{M}_\omega \\ \simeq {} & \left\{ \begin{array}{ll} 0.95507, & \epsilon_1 = \frac{1}{7}, \\ 0.96875, & \epsilon_1 = \frac{1}{2}, \\ 0.93475, & \epsilon_1 = \frac{7}{8}, \end{array} \right\} < \left\{ \begin{array}{ll} 1, & z \in [1.1, 1.45], \\ 1, & z \in [1.1, 1.55], \\ 1, & z \in [1.1, 1.60]. \end{array} \right. \end{aligned} \tag{110}$$

These results show in Table 6*. We can see graphical representation of* Λ_i *for* $\epsilon_1 \in \left\{\frac{1}{7}, \frac{1}{2}, \frac{7}{8}\right\}$ *and* $z \in [1.1, 1.8]$*, in Figure* 3*. In table* 6 *the intervals that in it* $\Lambda_i < 1$ *are marked with a circle for* $\epsilon_1 \in \{\frac{1}{7}, \frac{1}{2}, \frac{7}{8}\}$*. Actually,* $\Lambda_i < 1$ *for* $z \in [1.1, 1.45]$*,* $[1.1, 1.55]$*,* $[1.1, 1.60]$ *whenever* $\epsilon_1 = \frac{1}{7}$*,* $\epsilon_1 = \frac{1}{2}$*,* $\epsilon_1 = \frac{7}{8}$ *respectively. Using the given values there exists a solution for the Langevin and Sturm-Liouville* $\mathbb{HFDE}$ (104) *on* J *by Theorem* 3.4*. Furthermore, Theorem* 3.7 *implies that the solution of Langevin and Sturm-Liouville* $\mathbb{HFDE}$ (104) *is* UH *and generalized* UH *stable.*

Table 6. Numerical results of Λ_i, for $\epsilon_1 \in \{\frac{1}{7}, \frac{1}{2}, \frac{7}{8}\}$ and $z \in [1.1, 1.8]$ of Langevin and Sturm-Liouville $\mathbb{HFDE}$ (104) with $\psi_2(z) = z$.

z	$\psi(z) = z$		
	$\Lambda_i < 1$		
	$\epsilon_1 = \frac{1}{7}$	$\epsilon_1 = \frac{1}{2}$	$\epsilon_1 = \frac{7}{8}$
1.10	0.00000	0.00000	0.00000
1.15	0.35013	0.18637	0.12943
1.20	0.47538	0.30604	0.23078
1.25	0.58388	0.41296	0.32520
1.30	0.68371	0.51316	0.41604
1.35	0.77791	0.60907	0.50471
1.40	0.86806	0.70194	0.59199
1.45	<u>0.95507</u>	0.79253	0.67836
1.50	1.03955	0.88136	0.76414
1.55	1.12194	<u>0.96875</u>	0.84955
1.60	1.20254	1.05498	<u>0.93475</u>
1.65	1.28158	1.14023	1.01987
1.70	1.35927	1.22465	1.10501
1.75	1.43574	1.30836	1.19025
1.80	1.51112	1.39146	1.27564

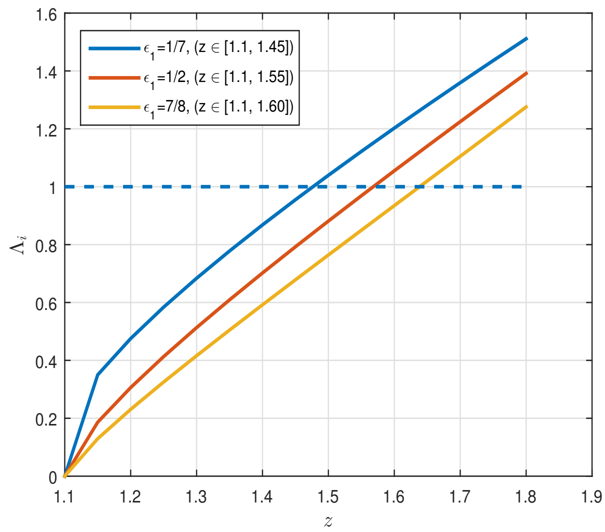


FIGURE 3. Graphical representation of Λ_i for $\epsilon_1 \in \{\frac{1}{7}, \frac{1}{2}, \frac{7}{8}\}$ and $z \in [1.1, 1.8]$ of Langevin and Sturm-Liouville HFDE (104) with $\psi_2(z) = z$ in Example 5.2.

In the next Example 5.3, we consider the Theorem 4.3.

Example 5.3. *Let us consider the Sturm-Liouville and Langevin* HFDI

$$\begin{aligned} &{}^{\psi}_{\mathrm{H}}\mathcal{D}_0^{2/9,4/11}\left[\tfrac{3\cos^2 z}{3\cos^2 z+1}\,{}^{\psi}_{\mathrm{H}}\mathcal{D}_0^{8/13,4/11}\left[\tfrac{2\pi\varrho(z)}{(z^2+\frac{\pi}{4})\arctan(|\varrho(z)|)}\right]\right.\\ &\left.-\tfrac{1}{1+\sin^2 z}\varrho(z)\right] \in \left[0,\ \tfrac{\tan^2|\varrho(z)|}{\tan^2|\varrho(z)|+4}+\tfrac{3}{15z^2+5}+\tfrac{4}{9}\right], \end{aligned} \tag{111}$$

$\forall z \in \mathrm{J} = [\mathfrak{s}_1, \mathfrak{s}_2] = [\frac{\pi}{56}, \frac{2\pi}{7}]$, *with hybrid boundary conditions*

$$\begin{cases} \varrho(\frac{\pi}{56}) = 0, \\ \mathrm{p}\left(\frac{2\pi}{7}\right)\,{}^{\psi}_{\mathrm{H}}\mathcal{D}_0^{8/13,4/11}\left[\frac{\varrho(z)}{\omega(z,\varrho(z))}\right]_{z=\frac{2\pi}{7}} - \mathrm{q}\left(\frac{2\pi}{7}\right)\varrho\left(\frac{2\pi}{7}\right) = 0. \end{cases} \tag{112}$$

Clearly, $\epsilon_1 = \frac{2}{9} \in \Delta$, $\zeta = \frac{4}{11} \in \overline{\Delta}$, $\epsilon_2 = \frac{8}{13} \in \Delta$, $\mathrm{p}(z) = \frac{3\cos^2 z}{3\cos^2 z+1}$, $\mathrm{q}(z) = \frac{1}{1+\sin^2 z}$,

$$\begin{aligned} \omega(z,\varrho(z)) &= \tfrac{1}{2\pi}\left(z^2+\tfrac{\pi}{4}\right)\arctan(|\varrho(z)|),\\ \Theta(z,\varrho(z)) &= \tfrac{4}{9}+\tfrac{3}{15z^2+5}+\tfrac{\tan^2|\varrho(z)|}{4+\tan^2|\varrho(z)|}, \end{aligned} \tag{113}$$

and

$$\gamma = (\epsilon_1+\epsilon_2)(1-\zeta)+\zeta = \left(\tfrac{2}{9}+\tfrac{8}{13}\right)\left(1-\tfrac{4}{11}\right)+\tfrac{4}{11} \simeq 1.672077.$$

Observe that Θ *has* $\mathcal{L}^1$*-Carathéodory property and satifies in Condition* (C1), *and*

$$\begin{aligned} |\omega(z,\varrho(z))| &= \left|\tfrac{1}{2\pi}\left(z^2+\tfrac{\pi}{4}\right)\arctan(|\varrho(z)|)\right| \le \tfrac{4z^2+\pi}{8\pi}|\varrho(z)| =: \breve{\mathrm{M}}|\varrho(z)|,\\ |\Theta(z,\varrho(z))| &= \left|\tfrac{4}{9}+\tfrac{3}{15z^2+5}+\tfrac{\tan^2|\varrho(z)|}{4+\tan^2|\varrho(z)|}\right| \le \tfrac{13}{9}+\tfrac{3}{15z^2+5} =: \Upsilon(z). \end{aligned} \tag{114}$$

So, conditions (C2) *and* (C3) *hold. By using Eqs.* (41), *we obtain*

$$\begin{aligned} \mathrm{p}^* &= \sup_{z\in\mathrm{J}}\mathrm{p}(z) = \sup_{z\in\mathrm{J}}\tfrac{3\cos^2 z}{3\cos^2 z+1} \simeq 0.7494,\\ \mathrm{q}^* &= \sup_{z\in\mathrm{J}}\mathrm{q}(z) = \sup_{z\in\mathrm{J}}\tfrac{1}{1+\sin^2 z} = 0.9968, \end{aligned} \tag{115}$$

and $\omega^* = \sup_{z\in J} |\omega(z,0)| = 0$. *Now, we consider four cases for* ψ *as*

$$\psi \in \left\{\psi_1 = 2^z, \psi_2 = z, \psi_3 = \ln z, \psi_4 = \sqrt{z}\right\}, \tag{116}$$

the same in Example 5.1. Hence, from Eqs. (41) and (64), we have

$$\mathcal{B}_i = \frac{q^*\left(\psi_i\left(\frac{2\pi}{7}\right)-\psi_i(0)\right)^{8/13}}{p^*\Gamma\left(\frac{8}{13}+1\right)} \simeq \begin{cases} 1.42980, & \psi_1(z) = 2^z, \\ 1.42980, & \psi_2(z) = z, \\ 1.87256, & \psi_3(z) = \ln z, \\ 1.38942, & \psi_4(z) = \sqrt{z}, \end{cases} \tag{117}$$

and

$$\begin{aligned} \mu_i &= \frac{1}{p^*}\left\{\frac{\left(\psi_i\left(\frac{2\pi}{7}\right)-\psi_i(0)\right)^{2/9+8/13}}{\Gamma\left(\frac{2}{9}+\frac{8}{13}+1\right)} + \frac{(\Gamma(\gamma))^2\left(\psi_i\left(\frac{2\pi}{7}\right)-\psi_i(0)\right)^{2/9+\frac{8}{13}}}{\Gamma\left(\gamma+\frac{8}{13}\right)\Gamma\left(\frac{2}{9}+1\right)}\right\} \\ &\simeq \begin{cases} 2.86407, & \psi_1(z) = 2^z, \\ 2.86407, & \psi_2(z) = z, \\ 2.43463, & \psi_3(z) = \ln z, \\ 2.57092, & \psi_4(z) = \sqrt{z}, \end{cases} \end{aligned} \tag{118}$$

One can see these results in Tables 7, 8, 9 and 10. We can see graphical representation of $\mathcal{B}$, μ,

$$\breve{M}^*\left(\mu\|\Upsilon\|_{\mathcal{L}^1} + \mathcal{B}\tilde{\kappa}\right) \simeq \begin{cases} 0.49282, & \psi_1(z) = 2^z, \\ 0.49282, & \psi_2(z) = z, \\ 0.39104, & \psi_3(z) = \ln z, \\ 0.44686, & \psi_4(z) = \sqrt{z}, \end{cases} \tag{119}$$

and

$$\begin{aligned} \tilde{\kappa} = \left\{\begin{array}{ll} 0.5, & \psi_1(z) = 2^z, \\ 0.5, & \psi_2(z) = z, \\ 0.2, & \psi_3(z) = \ln z, \\ 0.5, & \psi_4(z) = \sqrt{z}, \end{array}\right\} &> \left[\breve{M}^*\tilde{\kappa} + \omega^*\right]\left[\mu\|\Upsilon\|_{L^1} + \mathcal{B}\tilde{\kappa}\right] \\ &\simeq \begin{cases} 0.24644, & \psi_1(z) = 2^z, \\ 0.24644, & \psi_2(z) = z, \\ 0.07821, & \psi_3(z) = \ln z, \\ 0.22343, & \psi_4(z) = \sqrt{z}, \end{cases} \end{aligned} \tag{120}$$

for $\psi_i \in \{2^z, z, \ln z, \sqrt{z}\}$ *in Figurs 4 and 5. According to the numerical results, all conditions of Theorem 4.3 holds. So the Sturm-Liouville and Langevin* HFDI *(111) has at least one solution on* $\left[\frac{\pi}{56}, \frac{2\pi}{7}\right]$.

TABLE 7. Numerical results of $\mathcal{B}$, μ, and $\left[\breve{\mathrm{M}}^{*}\tilde{\kappa}+\omega^{*}\right][\mu\|\Upsilon\|_{L^1}+\mathcal{B}\tilde{\kappa}]$, $\breve{\mathrm{M}}^{*}\left(\mu\|\Upsilon\|_{\mathcal{L}^1}+\mathcal{B}\tilde{\kappa}\right)$ of Sturm-Liouville and Langevin HFDI (111) for $z \in [\frac{\pi}{56}, \frac{2\pi}{7}]$ whenever $\psi_1(z) = 2^z$.

z	$\psi_1(z) = 2^z$, $\kappa = 0.5$			
	$\mathcal{B}$	μ	$\tilde{\kappa} >$	$\breve{\mathrm{M}}^{*}\left(\mu\|\Upsilon\|_{\mathcal{L}^1}+\mathcal{B}\tilde{\kappa}\right) < 0.5$
0.06	0.00000	0.00000	0.00000	0.00000
0.11	0.78329	0.29640	0.03879	0.07758
0.17	0.91373	0.52969	0.05916	0.11832
0.22	0.99988	0.74391	0.07716	0.15431
0.28	1.06589	0.94661	0.09386	0.18772
0.34	1.12008	1.14115	0.10970	0.21940
0.39	1.16639	1.32943	0.12490	0.24981
0.45	1.20704	1.51266	0.13961	0.27921
0.50	1.24339	1.69167	0.15390	0.30780
0.56	1.27637	1.86707	0.16785	0.33570
0.62	1.30660	2.03933	0.18151	0.36301
0.67	1.33457	2.20881	0.19491	0.38981
0.73	1.36063	2.37580	0.20808	0.41615
0.79	1.38505	2.54055	0.22104	0.44208
0.84	1.40804	2.70325	0.23382	0.46764
<u>0.90</u>	1.42980	2.86407	0.24644	0.49287
0.95	1.45045	3.02315	0.25889	0.51779

TABLE 8. Numerical results of $\mathcal{B}$, μ, and $\left[\breve{\mathrm{M}}^{*}\tilde{\kappa}+\omega^{*}\right][\mu\|\Upsilon\|_{L^1}+\mathcal{B}\tilde{\kappa}]$, $\breve{\mathrm{M}}^{*}\left(\mu\|\Upsilon\|_{\mathcal{L}^1}+\mathcal{B}\tilde{\kappa}\right)$ of Sturm-Liouville and Langevin HFDI (111) for $z \in [\frac{\pi}{56}, \frac{2\pi}{7}]$ whenever $\psi_2(z) = z$.

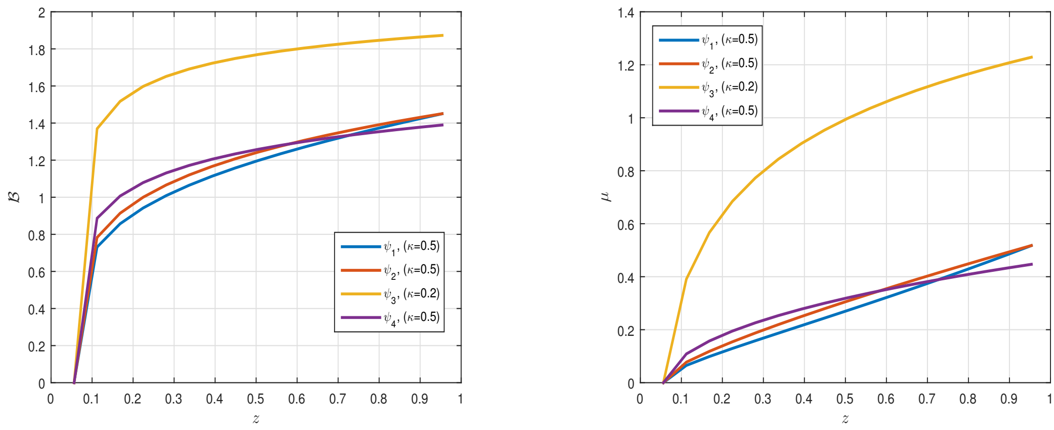
z	$\psi_2(z) = z$, $\kappa = 0.5$			
	$\mathcal{B}$	μ	$\tilde{\kappa} >$	$\breve{\mathrm{M}}^{*}\left(\mu\|\Upsilon\|_{\mathcal{L}^1}+\mathcal{B}\tilde{\kappa}\right) < 0.5$
0.06	0.00000	0.00000	0.00000	0.00000
0.11	0.78329	0.29640	0.03879	0.07758
0.17	0.91373	0.52969	0.05916	0.11832
0.22	0.99988	0.74391	0.07716	0.15431
0.28	1.06589	0.94661	0.09386	0.18772
0.34	1.12008	1.14115	0.10970	0.21940
0.39	1.16639	1.32943	0.12490	0.24981
0.45	1.20704	1.51266	0.13961	0.27921
0.50	1.24339	1.69167	0.15390	0.30780
0.56	1.27637	1.86707	0.16785	0.33570
0.62	1.30660	2.03933	0.18151	0.36301
0.67	1.33457	2.20881	0.19491	0.38981
0.73	1.36063	2.37580	0.20808	0.41615
0.79	1.38505	2.54055	0.22104	0.44208
0.84	1.40804	2.70325	0.23382	0.46764
<u>0.90</u>	1.42980	2.86407	0.24644	0.49287
0.95	1.45045	3.02315	0.25889	0.51779

TABLE 9. Numerical results of $\mathcal{B}$, μ, and $\left[\breve{\mathrm{M}}^*\tilde{\kappa}+\omega^*\right][\mu\|\Upsilon\|_{L^1}+\mathcal{B}\tilde{\kappa}]$, $\breve{\mathrm{M}}^*\left(\mu\|\Upsilon\|_{\mathcal{L}^1}+\mathcal{B}\tilde{\kappa}\right)$ of Sturm-Liouville and Langevin HFDI (111) for $z\in[\frac{\pi}{56},\frac{2\pi}{7}]$ whenever $\psi_3(z)=\ln z$.

z	$\psi_3(z)=\ln z$, $\kappa=0.2$			
	$\mathcal{B}$	μ	$\tilde{\kappa}>$	$\breve{\mathrm{M}}^*\left(\mu\|\Upsilon\|_{\mathcal{L}^1}+\mathcal{B}\tilde{\kappa}\right)<0.5$
0.06	0.00000	0.00000	0.00000	0.00000
0.11	1.36949	2.43463	0.07821	0.39104
0.17	1.51707	3.58072	0.11336	0.56680
0.22	1.59754	4.35089	0.13692	0.68460
0.28	1.65142	4.93027	0.15462	0.77311
0.34	1.69127	5.39396	0.16878	0.84389
0.39	1.72258	5.78003	0.18056	0.90279
0.45	1.74817	6.11045	0.19064	0.95318
0.50	1.76971	6.39905	0.19943	0.99717
0.56	1.78822	6.65508	0.20724	1.03620
0.62	1.80441	6.88505	0.21425	1.07124
0.67	1.81876	7.09371	0.22061	1.10303
0.73	1.83162	7.28461	0.22642	1.13211
0.79	1.84325	7.46049	0.23178	1.15890
0.84	1.85385	7.62351	0.23675	1.18373
0.90	1.86358	7.77540	0.24137	1.20685
0.95	1.87256	7.91756	0.24570	1.22850

TABLE 10. Numerical results of $\mathcal{B}$, μ, and $\left[\breve{\mathrm{M}}^*\tilde{\kappa}+\omega^*\right][\mu\|\Upsilon\|_{L^1}+\mathcal{B}\tilde{\kappa}]$, $\breve{\mathrm{M}}^*\left(\mu\|\Upsilon\|_{\mathcal{L}^1}+\mathcal{B}\tilde{\kappa}\right)$ of Sturm-Liouville and Langevin HFDI (111) for $z\in[\frac{\pi}{56},\frac{2\pi}{7}]$ whenever $\psi_4(z)=\sqrt{z}$.

z	$\psi_4(z)=\sqrt{z}$, $\kappa=0.5$			
	$\mathcal{B}$	μ	$\tilde{\kappa}>$	$\breve{\mathrm{M}}^*\left(\mu\|\Upsilon\|_{\mathcal{L}^1}+\mathcal{B}\tilde{\kappa}\right)<0.5$
0.06	0.00000	0.00000	0.00000	0.00000
0.11	0.88688	0.47337	0.05434	0.10868
0.17	1.00652	0.76271	0.07872	0.15743
0.22	1.07876	0.99042	0.09744	0.19488
0.28	1.13078	1.18281	0.11308	0.22615
0.34	1.17152	1.35162	0.12669	0.25338
0.39	1.20505	1.50331	0.13886	0.27772
0.45	1.23357	1.64187	0.14993	0.29986
0.50	1.25841	1.76996	0.16013	0.32027
0.56	1.28041	1.88949	0.16963	0.33926
0.62	1.30018	2.00182	0.17854	0.35707
0.67	1.31814	2.10803	0.18694	0.37388
0.73	1.33459	2.20892	0.19491	0.38983
0.79	1.34978	2.30517	0.20251	0.40502
0.84	1.36388	2.39730	0.20977	0.41954
0.90	1.37706	2.48576	0.21673	0.43346
0.95	1.38942	2.57092	0.22343	0.44686



(A) $\mathcal{B}$ (B) μ

FIGURE 4. Graphical representation of $\mathcal{B}$, μ for ψ_i and $z \in [\frac{\pi}{56}, \frac{2\pi}{7}]$ in Example 5.3.

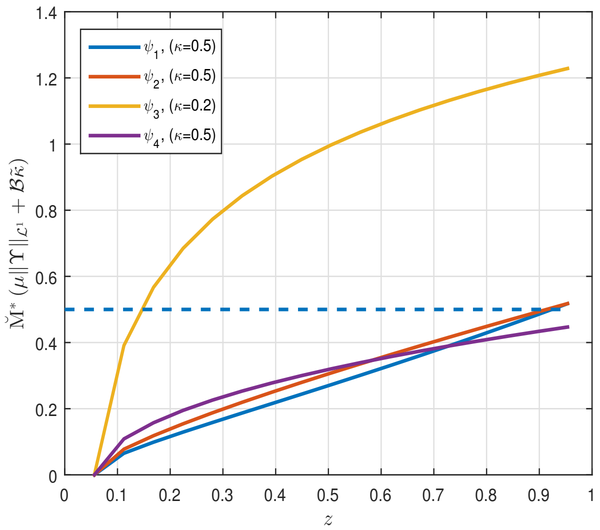


FIGURE 5. Graphical representation of $\breve{M}^* \left(\mu \|\Upsilon\|_{\mathcal{L}^1} + \mathcal{B}\tilde{\kappa}\right)$ for ψ_i and $z \in [\frac{\pi}{56}, \frac{2\pi}{7}]$ in Example 5.3.

6. Conclusion

In this paper, we have defined a new fractional mathematical model of a boundary value problem to the nonlinear Sturm-Liouville and Langevin $\mathbb{HFDI}$ (14) and Langevin and Sturm-Liouville $\mathbb{HFDE}$ (16) under condition (15) in ψ–Hilfer fractional operators, and turned to the investigation of the qualitative behaviors of its solutions including existence, uniqueness and stability. To confirm the existence criterion, we utilize the presumptions of Dhage's fixed point for the operator within the hybrid case. Moreover, the stability analysis within the UH sense of a given system is considered.

We have observed that, with our discussions, Problem (16) is not just incorporated the previously specified boundary value problems in the literature, yet additionally non trivially stretches out the circumstance to a lot more extensive class of boundary value problems for $\mathbb{FDE}$s, i.e., for various values of ζ and ψ, our considered problem covers the problems referenced in Remark 1.1. At last, illustrations are provided to guarantee the legitimacy of our obtained results. This topic can be used in mathematical modeling of applied problems in science, engineering, and the real-world phenomena [45–48].

Availability of Data and Materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

AB: Actualization, methodology, formal analysis, validation, investigation, and initial draft. **MKAK**: Actualization, methodology, validation, investigation, initial draft, formal analysis and supervision of the original draft, editing. **MES**: Actualization, methodology, formal analysis, validation, investigation, software, simulation, initial draft and was a major contributor in writing the manuscript. All authors read and approved the final manuscript.

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