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EXISTENCE OF POSITIVE WEAK SOLUTION FOR A WEIGHTED SYSTEM OF AUTOCATALYTIC REACTION STEADY STATE TYPE

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Abstract. We establish the existence results of positive weak solution for the weighted p-Laplacian autocatalytic reaction problem $-\Delta_{P,p}u = \lambda m(x)[\nu a(x)u^{\alpha} - \upsilon u^{\beta}]$ in Ω, $u = 0$ on $\partial\Omega$, where $\Delta_{P,p}$ with $p > 1$ and $P = P(x)$ is a weight function, denotes the weighted p-Laplacian defined by $\Delta_{P,p}u \equiv div[P(x)|\nabla u|^{p-2}\nabla u], m(x), a(x)$ are weight functions, λ, ν, υ are positive parameters, $\alpha + 1 \leq p < \beta + 1$, and $\Omega \subset R^n$ is a bounded domain with smooth boundary $\partial\Omega$. We establish that there exists positive constant $\lambda^*(\Omega)$ such that the above system has a positive weak solution for $\lambda \geq \lambda^*$. We use the method of sub-supersolutions to establish our results.

1. Introduction

In this paper, we are concerned with the existence and nonexistence results of positive weak solution for the weighted p-Laplacian autocatalytic reaction problem

$$(1) \qquad \begin{cases} -\Delta_{P,p}u = \lambda m(x)[\nu a(x)u^{\alpha} - \upsilon u^{\beta}] & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where $\Delta_{P,p}$ with $p > 1$ and $P = P(x)$ is a weight function, denotes the weighted p-Laplacian which is defined by $\Delta_{P,p}u \equiv div[P(x)|\nabla u|^{p-2}\nabla u]$, λ is a positive parameter, $m(x), a(x)$ are weight function and that there exist positive constant m_0 such that $m(x) \geq m_0$, λ, ν, υ are positive parameters, $\alpha+1 \leq p < \beta+1$ and $\Omega \subset R^n$ is a bounded domain with smooth boundary $\partial\Omega$. We establish that there exists positive constant $\lambda^*(\Omega)$ such that the above system have a positive weak solution for $\lambda \geq \lambda^*$. We use the method of sub-supersolutions to establish our results (see e.g. [4] and [6]).

When $P(x) = m(x) = 1$, problems of the form given by (1) arise from many branches of pure mathematics as in the theory of quasiregular and quasiconformal mappings (see [29]) as well as from various problems in mathematical physics notably the flow of non-Newtonian fluids. In the latter case, the quantity p is a characteristic of the medium. The situation $p > 2$ corresponds to dilatant fluids, while the situation $1 < p < 2$ describes pseudo-plastic fluid (see [2]). The case $p = 2$ describes Newtonian fluid (see [30]).

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Systems of type (1), with certian values for $P(x), p, \alpha, \beta$, have received considerable attention in the last decade (see, e.g., [12, 21, 22, 24] and the references therein). It has been shown that for some certian values of α, β, the system (1) has a rich mathematical structure. In [7], the system (1) is considered under the conditions $P(x) = \lambda = 1, p = 2$ and $f(u) = u^{\alpha}$,where $\alpha \geq 1$ is called the polytropic index. This corresponds to the Emden-Fowler nonlinear steady-state problem. While in [8], system (1) is considered under the hypothesis $P(x) = \lambda = 1, p = 2$ and $f(u) = u - u^{\beta}$,where u is the population denisty. This corresponds to the Logestic nonlinear steady-state problem. In [9], system (1) is considered under the hypothesis $P(x) = \lambda = 1$, and $f(u) = u^{\alpha} - u^{\beta}$, with $1 \leq \alpha < \beta$. This corresponds to the p-autocatalytic reaction nonlinear steady-state problem. Additionally, in [22], system (1) is considered under the conditions $P(x) = 1, \lambda = k$ and $f(u) = ku(1 - u^2)$ which corresponds to the p-Fisher–Kolmogoroff nonlinear steady-state problem. Due to the appearance of weighted p-Laplacian operator in (1) and the particular cases; the extensions are challenging and nontrivial.

On the other hand, the existence of weak solutions for nonlinear elliptice systems involving p-Laplacian operators with different weights has been studied using an approximation method (see [25]), the theory of nonlinear monotone operators method (see [26]) and the sub-supersolutions method (see [10, 11, 14, 19, 20]). Recently, the behaviours and properties of the weak solution for some nonlinear systems have received considerable attention (see [1, 15–18])

This paper is organized as follows:

In section 2, we introduce some technical results and notations, that are established in [5]. In section 3, we prove the existence of a positive weak solution for system (1) using the method of sub-supersolutions. Additionally, we consider the nonexistence result.

2. Technical results

Now, we introduce some technical results [5] concerning the degenerated homogeneous eigenvalue problem

$$\left.\begin{array}{ll} -\Delta_{P,p} u = -div[P(x)|\nabla u|^{p-2}\nabla u] = \lambda a(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array}\right\} \tag{2}$$

where $P(x)$ and $a(x)$ are measurable functions satisfying

$$\frac{\nu(x)}{c_1} \leq P(x) \leq c_1 \nu(x), \tag{3}$$

for a.e. $x \in \Omega$ with some constant $c_1 \geq 1$, where $\nu(x)$ is a weight function in Ω satisfying the conditions

$$\nu, \nu^{-\frac{1}{p-1}} \in L^1_{Loc}(\Omega),\ \nu^{-s} \in L^1(\Omega),\ \text{with } s \in (\frac{n}{p}, \infty) \cap [\frac{1}{p-1}, \infty), \tag{4}$$

and

$$\|a(x)\|_{\infty} = \overline{a}, 0 \leq a(x) \in L^{\frac{k}{k-p}}(\Omega) \qquad \text{for a.e. } x \in \Omega, \tag{5}$$

with some k satisfies $p < k < p_s^*$ where $p_s^* = \frac{np_s}{n-p_s}$ with $p_s = \frac{ps}{s+1} < p < p_s^*$ and *meas* $\{x \in \Omega : a(x) > 0\} > 0$. Examples of functions satisfying (4) are mentioned in [5].

Lemma 2.1. *There exists the least(i.e. the first or principal) eigenvalue $\lambda_1 > 0$ and precisely one corresponding eigenfunction $\phi_1 \geq 0$ a.e. in Ω (ϕ_1 not identical to 0) of the eigenvalue problem (2). Moreover, it is characterized by*

$$\lambda_1 \int_\Omega a(x)\phi_1^p \leq \int_\Omega P(x)|\nabla \phi_1|^p. \tag{6}$$

Lemma 2.2. *Let $\phi_1 \in W_0^{1,p}(P,\Omega)$, $\phi_1 \geq 0$ a.e. in Ω, be the eigenfunction corresponding to the first eigenvalue $\lambda_1 > 0$ of the eigenvalue problem (2). Then $\phi_1 \in L^\infty(\Omega)$.*

Now, let us introduce the weighted Sobolev space $W^{1,p}(\nu,\Omega)$ which is the set of all real valued functions u defined in Ω for which (see [5])

$$\|u\|_{1,p,\nu} = \left[\int_\Omega |u|^p + \int_\Omega \nu(x)|\nabla u|^p\right]^{\frac{1}{p}} < \infty. \tag{7}$$

Since we are dealing with the Dirichlet problem, we introduce also the space $W_0^{1,p}(\nu,\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\nu,\Omega)$ with respect to the norm

$$\|u\|_{1,p,\nu} = \left[\int_\Omega \nu(x)|\nabla u|^p\right]^{\frac{1}{p}} < \infty, \tag{8}$$

which is equivalent to the norm given by (7). Both spaces $W^{1,p}(\nu,\Omega)$ and $W_0^{1,p}(\nu,\Omega)$ are well defined reflexive Banach Spaces.

In this paper, we shall take $c_1 = 1$ in (3) i. e. $\nu(x) = P(x)$.

3. Existence and nonexistence results

In this section, we shall prove the existence of positive weak solution for system (1) by constructing a positive weak subsolution $\psi \in W_0^{1,p}(P,\Omega)$ and supersolution $z \in W_0^{1,p}(P,\Omega)$ of (1) such that $\psi \leq z$. That is, ψ satisfies $\psi = 0$ on $\partial\Omega$ and

$$\int_\Omega P(x)|\nabla\psi|^{p-2}\nabla\psi\nabla\zeta dx \leq \lambda m(x)\int_\Omega [\nu a(x)\psi^\alpha - \upsilon\psi^\beta]\zeta dx, \tag{9}$$

and z satisfies $z = 0$ on $\partial\Omega$ and

$$\int_\Omega P(x)|\nabla z|^{p-2}\nabla z\nabla\zeta dx \geq \lambda m(x)\int_\Omega [\nu a(x)z^\alpha - \upsilon z^\beta]\zeta dx, \tag{10}$$

for all test function $\zeta \in W_0^{1,p}(P,\Omega)$ with $\zeta \geq 0$.

Then the following result holds:

Lemma 3.1. *(see [3, 23]) Suppose there exist a weak subsolution ψ and a weak supersolution z of (1) such that $\psi \leq z$; then there exists a weak solution u of (1) such that $\psi \leq u \leq z$.*

Our main results of this paper are the following theorems.

Theorem 3.2. *There exists positive constant $\lambda^* = \lambda^*(\Omega)$ such that system (1) has a positive weak solution u for $\lambda \geq \lambda^*$.*

Proof. Let λ_1 be the first eigenvalue of the eigenvalue problem (2) and ϕ_1 the corresponding positive eigenfunction satisfying $\phi_1 > 0$ in Ω and $|\nabla\phi_1| > 0$ on $\partial\Omega$ with $\|\phi_1\|_\infty = 1$. Then we have

$$\begin{cases} -\Delta_{P,p}\phi_1 = \lambda_1 a(x)\phi_1^{p-1} & \text{in } \Omega \\ \phi_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{11}$$

Also, let $k, \delta, \sigma > 0$ be such that $P(x)|\nabla\phi_1|^p - \lambda_1 a(x)\phi_1^p \geq k$ on $\overline{\Omega}_\delta = \{x \in \Omega : d(x, \partial\Omega) \leq \delta\}$ and $\phi_1 \geq \sigma$.

We shall verify that $\psi = m_0^{\frac{1}{p-1}}(\frac{p-1}{p})\ \phi_1^{\frac{p}{p-1}}$ is a weak subsolution of (1). Let $\zeta \in W_0^{1,p}(P,\Omega)$ with $\zeta \geq 0$.

A calculation shows that

$$\begin{aligned}\int_\Omega P(x)|\nabla\psi|^{p-2}\nabla\psi\cdot\nabla\zeta dx &= m_0\int_\Omega P(x)\phi_1|\nabla\phi_1|^{p-2}\nabla\phi_1\cdot\nabla\zeta dx \\ &= m_0\int_\Omega (P(x)|\nabla\phi_1|^{p-2}\nabla\phi_1\nabla(\phi_1\zeta) - P(x)|\nabla\phi_1|^p\zeta)dx \\ &= m_0\int_\Omega (\lambda_1 a(x)\phi_1^p - P(x)|\nabla\phi_1|^p)\zeta dx.\end{aligned}$$

Now , in $\overline{\Omega}_\delta$ we have $\lambda_1 a(x)\phi_1^p - P(x)|\nabla\phi_1|^p \leq -k$. We choose υ such that $-k \leq -\upsilon\lambda u^\beta \leq \lambda[\nu a(x)\psi^\alpha - \upsilon\psi^\beta]$, for all x$\in \overline{\Omega}_\delta$. Then we have

$$\int_{\overline{\Omega}_\delta} P(x)|\nabla\psi|^{p-2}\nabla\psi\nabla\zeta dx \leq -m_0 k \leq \lambda m(x)\int_{\overline{\Omega}_\delta}[\nu a(x)\psi^\alpha - \upsilon\psi^\beta]\zeta dx.$$

Next, in $\Omega - \overline{\Omega}_\delta$ we have $\lambda_1 a(x)\phi_1^p - P(x)|\nabla\phi_1|^p \leq \lambda_1$ and $\phi_1 \geq \sigma$. Now if we take

$$\lambda \geq \lambda^* = \frac{m_0\lambda_1}{a_0\nu[m_0^{\frac{1}{p-1}}(\frac{p-1}{p})\sigma^{\frac{p}{p-1}}]^\alpha - \upsilon[m_0^{\frac{1}{p-1}}(\frac{p-1}{p})\sigma^{\frac{p}{p-1}}]^\beta}, \tag{12}$$

then we have

$$\begin{aligned}\int_{\Omega-\overline{\Omega}_\delta} P(x)|\nabla\psi|^{p-2}\nabla\psi\cdot\nabla\zeta dx &= m_0\int_{\Omega-\overline{\Omega}_\delta}(\lambda_1 a(x)\phi_1^p - P(x)|\nabla\phi_1|^p)\zeta dx \\ &\leq m_0\lambda\int_{\Omega-\overline{\Omega}_\delta} a_0\nu[m_0^{\frac{1}{p-1}}(\frac{p-1}{p})\sigma^{\frac{p}{p-1}}]^\alpha\zeta dx \\ &- m_0\lambda\upsilon\int_{\Omega-\overline{\Omega}_\delta}[m_0^{\frac{1}{p-1}}(\frac{p-1}{p})\sigma^{\frac{p}{p-1}}]^\beta\zeta dx \\ &\leq \lambda m(x)\int_{\Omega-\overline{\Omega}_\delta}[\nu a(x)\psi^\alpha - \upsilon\psi^\beta]\zeta dx\end{aligned}$$

So, equation (9) is satisfy and ψ is a weak subsolution of (1).

Next, we construct a weak supersolution z of system (1). Let $e_p = e_p(x)$ be the positive weak solution of (see [25])

$$\left.\begin{array}{ll} -\Delta_{P,p}e_p = 1 & \text{in } \Omega, \\ e_p = 0 & \text{on } \partial\Omega. \end{array}\right\} \tag{13}$$

We denote $z(x) = Ae_p$ where the constant $A > 0$ is sufficiently large and to be chosen later. We shall verify that z is the weak supersolution of (1). To do this, let $\zeta \in W_0^{1,p}(P,\Omega)$ with $\zeta \geq 0$. Then, using (13), we have

$$\begin{aligned} \int_\Omega P(x)|\nabla z|^{p-2}\nabla z \cdot \nabla\zeta dx &= A^{p-1}\int_\Omega P(x)|\nabla e_p|^{p-2}\nabla e_p \cdot \nabla\zeta dx \\ &= A^{p-1}\int_\Omega \zeta dx. \end{aligned}$$

Since $0 < \alpha \leq p-1 < \beta$, then it is easy to prove that there exists positive large constant A such that

$$A^{p-1-\alpha} = \lambda\overline{m}(\nu\bar{a}e_p^\alpha - \upsilon A^{\beta-\alpha}e_p^\beta),$$

where $\overline{m} = \|m(x)\|_\infty$. Hence, we have

$$\begin{aligned} \int_\Omega P(x)|\nabla z|^{p-2}\nabla z \cdot \nabla\zeta dx &= A^{p-1}\int_\Omega \zeta dx = \int_\Omega \lambda\overline{m}\nu\bar{a}A^\alpha e_p^\alpha\zeta dx \\ &\geq \lambda m(x)\int_\Omega (\nu a(x)z^\alpha\zeta dx \\ &> \lambda m(x)\int_\Omega [\nu a(x)z^\alpha - \upsilon z^\beta]\zeta dx \end{aligned}$$

So, equation (10) is satisfy and z is the weak supersolution of (1). Thus, there exists a weak solution u of (1) with $\psi \leq u \leq z$. This completes the proof of Theorem 3.1.
Finally, we discuss the case in which there exists no positive weak solution for system through the following theorem.

Theorem 3.3. *When $\overline{m}\nu\lambda \leq \lambda_1$, system (1) has no positive weak solution.*

Proof. Suppose $u(x) \in W_0^{1,p}(P,\Omega)$ be a positive weak solution of (1). We prove Theorem 3.2 by arriving at a contradiction.

Multiplying (1) by u, we have

$$\begin{aligned} \int_\Omega P(x)|\nabla u|^p dx &= \lambda m(x)\int_\Omega (\nu a(x)u^{\alpha+1} - \upsilon u^{\beta+1})dx \\ &< \lambda m(x)\int_\Omega \nu a(x)u^{\alpha+1} \leq \lambda\overline{m}\int_\Omega \nu a(x)u^p dx. \end{aligned} \tag{14}$$

Also, we have

$$\lambda_1\int_\Omega a(x)u^p \leq \int_\Omega P(x)|\nabla u|^p. \tag{15}$$

Combining (14) and (15), we obtain

$$(\lambda_1 - \overline{m}\nu\lambda) \int_{\Omega} a(x)u^p \leq 0,$$

which is a contradiction if $\overline{m}\nu\lambda \leq \lambda_1$. Thus system (1) has no positive weak solution for $\overline{m}\nu\lambda \leq \lambda_1$, and we finish the proof of Theorem 3.2.

4. Conclusion

In this paper, on the one hand, we have proved the existence of positive weak solution for autocatalytic reaction steady state problem involving the weighted p-Laplacian operator using the sub-super solutions method. On the other hand, we discussed the case in which there exists no positive weak solution for the considered system.

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References

[1] G. Afrouzi, Z. Sadeeghi, Stability results for a class of elliptic problems, Int. J. Nonlinear Sci. 6 (2008), 114–117.
[2] C. Atkinson, K. Ali, Some boundary value problems for the bingham model, J. Non-Newton. Fluid Mech. 41 (1992), 339–363.
[3] A. Canada, P. Dravek, J. Gamez, Existence of positive solutions for some problems with nonlinear diffusion, Trans. Amer. Math. Soc. 349 (1997), 4231–4249.
[4] P. Drabek, J. Hernandez, Existence and uniqueness of positive solutions for some quasilinear elliptic problem, Nonlinear Anal., TMA. 44 (2001), 189–204.
[5] P. Drabek, A. Kufner, F. Nicolosi, Quasilinear elliptic equation with degenerations and singularities, Walter de Gruyter, Berlin, 1997.
[6] P. Drábek, P. Krejcí, P. Takác, Nonlinear differential equations, CRC Press, 1999.
[7] I. Flores, A resonance phenomenon for ground states of an elliptic equation of Emden-Fowler type, J. Diff. Equ. 198 (2004), 1–15.
[8] J. Goddard II, R. Shivaji, Diffusive logistic equation with constantyield harvesting and negative density dependent emigration on the boundary, J. Math. Anal. Appl. 4147 (2014), 561–573.
[9] S. Junping, Z. Zhitao, Lectures on solution set of semilinear elliptic equations, Tokyo Metropolitan University, 2005.
[10] S. Khafagy, Existence results for weighted (p, q)-Laplacian nonlinear system, Appl. Math. E-Notes, 17 (2017), 242–250.
[11] S. Khafagy, On positive weak solutions for a nonlinear system involving weighted (p, q)-Laplacian operators, J. Math. Anal. 9 (2018), 86–96.
[12] S. Khafagy, M. Herzallah, Maximum principle and existence of weak solutions for nonlinear system involving weighted (p, q)-Laplacian, Southeast Asian Bull. Math. 40 (2016), 353–364.
[13] S. Khafagy, H. Serag, Existence of weak solutions for $n \times n$ nonlinear systems involving different p-Laplacian operators, Elec. J. Diff. Eqn. 2009 (2009), 1–14.
[14] S. Khafagy, H. Serag, On the existence of positive weak solution for nonlinear system with singular weights, J. Contemp. Math. Anal. 55 (2020), 259–267.
[15] S. Khafagy, A. Ezzat, Existence and stability of positive weak solutions for a class of chemically reacting systems, Eur. J. Math. Appl. 4 (2024), 2.
[16] S. Khafagy, H. Serag, On the stability of positive weak solution for (p, q)-Laplacian nonlinear system, Appl. Math. E-Notes, 20 (2020), 108–114.

[17] S. Khafagy, H. Serag, Stability results of positive weak solution for singular p-Laplacian nonlinear system, J. Appl. Math. Inf. 36 (2018), 173–179.

[18] S. Khafagy, H. Serag, Stability of positive weak solution for generalized weighted p-Fisher-Kolmogoroff nonlinear stationary-state problem, Eur. J. Math. Anal. 2 (2022), 1–8.

[19] S. Khafagy, E. El-Zahrani, H. Serag, Existence and uniqueness of weak solution for nonlinear weighted (p, q)-laplacian system with application on an optimal control problem, Jordan J. Math. Stat. 15 (2022), 983–998.

[20] S. Khafagy, S. Rasouli, H. Serag, Existence results for fractional Fisher-Kolmogoroff steady state problem, Eur. J. Math. Appl. 3 (2023), 1–7.

[21] M.K. Kwong, Uniqueness of positive radial solutions of $\triangle u - u + u^p = 0$ in R^N, Arch. Ration. Mech. Anal. 105 (1989), 243–266.

[22] K.Q. Lan, J.H. Wu, Travelling wavefronts of scalar reaction–diffusion equations with and without delays, Nonlinear Anal., RWA. 4 (2003), 173–188.

[23] E. Lee, R. Shivaji, J. Ye, Classes of infinite semipositone systems, Proc. R. Soc. Edinburgh, 139A (2009), 853–865.

[24] H.P. McKean, Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov, Comm. Pure Appl. Math. 28 (1975), 323–331.

[25] H. Serag, S. Khafagy, On maximum principle and existence of positive weak solutions for $n \times n$ nonlinear systems involving degenerated p-Laplacian operator, Turk. J. Math. 34 (2010), 59–71.

[26] H. Serag, S. Khafagy, Existence of weak solutions for $n \times n$ nonlinear systems involving different degenerated p-Laplacian operators, N. Z. J. Math. 38 (2008), 75–86.

[27] A. Anane, Simplicité et isolation de la première valeur propre du p-laplacien avec poids, C. R. Acad. Sei. Paris Sér. I Math. 305 (1987), 725–728.

[28] P. Lindqvis, On the equation $div(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$, Proc. Amer. Math. Soc. 109 (1990), 57–164.

[29] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, J. Diff. Equ. 51 (1984), 126–150.

[30] Z. Zhang, On a Dirichelt problem with a singular nonliarity, J. Math. Anal. Appl. 194 (1995), 103–113.