DOI: 10.28919/ejma.2024.4.14

Eur. J. Math. Appl. (2024)4:14
URL: http://ejma.euap.org

QUASI-INTERIOR CO-IDEALS IN Γ-SEMIRING WITH APARTNESS

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Abstract. In this paper, as a continuation of research on Γ-semirings with apartness in Bishop's constructive framework, we determine the concepts of (left, right) quasi interior co-ideals (i.e. a constructive dual of (left, right, res.) quasi interior ideals) in Γ-semirings with apartness as a generalization of the concept of co-ideals and the concept of interior co-ideals in Γ-semiring with apartness and analyze some of their fundamental properties.

1. Introduction

The concept of Γ-semirings were first introduced and studied by M. M. Krishna Rao [8,9] as a generalization of notion of Γ-rings. Many authors have studies on these algebraic structures. For example: H. Hedayati and K. P. Shum [5] (2011), R. D. Jagatap [7] (2018) and W. M. Fakieh and F. A. Alhawiti [4] (2021). In that algebraic structure, the subject of study, among other things, are ideals and filters in it. Thus interior, weak interior and quasi-interior ideals and filters were studied (for example, see [7, 10, 22]).

The settings of this article is the Bishop's constructive mathematics in the sense of books [1,2,12,14] and articles [3,15,17] including the Intuitionistic logic. Within Bishop's constructive orientation, some algebraic structures with apartness are treated, where the carriers of those algebraic structures are relational structures $(A, =, \neq)$ with apartness as an additional relation. This additional relation has been the subject of study by many authors for more than 40 years (for example, see [3, 16, 23]).

In articles [18, 19] the concepts of Γ-semirings with apartness and co-ordered Γ-semirings with apartness were introduced and analyzed within Bishop's constructive orientation. In addition to previous constructive duals, the concept of interior co-ideals and the concept of (left, right) weak-interior co-ideals in a Γ-semiring with apartness were introduced and analized in papers [20, 21].

In this paper, as a continuation of the previously mentioned reports [18–21], we offer our reflections on establishing a constructive dual of classical concept of (left, right) quasi-interior ideals in a Γ-semiring with apartness. It was shown that any co-ideal in Γ-semiring with apartness R is a quasi-interior co-ideal in R what means that the notion of quasi-interior co-ideals in Γ-semiring with apartness is a generalization of the concept of co-ideal. Also, this newly

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Key words and phrases. Bishop's constructive mathematics, Γ-semiring with apartness, co-ideal, interior co-ideal, weak interior co-ideal, quasi-interior co-ideal in Γ-semiring with apartness.

Received 26/12/2023.

defined notion is a generalization of the concept of interior co-ideals in such structures. The family of all (left, right) quasi-interior co-ideals in Γ-semiring with aparness forms a complete lattice. In addition to the above, the paper also analyzes the interrelationships of co-ideals, interior co-ideals and (left, right) weak-interior co-ideals with this newly introduced class of (left, right) quasi-interior co-ideals in Γ-semirings with apartness in the case when Γ-semiring with apartness is regular.

2. Short preliminaries

The notions and notations used in this article but not determined in it, we are take over from previously published articles [17–21].

Looking at the definition of Γ-semigring in the classical sense ([8, 9]), we first introduce the concept of Γ-semirings with apartness which will be used throughout this paper. Let $(R, +, \cdot)$ and $(\Gamma, +, \cdot)$ be commutative semigroups with apartness. About the 'apartness' a reader can consult the following books [1, 2, 12, 14] and article [3, 17, 18]. By this we mean that the sets $R \equiv (R, =_R, \neq_R)$ and $\Gamma \equiv (\Gamma, =_\Gamma, \neq_\Gamma)$ are supplied by apartness relations and that the internal operations in them are strongly extensional total functions. In the following, we do not use indices in the equation relations and apartness relations, except in cases where it is necessary to distinguish them so as not to cause confusion. About the relations, functions and operations a reader can consult some of our previously published articles such as [17, 18].

Definition 2.1 ([18], Definition 2.1)**.** *We call R a* Γ-semiring with apartness *if there exists a se-map $R \times \Gamma \times R \longrightarrow R$, written image of (x, a, y) by xay, such that it satisfies the following axioms:*

(1) $(\forall x, y, z \in R)(\forall a \in \Gamma)(xa(y + z) = xay + xaz$ *and* $(x + y)az = xaz + yaz)$,
(2) $(\forall x, y \in R)(\forall a, b \in \Gamma)(x(a + b)y = xay + xby)$,
(3) $(\forall x, y, z \in R)(\forall a, b \in \Gamma)((xay)bz = xa(ybz))$.

In this environment, the following implications are valid

$$(\forall x, y, u, v \in R)(x + u \neq y + v \implies (x \neq y \lor u \neq v)) \text{ and}$$

$$(\forall x, y, u, v \in R)(\forall a, b \in \Gamma)(xay \neq ubv \implies (x \neq u \lor a \neq b \lor y \neq v))$$

where $\neq$ is an apartness in R, that is, it is a relation on R with properties of consistency, symmetry and co-transitivity.

- For the subset B of the set R it is said to be strongly extensional in R (shorter, *se-subset*) if valid

$$(\forall u, v \in R)(u \in B \implies (u \neq v \lor v \in B)).$$

Also, the following notions, taken from [18, 19], will play an important role. Let R be a Γ-semiring with apartness:

- A se-subset B of R is a *cosub-Γ-semiring* of R if B is an additive cosub-semigroup of R and the following holds

$$(\forall x, y \in R)(\forall a \in \Gamma)(xay \in B \implies (x \in B \lor y \in B));$$

- A se-subset B of R is a *right* Γ-*coideal* of R if B is a additive cosub-semigroup of R and the following holds

$$(\forall x, y \in R)(\forall a \in \Gamma)(xay \in B \implies y \in B).$$

- A se-subset B of R is a *left* Γ-*coideal* of R if B is a additive cosub-semigroup of R and the following holds

$$(\forall x, y \in R)(\forall a \in \Gamma)(xay \in B \implies x \in B).$$

- A se-subset B of R is a Γ-*coideal* of R if B is a additive cosub-semigroup of R and the following holds

$$(\forall x, y \in R)(\forall a \in \Gamma)(xay \in B \implies (x \in B \wedge y \in B)).$$

- An element $1 \in S$ is said to be unity if for each $x \in S$ there exists $a \in \Gamma$ such that $xa1 = 1ax = x$.

From the determinations of the notions of left co-ideal, right co-ideal and (bilateral) co-ideal and Γ-semiring with apartness R it follows that these substructures also satisfy the condition

$$(\forall x, y \in R)(\forall a \in \Gamma)(xay \in K \implies (x \in K \vee y \in K)),$$

that is, a (left, right) co-ideal in a Γ-semiring with apartness R is a cosub-Γ-semiring in R.

Example 2.2. *Let $R = M_{2\times 2}(\mathbb{R})$ be the set of matrices of type 2×2 over the real numbers field $\mathbb{R}$ and $\Gamma = M_{2\times 2}(\mathbb{R})$. If the ternary operation $R \times \Gamma \times R \ni (A, \alpha, B) \longmapsto A\alpha B \in R$ is defined as usual matrix multiplication, then R is Γ-semiring (with apartness).*

The apartness relation on $M_{2\times 2}(\mathbb{R})$ is determined in the following way:

$$(a_{ij}) \neq (b_{ij}) \iff (\exists\, i, j \in \{1, 2\})(a_{ij} \neq b_{ij}).$$

Remark 2.3. *Although the determination of the concept of Γ-semirings with apartness is identical to the determination of the concept of Γ-semirings in the classical sense, the differences between them appear both due to the specificity of the carriers of these structures (sets wuth apartness) and the uniqueness of the logical environment (Intuicionistic logic) in which they are observed.*

3. The main results

This section is the central part of this report. In the first subsection, we recall the determinations of interior, (left, right) weak-interior and (left, right) quasi-interior ideals in Γ-semiring in classical algebra. In the next subsection, the notion of (left, right) quasi-interior co-ideal in Γ-semiring with separateness is introduced and its basic properties are analyzed. In addition to the previous one, connections are established between the mentioned types of co-ideals in Γ-semiring with separateness.

3.1. **Types of interior ideals in the classic case.**

M. M. Krishna Rao [8, 9] defined and studied Γ-semiring. The concept of interior ideals in a Γ-semiring is introduced in paper [7] (Definition 2.1) by R. D. Jagatap:

- A non-empty subset J of a Γ-semiring S is an *interior ideal* of S if J is an additive subsemigroup of S and $S\Gamma J\Gamma S \subseteq J$.

In other words, the following formulas

(a) $(\forall u, v \in S)((u \in J \wedge v \in J) \implies u + v \in J)$,

(b) $(\forall u, v, x \in S)(\forall a, b \in \Gamma)(x \in J \implies uaxbv \in J)$

are valid formulas. Note here that the requirement that J is a Γ-subsemiring of R is not included in this determination. A subset A of a Γ-semiring S is a Γ–subsemiring of S if $(A, +)$ is a subsemigroup of $(S, +)$ and $A\Gamma A \subseteq A$ holds. Every ideal J on a Γ-semiring S is an interior ideal of S but not conversely ([7], Remark 2.2).

The concept of weak-interior ideals in a Γ-semiring is introduced in paper [11] (Definitions 3.1, 3.2 and 3.3) by M. M. Krishna Rao:

- A non-empty subset J of a Γ-semiring S is said to be a *left weak-interior ideal* of S if J is a Γ-subsemiring of S and $S\Gamma J\Gamma J \subseteq J$. A non-empty subset J of a Γ-semiring S is said to be a *right weak-interior ideal* of S if J is a Γ-subsemiring of S and $J\Gamma J\Gamma JS \subseteq J$. A non-empty subset J of a Γ-semiring S is said to be a *weak-interior ideal* of S if J is left and right weak-interior ideal of S.

In other words, in addition to formula (a), the following formulas are also observed:

(c) $(\forall u, v \in S)(\forall a \in \Gamma)((u \in J \wedge v \in J) \implies uav \in J)$,

(d) $(\forall x, u, v \in S)(\forall a, b \in \Gamma)((u \in J \wedge v \in J) \implies xaubv \in J)$,

(e) $(\forall x, u, v \in S)(\forall a, b \in \Gamma)((u \in J \wedge v \in J) \implies uavbx \in J)$.

As it was already done in the paper [20], will consider also and in this paper the substructures of (left, right) weak interior ideals in a Γ-semiring without the condition that J must be a Γ-subsemiring in R.

The concept of quasi-interior ideals in a Γ-semiring is introduced in paper [10] (Definitions 3.1, 3.2 and 3.3) by M. M. Krishna Rao:

- A non-empty subset J of a Γ-semiring R is said to be left quasi-interior ideal of R, if J is a Γ-subsemiring of R and $R\Gamma J\Gamma R\Gamma J \subseteq J$. A non-empty subset J of a Γ-semiring T is said to be right quasi-interior ideal of R, if J is a Γ-subsemiring of R and $J\Gamma R\Gamma J\Gamma R \subseteq J$. A non-empty subset J of a Γ-semiring R is said to be quasi-interior ideal of R, if J is a left quasi-interior ideal and a right quasi-interior ideal of R.

In other words, in addition to formula (a), the following formulas are also observed:

(c) $(\forall u, v \in S)(\forall a \in \Gamma)((u \in J \wedge v \in J) \implies uav \in J)$,

(f) $(\forall x, y, u, v \in S)(\forall a, b, c \in \Gamma)((u \in J \wedge v \in J) \implies xaubycv \in J)$,

(g) $(\forall x, y, u, v \in S)(\forall a, b, c \in \Gamma)((u \in J \wedge v \in J) \implies uaxbvcy \in J)$.

As it was already done in the papers [20, 21], will consider also and in this paper the substructures of (left, right) quasi-interior ideals in a Γ-semiring without the condition that J must be a Γ-subsemiring in R.

3.2. **Quasi-interior co-ideals in Γ-semiring with apartness.** Before we get into the determination of the concept of (left, right) quasi-interior co-ideals in Γ-semiring with apartness, let us recall the notions of interior co-ideals ([20], Definition 3.1) and (left, right) weak interior co-ideals ([21], Definition 3.1) in a Γ-semiring with apartness:

- A se-subset K of a Γ-semiring R is an *interior co-ideal* of R if

(4) K is an additive co-subsemigroup of R, that is, it satisfies the condition

$$(\forall u, v \in R)(u + v \in K \implies (u \in K \vee v \in K)),$$

(5) $(\forall u, v, x \in R)(\forall a, b \in \Gamma)(uaxbv \in K \implies x \in K)$

holds.

- A se-subset K of a Γ-semiring R is a *left weak interior co-ideal* of R if it satisfies (4) and the following condition

(6) $(\forall u, v, x \in R)(\forall a, b \in \Gamma)(xaubv \in K \implies (u \in K \vee v \in K))$.

- A se-subset K of a Γ-semiring R is a *right weak interior co-ideal* of R if it satisfies (4) and the following condition

(7) $(\forall x, u, v \in R)(\forall a, b \in \Gamma)(uavbx \in K \implies (u \in K \vee v \in K))$.

- A se-subset K of a Γ-semiring R is said to be *weak interior co-ideal* of R if K is a left and right weak interior co-ideal of R.

Here we define the concept of (left, right) quasi-interior co-ideals of a Γ-semiring with apartness.

Definition 3.1. *Let $R := (R, =, \neq, +, \cdot)$ be a Γ-semiring with apartness over the set $\Gamma := (\Gamma, =, \neq)$ with apartness.*

(i) *A se-subset K of a Γ-semiring R is a left quasi-interior co-ideal of R if*

(4) *K is an additive co-subsemigroup of R, that is, it satisfies the condition*

$$(\forall u, v \in R)(u + v \in K \implies (u \in K \vee v \in K)),$$

(8) $(\forall u, v, x, y \in R)(\forall a, b, c \in \Gamma)(xaubycv \in K \implies (u \in K \vee v \in K))$.

(ii) *A se-subset K of a Γ-semiring R is a right quasi-interior co-ideal of R if it satisfies (4) and the following condition*

(9) $(\forall u, v, x, y \in R)(\forall a, b, c \in \Gamma)(uaxbvcy \in K \implies (u \in K \vee v \in K))$.

(iii) *A se-subset K of a Γ-semiring R is said to be quasi-interior co-ideal of R if K is a left and right quasi-interior co-ideal of R.*

Example 3.2. *Let $\mathbb{R}$ be a field of real numbers, $R := \left\{ \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \mid b, d \in \mathbb{R} \right\}$ be the additive semigroup of matrices over the filed $\mathbb{R}$ and $\Gamma = R$. The ternary operation in R over Γ is the standard multiplication of matrices. Then R is an Γ-semiring with apartness. Then $K =: \left\{ \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \mid c \in \mathbb{R} \wedge c \neq 0 \right\}$ is a right quasi-interior co-ideal in the Γ-semiring R. Indeed. Let $\begin{pmatrix} u & 0 \\ v & 0 \end{pmatrix}, \begin{pmatrix} s & s \\ t & t \end{pmatrix}, \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix}, \begin{pmatrix} z & 0 \\ w & 0 \end{pmatrix} \in R$ and $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}, \begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix}, \begin{pmatrix} e & 0 \\ f & 0 \end{pmatrix} \in \Gamma$ be arbitrary elements. Assume that the product*

$$\begin{pmatrix} u & 0 \\ v & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix} \begin{pmatrix} z & 0 \\ w & 0 \end{pmatrix} \begin{pmatrix} e & 0 \\ f & 0 \end{pmatrix} \begin{pmatrix} s & 0 \\ t & 0 \end{pmatrix} = \begin{pmatrix} uaxczes & 0 \\ vaxcwes & 0 \end{pmatrix}$$

has the form $\begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix}$. This is possible if $u = 0$ or $s = 0$ since a, c, e, x, z do not have to be 0 by assumption. So, $\begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix} \in K$ or $\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} \in K$. This shows that K is a right quasi-interior co-ideal in R.

Let us show that the concept of (left, right) quasi-interior co-deals is well-defined. As usual, we will show this by showing that the strongly complement $K^{\lhd}$ of (left, right) quasi-interior co-ideal K in Γ-semiring with apartness R is a (left, right, res.) quasi-interior ideal in R.

Theorem 3.3. *If K ($\neq R$) is a left (right) quasi-interior co-ideal of a Γ-semiring with apartness R, then the set $K^{\lhd}$ is a left (right, res.) quasi-interior ideal in R.*

Proof. Proof that $K^{\lhd}$ is a left quasi-interior ideal in R if K is a left interior co-ideal in R we will demonstrate. The proof that $K^{\lhd}$ is a right quasi-interior ideal in R if K is a right quasi-interior co-ideal in R can be obtained analogously to the first one.

The condition $K \neq R$ ensures that the set $K^{\lhd}$ is inhabited.

Let $x, y, u \in R$ be arbitrary elements such that $u \in K$, $x \lhd K$ and $y \lhd K$. Then $u \neq x + y$ or $x + y \in K$ by strongly extensionality of K in R. The second option $x + y \in K$ would give $x \in K \vee y \in K$ by (4), which contradicts the assumptions. Therefore, it must be $x+y \neq u \in K$. This means that $x + y \lhd K$ holds.

Let $u, v, x, y, t \in R$ and $a, b, c \in \Gamma$ be arbitrary elements such that $u \lhd K$, $v \lhd K$ and $t \in K$. Then $xaubycv \neq t$ or $xaubycv \in K$ by strongly extensionality of K in R. Assume that $xaubycv \in K$. Then it would be $u \in K$ or $v \in K$ according to (8). We got a contradiction. Thus, it must be $xaubycv \neq t \in K$. This means that $xaubycv \lhd K$ is valid. This proves (f).

So, the subset $K^{\lhd}$ is a left quasi-interior ideal in R. □

The following theorem connects the concepts of (right, left) co-ideals and (left, right) quasi-interior co-ideals in a Γ-semiring with apartness.

Theorem 3.4. *Every (right, left) co-ideal is a (left, right, res.) quasi-interior co-ideal in a Γ-semiring.*

Proof. Let K be a right co-ideal in a Γ-semiring R. This means that K satisfies condition (4) and the condition $xay \in K \implies y \in K$ for any $x, y \in R$ and $a \in \Gamma$. It should be proved that K satisfies conditions (4) and (8):

(4) is a valid formula by assumption. Let us prove the validity of (8).

Let $x, y, u, v \in R$ and $a, b, c \in \Gamma$ be arbitrary elements such that $xaubycv \in K$. Then $xaub(ycv) \in K$ and $ycv \in K$ because K is a right co-ideal of R. Thus $v \in K$ for the same reason as before. So, the disjunction $u \in K \vee v \in K$ is valid. This proves that K is a left quasi-interior co-ideal in R.

The proof that any left co-ideal on a Γ-semiring with apartness is a right quasi-interior co-ideal can be demonstrated in an analogous way. □

The following corollary directly follows from the previous theorem:

Corollary 3.5. *Any co-ideal in Γ-semiring with apartness is a quasi-interior co-ideal in a Γ-semiring with apartness.*

The assertion of the following theorem in which the concepts of interior co-ideals and (left, right) quasi-interior co-ideals in a Γ-semiring with apartness are connected can be demonstrated without difficulty.

Theorem 3.6. *Let K be an interior co-ideal of a Γ-semiring with apartness R. Then K is a (left, right) quasi-interior co-ideal in R.*

Proof. Let K be an interior co-ideal in a Γ-semiring R. One only needs to prove the validity of the formula (8). Let $x, y, u, v \in R$ and $a, b, c \in \Gamma$ be arbitrary elements such that $xaubycv \in K$. This means $xaub(ycv) \in K$. Thus $u \in K$ by (5). Therefore, the disjunction $u \in K$ or $v \in K$ is valid which means that K is a left quasi-interior co-ideal in R.

Analogously, it can be shown that any interior co-ideal in R is a right quasi-interior co-ideal in R. □

A (left, right) quasi-interior co-ideal K is said to be maximal (left, right) quasi- co-ideal in a Γ-semiring with apartness R if for any (left, right, res.) quasi-interior co-ideal G in R holds $K \subseteq G \implies K = G$.

Theorem 3.7. *The family $\mathfrak{qIntc}(R)$ of all (left, right) quasi-interior co-ideals in a Γ-semiring with apartness R forms a complete lattice.*

Proof. Let $\{K_i\}_{i\in I}$ be a family of left quasi-interior co-ideals in a Γ-semiring with apartness R.

(a) (i) Let $u, v \in R$ be such that $u \in \bigcup_{i\in I}$. Then there exists an index $k \in I$ such that $u \in K_k$. Thus $u \neq v$ or $v \in K_k \subseteq \bigcup_{i\in I} K_i$ by strongly extensionality of the left quasi-interior co-ideal K_k in R. This means that the set $\bigcup_{i\in I} K_i$ is a strongly extensional subset in R.

(ii) Let $x, y \in R$ be such that $x + y \in \bigcup_{i\in I} K_i$. Then there exists an index $k \in I$ such that $x + y \in K_k$. Thus $x \in K_k \subseteq \bigcup_{i\in I} K_i$ or $y \in K_k \subseteq \bigcup_{i\in I} K_i$ by (4). This means that the set $\bigcup_{i\in I} K_i$ is an additive co-subsemigroup of R.

(iii) Let $u, v, x, y \in R$ and $a, b, c \in \Gamma$ be such that $xaubycv \in \bigcup_{i\in I} K_i$. Then there exists an index $k \in I$ such that $xaubycv \in K_k$. Thus $u \in K_k \subseteq \bigcup_{i\in I} K_i$ or $v \in K_k \subseteq \bigcup_{i\in I} K_i$ by (8). This shows that the set $\bigcup_{i\in I} K_i$ satisfies the condition (8).

Based on (i), (ii) and (iii), we conclude that the set $\bigcup_{i\in I} K_i$ is a left quasi-interior co-ideal in R.

(b) Let $\mathfrak{X}$ be the family of all left quasi-interior co-ideals in Γ-semiring with apartness R contained in $\bigcap_{i\in I} K_i$. Then $\cup\mathfrak{X}$ is the maximal left quasi-interior co-ideal in R contained in $\bigcap_{i\in I} K_i$, according to (a).

(c) If we put $\sqcup_{i\in I} K_i = \bigcup_{i\in I} K_i$ and $\sqcap_{i\in I} K_i = \cup\mathfrak{X}$, then $(\mathfrak{qIntc}(R), \sqcup, \sqcap)$ a is a complete lattice.

The proof that the family of all right quasi-interior co-ideals in a Γ-semiring with apartness R forms a complete lattice can be demonstrated in an analogous way. □

Corollary 3.8. *For any subset X of Γ-semiring with apartness R there is the maximal (left, right) quasi-interior co-ideal contained in X.*

Proof. Let $\mathfrak{X}$ be the family of all (left, right) quasi-interior co-ideals in a Γ-semiring with apartness R contained in the set X. Then, according to part (a) of the proof in the previous theorem, $\cup\mathfrak{X}$ is the maximal (left, right) quasi-interior co-ideal in R contained in X. □

Corollary 3.9. *For any element x in a Γ-semiring with apartness R there is the maximal (left, right) quasi-interior co-ideal K_x in R such that $x \lhd K_x$.*

Proof. One should take $X = \{u \in R : u \neq x\}$ and apply the previous corollary. □

However, in regular Γ-semirings the reverse inclusion is also valid. In what follows we need the notion of regular Γ-semiring with apartness ([6]): An element x of a Γ-semiring (with apartness) R is said to be regular if $x \in x\Gamma R\Gamma x$. If all elements of Γ-semiring R are regular, then R is known as a regular Γ-semiring.

Theorem 3.10. *If R is a regular Γ-semiring with apartness, then any left (right) quasi-interior co-ideal in R is a right (left, res.) co-ideal in R.*

Proof. Let K be a left quasi-interior co-ideal of a regular Γ-semiring with apartness. This means that (4) and (8) are valid. Let us prove that K is a right co-ideal in R. Let $x, y \in R$ and $a \in \Gamma$ such that $xay \in K$. Since R is regular, there are $a_y, b_y \in \Gamma$ and $u_y \in R$ such that $y = ya_yu_yb_yy$. Then $xay = xaya_yu_yb_yy \in K$. It follows $y \in K$ from here according to (8).

The second part of the theorem can be demonstrated analogously to the previous one. □

Corollary 3.11. *If R is a regular Γ-semiring with apartness, then any quasi-interior co-ideal in R is a co-ideal in R.*

Corollary 3.12. *If R is a regular Γ-semiring with apartness, then any quasi-interior co-ideal in R is an interior co-ideal in R.*

Proof. In [20] (Theorem 3.3) it was shown that in a Γ-semiring with apartness R, any co-ideal in R is an interior co-ideal in R. Therefore, according to the previous corollary, in regular Γ-semiring with apartness any quasi-interior co-ideal in R is an interior co-ideal in R. □

Corollary 3.13. *If R is a regular Γ-semiring with apartness, then any quasi-interior co-ideal in R is a weak interior co-ideal in R.*

Proof. In [21] (Corollary 3.2) it was shown that in a Γ-semiring with apartness R, any interior co-ideal in R is a weak interior co-ideal in R. Therefore, according to the Corollary 3.12, in regular Γ-semiring with apartness any quasi-interior co-ideal in R is a weak interior co-ideal in R. □

4. Conclusion

Concepts of (left, right) quasi-interior ideals in a Γ-semiring were introduced and discussed in articles [10] by M. M. Krishna Rao.

In articles [18–21] the concept of Γ-semiring with apartness was introduced and analyzed. While in [18], the concept of Γ-semirings with apartness was introduced, in paper [19] the concept of Γ-semirings with apartness ordered by a co-order relation was introduced. While in paper [20], the concept of interior co-ideals in Γ-semiring with apartness was introduced and discussed as a constructive dual of interior ideals in such semirings, in article [21] , the author deals with defining of (left, right) weak interior co-ideals in a Γ-semiring with apartness. In this paper, as a direct continuation of previous research, the author deals with defining of (left, right) quasi interior co-ideals in a Γ-semiring with apartness as well as finding their basic properties.

The author assumes that it would be interesting to observe the interior, (left, right) weak interior and (left, right) quasi-interior co-ideals in a co-ordered Γ-semiring with apartness.

Acknowledgement. The author thanks to the reviewer(s) and the editor of the this magazine for fast and quality correspondence regarding the material presented in this paper.

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