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A BRIEF STUDY CONCERNING RATIONAL TYPE CONTRACTION VIA A DIGRAPH AND DEFORMATION OF AN ELASTIC BEAM

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Abstract. In this paper we firstly bring forward some findings for certain rational type expression in the sense of metric space with a digraph. Secondly, utilizing the proposed results, we assert a solution of elastic beam equations. Our results generalize the conclusions given by Banach, Kannan, Chatterjee, Fabiano so on.

1. Introduction and preliminaries

Fixed point (brief, fp) theory could be defines as a framework for searching and probing the existence of the solving of the eq $Ga = a$ for a particular selfmap G that is described on a set $\Theta \neq \emptyset$. Unsurprisingly, herein, a is implied the fp of the map G. Many worthwhile mathematical essentials can be stated by arguments which call that obvious maps have fp. This theory is a crucial branch of mathematics along with diverse numerical science fields such as engineering, management, chemistry, biology, economics, medical and computer sciences, physics and game theory [9]- [13].

$Z : \Theta^2 \to \Theta$ and $h : \Theta \to \Theta$, a notion of coupled coincidence point, $(x, y) \in \Theta^2$ such that $Z(y, x) = hy$ and $Z(x, y) = hx$ was initially put forward by [1].

Definition 1.1. *[1] Let $\Theta \neq \emptyset$ and $Z : \Theta^2 \to \Theta$ and $h : \Theta \to \Theta$. We call Z and h are commutative if $hZ(x, y) = Z(hx, hy)$ for $\forall x,\ y \in \Theta$.*

Researchers [14] presented a concept of compatible hereinafter.

Definition 1.2. *[14] Let (Θ, d) be a metric space. $Z : \Theta^2 \to \Theta$, $h : \Theta \to \Theta$ are called to be compatible if $\lim_{n\to\infty} d(hZ(y_n, x_n), Z(hy_n, hx_n)) = 0 = \lim_{n\to\infty} d(hZ(x_n, y_n), Z(hx_n, hy_n))$ whenevere $(x_n), (y_n) \subset \Theta$ such that $\lim_{n\to\infty} Z(y_n, x_n) = \lim_{n\to\infty} hy_n$, $\lim_{n\to\infty} Z(x_n, y_n) = \lim_{n\to\infty} hx_n$.*

Jachymski [2] obtained some fp results of contraction maps on metric space with a graph (briefly, $MSWG$). Many authors have generalized, enriched and complemented the results of [2] via some operators in abstract spaces ([3]- [7]).

Let (Θ, d) be a metric space, Δ be a diagonal of Θ^2, and G be a digraph without parallel edges such that $E(G) \supseteq \Delta$, $E(G)$ is the edges of the graph, and $V(G)$ of its vertices overlaps

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Θ. Videlicet, $(V(G), E(G)) = G$. Additionally, indicate by G^{-1} reaped from G by reversing the direction of the edges via G. Therefrom, $E(G^{-1}) = \{(x,y) \in \Theta^2 : (y,x) \in E(G)\}$.

Definition 1.3. *[8] We call that $Z : \Theta^2 \to \Theta$ and $h : \Theta \to \Theta$ are $G-$edge preserving if* $\{(hx, h\varpi), (hy, h\omega)\} \in E(G) \Longrightarrow \{(Z(x,y), Z(\varpi,\omega)), (Z(y,x), Z(\omega,\varpi))\} \in E(G)$.

Definition 1.4. *[8] Let (Θ, d) be a complete metric space. We call that $E(G)$ ensures the transitivity property* $\Longleftrightarrow$ $\forall \tau, \varrho, \xi \in \Theta$; $(\tau,\xi), (\xi,\varrho) \in E(G) \Longrightarrow (\tau,\varrho) \in E(G)$.

Definition 1.5. *[7] Let (Θ, d) be a complete $MSWG$, $Z : \Theta^2 \to \Theta$ be a map. Here,*

(i) Z is said $G-$continuous if for $\forall$ $(\tau,\upsilon), (x,y) \in \Theta^2$ and for $(n_k)_{k\in N}$ of $\mathbb{Z}^+$ such that $(x_{n_k}) \to \tau$, $(y_{n_k}) \to \upsilon$ as $k \to \infty$ and $(x_{n_k}, x_{n_k+1}) \in E(G)$, $(y_{n_k}, y_{n_k+1}) \in E(G^{-1})$ implies $Z(x_{n_k}, y_{n_k}) \to Z(\tau,\upsilon)$ for $n \in N$ and $Z(y_{n_k}, x_{n_k}) \to Z(\upsilon,\tau)$ as $k \to \infty$;

(ii) (Θ, d, G) hold property A if (a) $(x_n)_{n\in N} \subseteq \Theta$ with $(x_n) \to \tau$ when $n \to \infty$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in N$, in that case $(x_n, \tau) \in E(G)$; (b) $(y_n)_{n\in N} \subseteq \Theta$ with $(y_n) \to \upsilon$ as $n \to \infty$ and $(y_n, y_{n+1}) \in E(G^{-1})$, in that case $(y_n, \upsilon) \in E(G^{-1})$.

Suantai et al. [8] studied firstly coincidence coupled fp results of $\theta-\psi-$contractions maps in $MSWG$. Yolacan et al. [15] considered new findings for coupled coincidence point and coupled fp of $\varphi-\psi-$contraction maps on $MSWG$. Rao and Kalyani [18] derived coupled fp results for contractive condition of rational type on abstract space. Rao&Kalyani [16] established some existence and uniqueness results for a novel rational contractions on partially ordered metric space. Recently, Fabiano [17] showed that the Eq (1) defined by [16] was not well defined. He modified the term proportional to b of the Eq (1) in [16] and propounded verified proofs.

Inspired and motivated by these facts, we study Fabiano's contraction [17] in this paper. The condition is given as indicated below.

Definition 1.6. *Let (Θ, d) be a complete $MSWG$. The maps $Z : \Theta^2 \to \Theta$ and $h : \Theta \to \Theta$ are said a rational contraction if;*

1. Z, h are $G-$edge preserving;

2. there is $\varsigma_i \in [0,1)$ for $i = \overline{1,5}$ with $0 \le \varsigma_1 + \varsigma_2 + 2(\varsigma_3 + \varsigma_4) + \varsigma_5 < 1$ such that for $\forall x, y, \varpi, \nu \in \Theta$ supplying $(hx, h\varpi), (hy, h\nu) \in E(G)$,

$$
\begin{aligned}
& d(Z(x,y), Z(\varpi,\nu)) \\
\le\ & \varsigma_1 \frac{d(hx, Z(x,y))\,[1 + d(h\varpi, Z(\varpi,\nu))]}{1 + d(hx, h\varpi)} \\
& + \varsigma_2 \frac{d(hx, Z(x,y))\, d(h\varpi, Z(\varpi,\nu))}{1 + d(hx, Z(x,y))} \\
& + \varsigma_3 [d(hx, Z(x,y)) + d(h\varpi, Z(\varpi,\nu))] \\
& + \varsigma_4 [d(hx, Z(\varpi,\nu)) + d(h\varpi, Z(x,y))] + \varsigma_5 d(hx, h\varpi).
\end{aligned}
\tag{1.1}
$$

Determine the set $Fix_{coin(Zh)}$ of all coupled coincidence points of maps $Z : \Theta^2 \to \Theta$, $h : \Theta \to \Theta$ and the set $(\Theta^2)_{Zh}$ as noted below:

$$Fix_{coin(Zh)} = \left\{(x,y) \in \Theta^2 : hy = Z(y,x),\ hx = Z(x,y)\right\}$$

and

$$(\Theta^2)_{Zh} = \left\{(x,y) \in \Theta^2 : (hx, Z(x,y)) \in E(G),\ (hy, Z(y,x)) \in E\left(G^{-1}\right)\right\}.$$

Henceforward, we firstly bring forward some findings for certain rational type expression in the sense of $MSWG$. Secondly, utilizing the proposed results, we assert a solution of elastic beam equations. Our results generalize the conclusions given by Banach [19], Kannan [20], Chatterjee [21], Fabiano [17] so on.

2. Main results

Lemma 2.1. *Let (Θ, d) be complete $MSWG$, and let $Z : \Theta^2 \to \Theta$, $h : \Theta \to \Theta$ be a rational contraction. Supposing $Z(\Theta^2) \subseteq h(\Theta)$. Let (x_n), $(y_n) \subseteq \Theta$. If for each $(x, y) \in (\Theta^2)_{Zh}$, here $\lim_{n\to\infty} E_n = 0$.*

Proof. Let x_0, $y_0 \in \Theta$. Due to $Z(\Theta^2) \subseteq h(\Theta)$, we constitute x_1, $y_1 \in \Theta$ such that $Z(x_0, y_0) = hx_1$, $Z(y_0, x_0) = hy_1$. Again, we could contrive x_2, $y_2 \in \Theta$ such that $Z(x_1, y_1) = hx_2$ and $Z(y_1, x_1) = hy_2$. Repeating this process we acquire (x_n), $(y_n) \subseteq \Theta$ such that $x = x_0$, $y = y_0$

$$hx_n = Z(x_{n-1}, y_{n-1}), \; hy_n = Z(y_{n-1}, x_{n-1}) \text{ for } \forall n \geq 1. \tag{2.1}$$

Let $(hx_{n+1}, hx_n) \in E(G)$ and $(hy_{n+1}, hy_n) \in E(G^{-1})$ for $\forall n \in N$. Owing to the rational contraction (1.1)&(2.1), we get

$$\begin{aligned} d(hx_{n+1}, hx_n) &= d(Z(x_n, y_n), Z(x_{n-1}, y_{n-1})) \\ &\leq \varsigma_1 \frac{d(hx_n, Z(x_n, y_n))[1 + d(hx_{n-1}, Z(x_{n-1}, y_{n-1}))]}{1 + d(hx_n, hx_{n-1})} \\ &\quad + \varsigma_2 \frac{d(hx_n, Z(x_n, y_n))\, d(hx_{n-1}, Z(x_{n-1}, y_{n-1}))}{1 + d(hx_n, Z(x_n, y_n))} \\ &\quad + \varsigma_3 [d(hx_n, Z(x_n, y_n)) + d(hx_{n-1}, Z(x_{n-1}, y_{n-1}))] \\ &\quad + \varsigma_4 [d(hx_n, Z(x_{n-1}, y_{n-1})) + d(hx_{n-1}, Z(x_n, y_n))] \\ &\quad + \varsigma_5 d(hx_n, hx_{n-1}). \end{aligned}$$

So that

$$\begin{aligned} d(hx_{n+1}, hx_n) &= d(Z(x_n, y_n), Z(x_{n-1}, y_{n-1})) \\ &\leq \varsigma_1 \frac{d(hx_n, hx_{n+1})[1 + d(hx_{n-1}, hx_n)]}{1 + d(hx_n, hx_{n-1})} \\ &\quad + \varsigma_2 \frac{d(hx_n, hx_{n+1})\, d(hx_{n-1}, hx_n)}{1 + d(hx_n, hx_{n+1})} \\ &\quad + \varsigma_3 [d(hx_n, hx_{n+1}) + d(hx_{n-1}, hx_n)] \\ &\quad + \varsigma_4 [d(hx_n, hx_n) + d(hx_{n-1}, hx_{n+1})] \\ &\quad + \varsigma_5 d(hx_n, hx_{n-1}). \end{aligned}$$

Then,

$$d(hx_n, hx_{n+1}) \leq \left(\frac{\sum_{j=2}^{5} \varsigma_j}{1 - \varsigma_1 - \varsigma_3 - \varsigma_4} \right) d(hx_{n-1}, hx_n). \tag{2.2}$$

In the same way, we can testify that

$$d\left(hy_{n+1}, hy_{n}\right) \leq \left(\frac{\sum_{j=2}^{5} \varsigma_{j}}{1-\varsigma_{1}-\varsigma_{3}-\varsigma_{4}}\right) d\left(hy_{n-1}, hy_{n}\right). \tag{2.3}$$

Put $E_{n}:=d\left(hx_{n}, hx_{n+1}\right)+d\left(hy_{n}, hy_{n+1}\right).$

Adding (1.1)&(2.1), one can assert that

$$E_{n} \leq \left(\frac{\sum_{j=2}^{5} \varsigma_{j}}{1-\varsigma_{1}-\varsigma_{3}-\varsigma_{4}}\right) E_{n-1}.$$

Thereat, using inductive we acquire

$$E_{n} \leq zE_{n-1} \leq z^{2}E_{n-2} \leq \cdots \leq z^{n}E_{0},$$

where $z=\left(\frac{\sum_{j=2}^{5} \varsigma_{j}}{1-\varsigma_{1}-\varsigma_{3}-\varsigma_{4}}\right)<1$, thence $\lim_{n\to\infty} E_{n}=0$. □

Theorem 2.1. *Let (Θ, d) be complete MSWG, $Z: \Theta^{2} \to \Theta$, $h: \Theta \to \Theta$ be a rational contraction and $h(\Theta) \supseteq Z(\Theta^{2})$. Assume that:*

(i) $h(\Theta)$ is closed and h is continuous;

(ii) h and Z are compatible;

(iii) Z is $G-$continuous, or (Θ, d, G) hold a property A.

(iv) $E(G)$ ensures the transitivity property.

Then $Fix_{coin(Zh)} \neq \emptyset$ iff $(\Theta^{2})_{Zh} \neq \emptyset$.

Proof. Let $Fix_{coin(Zh)} \neq \emptyset$. Then there exists $(\tau, \upsilon) \in Fix_{coin(Zh)}$ such that $(h\tau, Z(\tau, \upsilon)) = (h\tau, h\tau) \in \Delta \subset E(G)$, $(h\upsilon, Z(\upsilon, \tau)) = (h\upsilon, h\upsilon) \in \Delta \subset E(G^{-1})$. It follows that $(\tau, \upsilon) \in (\Theta^{2})_{Zh}$, so that $(\Theta^{2})_{Zh} \neq \emptyset$.

Let $(\Theta^{2})_{Zh} \neq \emptyset$. So, $(x_{0}, y_{0}) \in (\Theta^{2})_{Zh}$, i.e., $(hx_{0}, Z(x_{0}, y_{0})) \in E(G)$, $(hy_{0}, Z(y_{0}, x_{0})) \in E(G^{-1})$. Owing to edge preserving of Z and h, by (2.1), we attain $(Z(x_{n-1}, y_{n-1}), Z(x_{n}, y_{n})) \in E(G)$ and $(Z(y_{n-1}, x_{n-1}), Z(y_{n}, x_{n})) \in E(G^{-1})$. Then $(hx_{n}, hx_{n+1}) \in E(G)$ and $(hy_{n}, hy_{n+1}) \in E(G^{-1})$ for $\forall n \geq 1$.

According to Lemma 2.1, we get

$$\lim_{n\to\infty} E_{n} = 0. \tag{2.4}$$

Now, we will evidence that $\{hx_{n}\}$ and $\{hy_{n}\}$ are Cauchy sequences. Appealing an analog assertion as in the demonstration of Theorem 1 in [16] and see [17] page 14, hypothesis (iv),

$\{hx_n\}$ and $\{hy_n\}$ are Cauchy sequences. Due to hypothesis (i), there exists $\kappa, \omega \in h(\Theta)$

$$\lim_{n\to\infty} Z(x_n, y_n) = \lim_{n\to\infty} hx_n = \kappa$$
$$\lim_{n\to\infty} Z(y_n, x_n) = \lim_{n\to\infty} hy_n = \omega.$$

Next, given that Z is $G-$continuous. Then,

$$d(h\kappa, Z(hx_n, hy_n)) \le d(h\kappa, hZ(x_n, y_n)) + d(hZ(x_n, y_n), Z(hx_n, hy_n)) \tag{2.5}$$

Letting $n \to \infty$ in (2.5) and inasmuch as Z and h are compatible, assumptions (i) and (iii), we have that $d(h\kappa, Z(\kappa, \omega)) = 0$, namely $h\kappa = Z(\kappa, \omega)$. With similar thought, we also get $h\omega = Z(\omega, \kappa)$. Consequently, $Fix_{coin(Zh)} \neq \emptyset$.

Supposing now (Θ, d, G) has a property A. Let $x, y \in \Theta$; $hx = \kappa$, $hy = \omega$. Here, we hold (hx_n, hx), $(hy_n, hy) \in E(G)$ for $\forall n \ge 1$. From (1.1), we derive

$$\begin{aligned}
& d(hx, Z(x,y)) + d(hy, Z(y,x)) \\
\le\ & d(hx, hx_{n+1}) + d(hx_{n+1}, Z(x,y)) \\
& + d(hy, hy_{n+1}) + d(hy_{n+1}, Z(y,x)) \\
\le\ & d(Z(x_n,y_n), Z(x,y)) + d(Z(y_n,x_n), Z(y,x)) \\
& + d(hy, hy_{n+1}) + d(hx, hx_{n+1}) \\
\le\ & \varsigma_1 \frac{d(hx_n, Z(x_n,y_n))[1 + d(hx, Z(x,y))]}{1 + d(hx_n, hx)} \\
& + \varsigma_2 \frac{d(hx_n, Z(x_n,y_n))\, d(hx, Z(x,y))}{1 + d(hx_n, Z(x_n,y_n))} \\
& + \varsigma_3 [d(hx_n, Z(x_n,y_n)) + d(hx, Z(x,y))] \\
& + \varsigma_4 [d(hx_n, Z(x,y)) + d(hx, Z(x_n,y_n))] \\
& + \varsigma_5 d(hx_n, hx) + \varsigma_1 \frac{d(hy_n, Z(y_n,x_n))[1 + d(hy, Z(y,x))]}{1 + d(hy_n, hy)} \\
& + \varsigma_2 \frac{d(hy_n, Z(y_n,x_n))\, d(hy, Z(y,x))}{1 + d(hy_n, Z(y_n,x_n))} \\
& + \varsigma_3 [d(hy_n, Z(y_n,x_n)) + d(hy, Z(y,x))] \\
& + \varsigma_4 [d(hy_n, Z(y,x)) + d(hy, Z(y_n,x_n))] \\
& + \varsigma_5 d(hy_n, hy) + d(hy, hy_{n+1}) + d(hx, hx_{n+1}) \\
\to\ & 0 \text{ when } n \to \infty.
\end{aligned} \tag{2.6}$$

Ultimately, $hy = Z(y,x)$ and $hx = Z(x,y)$. □

Theorem 2.2. *Along with assumptions of Theorem 2.1, supposing for $(\tau, \upsilon), (x,y) \in \Theta^2$, there exists $(\eta, \vartheta) \in \Theta^2$ such that*

$$(Z(x,y), Z(\eta,\vartheta)) \in E(G), \quad (Z(y,x), Z(\vartheta,\eta)) \in E(G^{-1})$$

and

$$(Z(\tau,\upsilon), Z(\eta,\vartheta)) \in E(G), \quad (Z(\upsilon,\tau), Z(\vartheta,\eta)) \in E(G^{-1}).$$

Here Z and h hold a unique coupled common fp.

Proof. Due to Theorem 2.1, we get $Fix_{coin(Zh)} \neq \emptyset$. Assume $(x,y), (\tau,\upsilon)$ are coupled fp of Z, viz,

$$h\tau = Z(\tau,\upsilon),\ h\upsilon = Z(\upsilon,\tau),\ hx = Z(x,y),\ hy = Z(y,x). \tag{2.7}$$

Take into consideration (η_n) and (r_n) as follows

$$\eta_0 = \eta,\ \vartheta_0 = \vartheta,\ \eta_{n+1} = Z(\eta_n,\vartheta_n) \text{ and } \vartheta_{n+1} = Z(\vartheta_n,\eta_n) \text{ for } \forall n \geq 0.$$

By hypothesis, we obtain

$$(Z(x,y), Z(\eta,\vartheta)) = (hx, h\eta_1) \in E(G),\ (Z(y,x), Z(\vartheta,\eta)) = (hy, h\vartheta_1) \in E(G^{-1}) \tag{2.8}$$

and

$$(Z(\tau,\upsilon), Z(\eta,\vartheta)) = (h\tau, h\eta_1) \in E(G),\ (Z(\upsilon,\tau), Z(\vartheta,\eta)) = (h\upsilon, h\vartheta_1) \in E(G^{-1}). \tag{2.9}$$

Owing to edge preserving, we hold

$$(Z(x,y), Z(\eta_1,\vartheta_1)) = (hx, h\eta_2) \in E(G),\ (Z(y,x), Z(\vartheta_1,\eta_1)) = (hy, h\vartheta_2) \in E(G^{-1}) \tag{2.10}$$

and

$$(Z(\tau,\upsilon), Z(\eta_1,\vartheta_1)) = (h\tau, h\eta_2) \in E(G),\ (Z(\upsilon,\tau), Z(\vartheta_1,\eta_1)) = (h\upsilon, h\vartheta_2) \in E(G^{-1}). \tag{2.11}$$

Repeating this technic foregoing, we acquire

$$(hx, h\eta_n) \in E(G),\ (hy, h\vartheta_n) \in E(G^{-1}) \tag{2.12}$$

and

$$(h\tau, h\eta_n) \in E(G),\ (h\upsilon, h\vartheta_n) \in E(G^{-1}). \tag{2.13}$$

By (1.1), we obtain

$$\begin{aligned}
& d(h\tau, \eta_{n+1}) \\
= \ & d(Z(\tau,\upsilon), Z(\eta_n,\vartheta_n)) \\
\leq \ & \varsigma_1 \frac{d(h\tau, Z(\tau,\upsilon))[1 + d(h\eta_n, Z(\eta_n,\vartheta_n))]}{1 + d(h\tau, h\vartheta_n)} \\
& + \varsigma_2 \frac{d(h\tau, Z(\tau,\upsilon))\, d(h\eta_n, Z(\eta_n,\vartheta_n))}{1 + d(h\tau, Z(\tau,\upsilon))} \\
& + \varsigma_3 [d(h\tau, Z(\tau,\upsilon)) + d(h\eta_n, Z(\eta_n,\vartheta_n))] \\
& + \varsigma_4 [d(h\tau, Z(\eta_n,\vartheta_n)) + d(h\eta_n, Z(\tau,\upsilon))] + \varsigma_5 d(h\tau, h\eta_n)
\end{aligned}$$

Then,

$$d(h\tau, \eta_{n+1}) \leq \left(\frac{\sum_{j=3}^{5} \varsigma_j}{1 - \varsigma_3 - \varsigma_4} \right) d(h\tau, \eta_n).$$

In the same way, we can testify that

$$d(\vartheta_{n+1}, h\upsilon) \leq \left(\frac{\sum_{j=3}^{5} \varsigma_j}{1 - \varsigma_3 - \varsigma_4} \right) d(h\tau, \vartheta_n).$$

Applying an analog argumentum as in the proof of Theorem 2 in [16] and see [17] page 15, we obtain

$$\lim_{n\to\infty} d(h\tau, h\eta_n) = 0 = \lim_{n\to\infty} d(h\upsilon, h\vartheta_n).$$

Similarly

$$\lim_{n\to\infty} d(hx, h\eta_n) = 0 = \lim_{n\to\infty} d(hy, h\vartheta_n).$$

By the triangular inequality we get

$$\begin{aligned} d(h\tau, hx) &\leq d(h\tau, h\eta_n) + d(h\eta_n, hx), \\ d(h\upsilon, hy) &\leq d(h\upsilon, h\vartheta_n) + d(h\vartheta_n, hy), \text{ for } \forall n \in N, \\ &\to 0 \text{ when } n \to \infty. \end{aligned} \tag{2.14}$$

Consequently,

$$h\tau = hx \text{ and } h\upsilon = hy. \tag{2.15}$$

Let $h\tau = hx = \rho$ and $h\upsilon = hy = s$.

As Z and h are commutativity, from (2.7), we obtain

$$h(h\tau) = h(Z(\tau, \upsilon)) = Z(h\tau, h\upsilon) \Rightarrow h\rho = Z(\rho, s)$$

and

$$h(h\upsilon) = h(Z(\upsilon, \tau)) = Z(h\upsilon, h\tau) \Rightarrow hs = Z(s, \rho).$$

Herewith, (ρ, s) is a coupled coincidence point. Hence, continuing foregoing assertion for (τ, υ), (ρ, s),

$$h\tau = h\rho \Rightarrow \rho = h\rho \text{ and } h\upsilon = hs \Rightarrow s = hs.$$

Consequently, $\rho = h\rho = Z(\rho, \sigma)$ and $\sigma = h\sigma = Z(\sigma, \rho)$. Thence, (ρ, σ) is a coupled common fp of Z and h.

To verify the uniqueness, assume that (κ, ω) is another coupled common fp of Z and h. Consequently,

$$\omega = h\omega = Z(\omega, \kappa) \text{ and } \kappa = h\kappa = Z(\kappa, \omega). \tag{2.16}$$

From (2.15), we obtain

$$h\kappa = h\rho = \rho \text{ and } h\omega = h\sigma = \sigma. \tag{2.17}$$

Thus, from (2.16) and (2.17), we acquire

$$\kappa = \rho \text{ and } \omega = \sigma.$$

Then, $\kappa = h\kappa = h\rho = \rho$ and $\omega = h\omega = h\sigma = \sigma$. □

Remark 2.1. *Letting $E(G) = \{(x, y) \in \Theta^2 : y \geq x\}$. Under the circumstances, our results advance and enhance the concerning findings of Banach [19], Kannan [20], Chatterjee [21], Fabiano [17] for $h = I_\Theta$, the identity map in Theorem 2.1-2.2.*

3. Application to deformation of an elastic beam

Timoshenko–Ehrenfest (brief, $T-E$) beam theory and Euler–Bernoulli beam theory which is a special case in $T-E$ beam theory, can be used to portray the relation between the direction and amount of the force, the deformation, the elastic characteristics of the beam such as curvature, slope and deflection. For more details on applicabilities of elastic beams, see [[22]- [25]].

It is worthy of note that a fourth-order two-point boundary value problem emerge as patterns examining the deformations of an elastic beam, which is one of the basic structures in architectural, frequently occupied in the design of buildings and a variety of structures.

In this part, we demonstrate applicability and significance of the achieved finds.

Give functional eq symbolizing the deformation of an elastic beam by

$$\begin{aligned} \zeta^{'''}(r) &= \phi(r,\zeta(r),\zeta(r)), \quad r\in\Xi; \\ \zeta(0) &= \zeta^{'}(0)=\zeta^{''}(1)=\zeta^{'''}(1)=0, \end{aligned} \tag{3.1}$$

where $\Xi := [0,1]$ and $\phi \in C(\Xi\times R^2, R)$.

Consider the following integral system:

$$\begin{aligned} x(r) &= \int_0^1 G(r,s)\,\phi(s,x(s),y(s))\,ds, \\ y(r) &= \int_0^1 G(r,s)\,\phi(s,y(s),x(s))\,ds, \end{aligned} \tag{3.2}$$

where $r\in\Xi$ and $x,y\in\Omega := C(\Xi,R)$. Here, G is the Green Function endowed with

$$G(r,s)=\frac{1}{6}\begin{cases} s^2(3r-s), & 0\le s\le r\le 1, \\ r^2(3s-r), & 0\le r\le s\le 1. \end{cases} \tag{3.3}$$

From here, we acquire $0\le G(r,s)\le 2^{-1}r^2 s$ for $\forall r,s\in\Xi$.

Describe G via partial order relation by $x,y\in\Omega$, $y\ge x \; iff \; y(r)\ge x(r)$ for $\forall r\in\Xi$.

Define $\|x-y\|_\infty := \max_{r\in\Xi}|x(r)-y(r)|$ for $\forall x,\, y\in\Omega$.

Therefore $(\Omega, \|x\|_\infty)$ is complete $MSWG$.

Reckon with $E(G) := \{(x,y)\in\Omega^2 : y\ge x\}$, then $E(G)\supseteq \Delta(\Omega^2)$. Conversely $E(G^{-1}) := \{(x,y)\in\Omega^2 : x\ge y\}$.

Moreover, $(\Omega, \|x\|_\infty, G)$ acquire property A.

Then $(\Omega^2)_{Zh} = \{(x,y)\in\Omega^2 : Z(y,x)\le hy,\, hx\le Z(x,y)\}$.

Theorem 3.1. *Suppose the following assumptions have:*

(1) $\phi\in C(\Xi\times R^2, R)$;

(2) *for* $\forall\varpi,\nu,x,y\in R$ *with* $\varpi\ge x$, $y\ge\nu$, *we get*

$$\begin{aligned} 0 &\le \phi(r,\varpi,\nu)-\phi(r,x,y) \\ &\le \varsigma_3\left[|h(x(r))-Z(x,y)(r)|+|h(\varpi(r))-Z(\varpi,\nu)(r)|\right] \\ &\quad +\varsigma_4\left[|h(x(r))-Z(\varpi,\nu)(r)|+|h(\varpi(r))-Z(x,y)(r)|\right]+\varsigma_5|h(x(r))-h(\varpi(r))| \end{aligned}$$

$\varsigma_3,\varsigma_4,\varsigma_5\in[0,1)$ *with* $0\le 2(\varsigma_3+\varsigma_4)+\varsigma_5<1$;

(3) *there is* $(x_0, y_0) \in \Omega^2$ *such that*

$$x_0(r) \leq \int_0^1 G(r,s)\,\phi(s, x_0(s), y_0(s))\,ds,$$
$$y_0(r) \geq \int_0^1 G(r,s)\,\phi(s, y_0(s), x_0(s))\,ds,$$

where $r \in \Xi$ *and* $x, y \in \Omega := C(\Xi, R)$.

Here, there is at least one solution of (3.1).

Proof. Let $Z : \Theta^2 \to \Theta$ and $h : \Theta \to \Theta$ be defined as

$$Z(x,y)(r) = \int_0^1 G(r,s)\,\phi(s, x(s), y(s))\,ds, \qquad (3.4)$$
$$h(x)(r) = x(r), \quad r \in \Xi.$$

Then (3.1) can be indicated as

$$hy = Z(y,x) \text{ and } hx = Z(x,y). \qquad (3.5)$$

By (3.5), the solution of this functional Eq is a coupled coincidence point of the maps Z and h, if we show hypothesis in Theorem 2.1.

Let $\varpi, \nu, x, y \in \Theta$ be such that $h\varpi \geq hx$, $hy \geq h\nu$. Using (3.1), we obtain the following expressions.

$$Z(x,y)(r) \leq Z(\varpi,\nu)(r);\ \forall r \in \Xi, \Rightarrow (Z(x,y), Z(\varpi,\nu)) \in E(G),$$

$$Z(\nu,\varpi)(r) \leq Z(y,x)(r);\ \forall r \in \Xi, \Rightarrow (Z(y,x), Z(\nu,\varpi)) \in E\left(G^{-1}\right).$$

In this case, Z is edge preserving.

Next, let $\varpi, \nu, x, y \in R$ with $x \leq \varpi$, $\nu \leq y$. By assumption (2), we get

$$\begin{aligned}
&|Z(x,y)(r) - Z(\varpi,\nu)(r)| \\
\leq\ & \int_0^1 G(r,s)\,|\phi(s, x(s), y(s)) - \phi(s, \varpi(s), \nu(s))| \\
\leq\ & \left(\int_0^1 G(r,s)\right)\left[\begin{array}{c} \varsigma_3 \|h(x(r)) - Z(x,y)(r)\|_\infty + \varsigma_3 \|h(\varpi(r)) - Z(\varpi,\nu)(r)\|_\infty \\ +\varsigma_4 \|h(x(r)) - Z(\varpi,\nu)(r)\|_\infty + \varsigma_4 \|h(\varpi(r)) - Z(x,y)(r)\|_\infty \\ +\varsigma_5 \|h(x(r)) - h(\varpi(r))\|_\infty \end{array}\right] \\
\leq\ & 4^{-1}\left[\begin{array}{c} \varsigma_3 \|h(x(r)) - Z(x,y)(r)\|_\infty + \varsigma_3 \|h(\varpi(r)) - Z(\varpi,\nu)(r)\|_\infty \\ +\varsigma_4 \|h(x(r)) - Z(\varpi,\nu)(r)\|_\infty + \varsigma_4 \|h(\varpi(r)) - Z(x,y)(r)\|_\infty \\ +\varsigma_5 \|h(x(r)) - h(\varpi(r))\|_\infty \end{array}\right], \quad \text{for } \forall r \in \Xi;
\end{aligned}$$

then,

$$|Z(x,y)(r) - Z(\varpi,\nu)(r)| \tag{3.6}$$

$$\leq \quad 4^{-1}\begin{bmatrix} \varsigma_3 \|h(x(r)) - Z(x,y)(r)\|_\infty \\ +\varsigma_3 \|h(\varpi(r)) - Z(\varpi,\nu)(r)\|_\infty \\ +\varsigma_4 \|h(x(r)) - Z(\varpi,\nu)(r)\|_\infty \\ +\varsigma_4 \|h(\varpi(r)) - Z(x,y)(r)\|_\infty \\ +\varsigma_5 \|h(x(r)) - h(\varpi(r))\|_\infty \end{bmatrix}, \quad \text{for } \forall r \in \Xi.$$

Applying maximum in (3.6), we have

$$\|Z(x,y) - Z(\varpi,\nu)\|_\infty \leq \begin{array}{c} \varsigma_3 \|h(x(r)) - Z(x,y)(r)\|_\infty + \varsigma_3 \|h(\varpi(r)) - Z(\varpi,\nu)(r)\|_\infty \\ +\varsigma_4 \|h(x(r)) - Z(\varpi,\nu)(r)\|_\infty + \varsigma_4 \|h(\varpi(r)) - Z(x,y)(r)\|_\infty \\ +\varsigma_5 \|h(x(r)) - h(\varpi(r))\|_\infty . \end{array}.$$

There is $(x_0, y_0) \in \Theta^2$ such that $hy_0 \leq Z(y_0, x_0)$, $hx_0 \leq Z(x_0, y_0)$ which satisfies $(\Theta^2)_{Zh} \neq \emptyset$. Also, Z and h are commutative.

On the one hand, $(\Theta, \|x\|_\infty, G)$ has property A. Taking $\varsigma_1 = \varsigma_2 \equiv 0$ in (1.1), we obtain Theorem 2.1 and 2.2 is fulfilled. Thus, we deduce Z and h hold a unique coupled common fp, which is the solution of (3.1). □

Conclusion 3.1. *This writing, we acquire some findings for a rational type expression in MSWG. We express a nonlinear problem of practical significance utilizing these findings: boundary value problems for fourth-order diferential Eq. Obtained results generalize the conclusions given by Banach [19], Kannan [20], Chatterjee [21], Fabiano [17] so on. The findings are important both mechanical/civil engineering and researchers who specialize in fp theory.*

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