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C^n-PSEUDO ALMOST PERIODIC SOLUTIONS UNDER THE LIGHT OF MEASURE THEORY

MICAILOU NAPO AND ISSA ZABSONRE*

ABSTRACT. The aim of this work is study some properties and the existence of solution of some C^n-(μ,ν)-pseudo almost periodic solutions of class r in a Banach space when the delay is distributed using the variation of constants formula and the spectral decomposition of the phase space.

1. Introduction

In this work, we study the existence and uniqueness of C^n-(μ,ν)-pseudo almost periodic solutions of class r for the following neutral partial functional differential equation

$$\frac{d}{dt}u_t = Au_t + L(u_t) + f(t) \text{ for } t \in \mathbb{R}, \tag{1.1}$$

where A is a linear operator on a Banach space X satisfying the Hille-Yosida condition, that is, there exist $M_0 \geq 1$ and $\beta \in \mathbb{R}$ such that $]\beta, +\infty[\subset \rho(A)$ and

$$|R(\lambda, A)^n| \leq \frac{M_0}{\lambda - \beta} \text{ for } n \in \mathbb{N} \text{ and } \lambda > \beta,$$

where $\rho(A)$ is the resolvent set of A and $R(\lambda, A) = (\lambda I - A)^{-1}$ for $\lambda \in \rho(A)$. In sequel, without lost of generality, we suppose that $M_0 = 1$. $C = C([-r, 0]; X)$ denotes the space of continuous functions from $[-r, 0]$ to X endowed with the uniform norm topology. For every $t \geq 0$, u_t denotes the history function of C defined by

$$u_t(\theta) = u(t + \theta) \text{ for } -r \leq \theta \leq 0.$$

L is a bounded linear operator from C into X and $f : \mathbb{R} \to X$ is a continuous function.
Some recent contributions concerning pseudo almost periodic solutions for abstract differential equations similar to equation (1.1) have been made. For example in [10] the authors studied the existence of C^n-almost periodic solutions and C^n-almost automorphic solutions ($n \geq 1$), for partial neutral functional differential equations. They proved that the existence of a bounded integral solution on $\mathbb{R}^+$ implies the existence of C^n-almost periodic and C^n-almost automorphic strict solutions. When the exponential dichotomy holds for the homogeneous linear equation, they shown the uniqueness of C^n-almost periodic and C^n-almost automorphic strict solutions. In [4], the authors proved the existence and uniqueness of C^n-almost periodic solutions to

Université Joseph KI-ZERBO, Unité de Recherche et de Formation en Sciences Exactes et Appliquées, Département de Mathématiques B.P.7021 Ouagadougou 03, Burkina Faso

*Corresponding author

E-mail addresses: napomicailou@yahoo.fr, zabsonreissa@yahoo.fr.

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the nonautonomous ordinary differential equation $x'(t) = A(t)x(t) + f(t),\ t \in \mathbb{R}$, where $A(t)$ generates an exponentially stable family of operators $(U(t,s))_{t\geq s}$ and f is a C^n-almost periodic function with values in a Banach space X. They also studied a Volterra-like equation with a C^n-almost periodic solution.

In [5], the authors present new approach to study weighted pseudo almost periodic functions using the measure theory. they present a new concept of weighted ergodic functions which is more general than the classical one. Then they establish many interesting results on the functional space of such functions like completeness and composition theorems. The theory of their work generalizes the classical results on weighted pseudo almost periodic functions.

The aim of this work is to prove the existence of C^n-(μ,ν)-pseudo almost periodic solutions of equation (1.1) when the delay is distributed on $[-r,0]$. Our approach is based on the variation of constants formula and the spectral decomposition of the phase space developed in [3] and a new approach developped in [5].

This work is organised as follow, in section 2 we recall some prelimary results on variation of constants formula and spectral decomposition. In section 3, we recall some prelimary results on C^n-(μ,ν)-pseudo almost periodic functions and neutral partial functional differential equations that will be used in this work. In section 4, we give some properties of C^n-(μ,ν)-pseudo almost periodic functions of class r. In section 5, we discuss the main result of this paper. Using the strict contraction principle we show the existence and uniqueness of C^n-(μ,ν)-pseudo almost periodic solution of class r for equation (1.1). Finally, for illustration, we propose to study the existence and uniqueness of C^n-(μ,ν)-pseudo almost periodic solution for some model arising in the population dynamics.

2. Variation of constants formula and spectral decomposition

To equation (1.1), we associate the following initial value problem

$$\begin{cases} \dfrac{d}{dt}u(t) = Au(t) + L(u_t) + f(t) \text{ for } t \geq 0 \\ \\ u_0 = \varphi \in C = C([-r,0];X), \end{cases} \tag{2.1}$$

where $f : \mathbb{R}^+ \to X$ is a continuous function.

Definition 2.1. We say that a continuous function u from $[-r,+\infty[$ into X is an integral solution of equation (2.1), if the following conditions hold:

i) $\displaystyle\int_0^t u(s)ds \in D(A)$ for $t \geq 0$,

ii) $u(t) = \varphi(0) + A\displaystyle\int_0^t u(s)ds + \int_0^t (L(u_s) + f(s))ds$ for $t \geq 0$,

iii) $u_0 = \varphi$.

If $\overline{D(A)} = X$, the integral solutions coincide with the known mild solutions.
One can see that if $u(t)$ is an integral solution of equation (2.1), then $u(t) \in \overline{D(A)}$ for all $t \geq 0$, in particular $\varphi(0) \in \overline{D(A)}$.

Let us introduce the part A_0 of the operator A in $\overline{D(A)}$ which defined by

$$\begin{cases} D(A_0) = \{x \in D(A) : \ Ax \in \overline{D(A)}\} \\ A_0 x = Ax \text{ ;for } \ x \in D(A_0) \end{cases}$$

We make the following assertion:

($\mathbf{H_0}$) A satisfies the Hille-Yosida condition.

Lemma 2.2. *[1] A_0 generates a strongly continuous semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$.*

Proposition 2.3. *[2] Assume that ($\boldsymbol{H_0}$) holds, then for all $\varphi \in C$ such that $\varphi(0) \in \overline{D(A)}$, equation (2.1) has a unique integral solution u on $[-r, +\infty[$. Moreover, u is given by*

$$u(t) = T_0(t)\varphi(0) + \lim_{\lambda \to +\infty} \int_0^t T_0(t-s)B_\lambda(L(u_s) + f(s))ds, \quad \textit{for } \ t \geq 0,$$

where $B_\lambda = \lambda R(\lambda, A)$, for $\lambda > \omega$.

The phase space C_0 of equation (2.1) is defined by

$$C_0 = \{\varphi \in C : \ \varphi(0) \in \overline{D(A)}\}.$$

For each $t \geq 0$, we define the linear operator $\mathcal{U}(t)$ on C_0 by

$$\mathcal{U}(t)\varphi = v_t(., \varphi)$$

where $v(., \varphi)$ is the solution of the following homogeneous equation

$$\begin{cases} \dfrac{d}{dt}v(t) = Av(t) + L(v_t) \text{ for } t \geq 0 \\ \\ v_0 = \varphi \in C. \end{cases}$$

Proposition 2.4. *[3] $(\mathcal{U}(t))_{t\geq 0}$ is a strongly continuous semigroup of linear operators on C_0. Moreover, $(\mathcal{U}(t))_{t\geq 0}$ satisfies, for $t \geq 0$ and $\theta \in [-r, 0]$, the following translation property*

$$(\mathcal{U}(t)\varphi)(\theta) = \begin{cases} (\mathcal{U}(t+\theta)\varphi)(0) \textit{ for } t+\theta \geq 0 \\ \\ \varphi(t+\theta) \textit{ for } t+\theta \leq 0. \end{cases}$$

Proposition 2.5. *[3] Let $\mathcal{A}_\mathcal{U}$ defined on C_0 by*

$$\begin{cases} D(\mathcal{A}_\mathcal{U}) = \Big\{\varphi \in C^1([-r, 0]; X); \ \varphi(0) \in D(A), \ \varphi(0)' \in \overline{D(A)} \ \textit{ and } \ \varphi(0)' = A\varphi(0) + L(\varphi)\Big\} \\ \\ \mathcal{A}_\mathcal{U}\varphi = \varphi' \ \textit{ for } \ \varphi \in D(\mathcal{A}_\mathcal{U}). \end{cases}$$

Then $\mathcal{A}_\mathcal{U}$ is the infinitesimal generator of the semigroup $(\mathcal{U}(t))_{t\geq 0}$ on C_0.

Let $\langle X_0 \rangle$ be the space defined by

$$\langle X_0 \rangle = \{X_0 c : \ c \in X\}$$

where the function X_0c is defined by

$$(X_0c)(\theta) = \begin{cases} 0 & \text{if} \quad \theta \in [-r, 0[, \\ \\ c & \text{if} \quad \theta = 0. \end{cases}$$

The space $C_0 \oplus \langle X_0 \rangle$ equipped with the norm $|\phi + X_0c| = |\phi|_C + |c|$ for $(\phi, c) \in C_0 \times X$ is a Banach space and consider the extension $\mathcal{A}_\mathcal{U}$ defined on $C_0 \oplus \langle X_0 \rangle$ by

$$\begin{cases} D(\widetilde{\mathcal{A}_\mathcal{U}}) = \Big\{\varphi \in C^1([-r, 0]; X) : \ \varphi(0) \in D(A) \ \text{ and } \varphi(0)' \in \overline{D(A)}\Big\} \\ \widetilde{\mathcal{A}_\mathcal{U}}\varphi = \varphi' + X_0(A\varphi(0) + L(\varphi) - \varphi(0)'). \end{cases}$$

Lemma 2.6. *[3] Assume that ($\boldsymbol{H_0}$) holds. Then, $\widetilde{\mathcal{A}_\mathcal{U}}$ satisfies the Hille-Yosida condition on $C_0 \oplus \langle X_0 \rangle$ there exist $\widetilde{M} \geq 0$, $\widetilde{\omega} \in \mathbb{R}$ such that $]\widetilde{\omega}, +\infty[\subset \rho(\widetilde{\mathcal{A}_\mathcal{U}})$ and*

$$|(\lambda I - \widetilde{\mathcal{A}_\mathcal{U}})^{-n}| \leq \frac{\widetilde{M}}{(\lambda - \widetilde{\omega})^n} \ \text{ for } n \in \mathbb{N} \text{ and } \lambda > \widetilde{\omega}.$$

Moreover, the part of $\widetilde{\mathcal{A}_\mathcal{U}}$ on $D(\widetilde{\mathcal{A}_\mathcal{U}}) = C_0$ is exactly the operator $\mathcal{A}_\mathcal{U}$.

Now, we can state the variation of constants formula associated to equation (2.1).

Proposition 2.7. *[3] Assume that ($\boldsymbol{H_0}$) holds. Then for all $\varphi \in C_0$, the solution u of equation (2.1) is given by the following formula*

$$u_t = \mathcal{U}(t)\varphi + \lim_{\lambda \to +\infty} \int_0^t \mathcal{U}(t-s)\widetilde{B}_\lambda(X_0 f(s))ds \ \text{ for } t \geq 0,$$

where $\widetilde{B}_\lambda = \lambda(\lambda I - \widetilde{\mathcal{A}}_\mathcal{U})^{-1}$ for $\lambda > \widetilde{\omega}$.

Definition 2.8. We say a semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic if

$$\sigma(\mathcal{A}_\mathcal{U}) \cap i\mathbb{R} =$$

For the sequel, we make the following assumption:

($\mathbf{H_1}$) $T_0(t)$ is compact on $\overline{D(A)}$ for every $t > 0$.

Proposition 2.9. *[3] Assume that ($\boldsymbol{H_0}$) and ($\boldsymbol{H_1}$). Then the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is compact for $t > r$.*

From the compactness of the semigroup $(\mathcal{U}(t))_{t\geq 0}$, we get the following result on the spectral decomposition of the phase space C_0.

Proposition 2.10. *[11] Assume that ($\boldsymbol{H_1}$) holds. If the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic, then the space C_0 is decomposed as a direct sum*

$$C_0 = S \oplus U$$

of two $\mathcal{U}(t)$ invariant closed subspaces S and U such that the restricted semigroup on $\mathcal{U}$ is a group and there exist positive constants $\overline{M}$ and ω such that

$$\begin{aligned} |\mathcal{U}(t)\varphi| &\leq \overline{M}e^{-\omega t}|\varphi| \ \text{ for } t \geq 0 \text{ and } \varphi \in S \\ \\ |\mathcal{U}(t)\varphi| &\leq \overline{M}e^{\omega t}|\varphi| \ \text{ for } t \leq 0 \text{ and } \varphi \in U, \end{aligned}$$

where S and U are called respectively the stable and unstable space, Π^s and Π^u denote respectively the projection operator on S and U.

3. C^n-(μ, ν)-Pseudo almost periodic functions

In this section, we recall some properties about μ-pseudo almost periodic functions. The notion of μ-pseudo almost periodicity is a generalization of the pseudo almost periodicity introduced by Zhang [14–16]; it is also a generalization of weighted pseudo almost periodicity given by Diagana [9]. Let $BC(\mathbb{R}; X)$ be the space of all bounded and continuous function from $\mathbb{R}$ to X equipped with the uniform norm topology.
We denote by $\mathcal{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a, b]) < \infty$, for all $a, b \in \mathbb{R}$ $(a \leq b)$.

Definition 3.1. A bounded continuous function $\phi : \mathbb{R} \to X$ is called almost periodic if for each $\varepsilon > 0$, there exists a relatively dense subset of $\mathbb{R}$ denote by $\mathcal{K}(\varepsilon, \phi, X)$ such that $|\phi(t+\tau)-\phi(t)| < \varepsilon$ for all $(t, \tau) \in \mathbb{R} \times \mathcal{K}(\varepsilon, \phi, X)$.

We denote by $AP(\mathbb{R}; X)$, the space of all such functions.

Definition 3.2. Let X_1 and X_2 be two Banach spaces. A bounded continuous function $\phi : \mathbb{R} \times X_1 \to X_2$ is called almost periodic in $t \in \mathbb{R}$ uniformly in $x \in X_1$ if for each $\varepsilon > 0$ and all compact $K \subset X_1$, there exists a relatively dense subset of $\mathbb{R}$ denote by $\mathcal{K}(\varepsilon, \phi, K)$ such that $|\phi(t + \tau, x) - \phi(t, x)| < \varepsilon$ for all $t \in \mathbb{R},\ x \in K,\ \tau \in \mathcal{K}(\varepsilon, \phi, K)$.

We denote by $AP(\mathbb{R} \times X_1; X_2)$, the space of all such functions.

The next lemma is also a characterization of almost periodic functions.

Lemma 3.3. *A function $\phi \in C(\mathbb{R}, X)$ is almost periodic if and only if the space of functions $\{\phi_\tau :\ \tau \in \mathbb{R}\}$, where $(\phi_\tau)(t) = \phi(t + \tau)$, is relatively compact in $BC(\mathbb{R}; X)$.*

Let $C^n(\mathbb{R}; X)$ be the space of all continuous function which have a continuous n-th derivative on $\mathbb{R}$ and $C_b^n(\mathbb{R}; X)$ be the subspace of $C^n(\mathbb{R}; X)$ of functions satisfying

$$\sup_{t \in \mathbb{R}} \sum_{i=0}^{n} |h^{(i)}(t)| < \infty,$$

where $h^{(i)}$ denotes the i-th derivative of h. Then $C_b^n(\mathbb{R}; X)$ is a Banach space provided with the following norm

$$|h|_n = \sup_{t \in \mathbb{R}} \sum_{i=0}^{n} |h^{(i)}(t)|.$$

Definition 3.4. [4] Let $\varepsilon > 0$ and $h \in C_b^n(\mathbb{R}; X)$. A number $\tau \in \mathbb{R}$ is said to be a $|\ .\ |_n - \varepsilon$ almost periodic of the function h if

$$|h_\tau - h|_n < \varepsilon$$

We denote by $E^{(n)}(\varepsilon, h)$ the space of $|\ .\ |_n - \varepsilon$ almost periodic of the function h.

Definition 3.5. [4] A function $h \in C_b^n(\mathbb{R}; X)$ is said to be C^n-almost periodic functions if for every $\varepsilon > 0$, the set $E^{(n)}(\varepsilon, h)$ is relatively dense in $\mathbb{R}$.

We denote by $AP^{(n)}(\mathbb{R};X)$ the space of the C^n-almost periodic functions. We can see that $AP^{(0)}(\mathbb{R};X)=AP(\mathbb{R};X)$ and for all $n\in\mathbb{N}$, $AP^{(n+1)}(\mathbb{R};X)\subset AP^{(n)}(\mathbb{R};X)$

Definition 3.6. Let X_1 and X_2 be two Banach spaces. A function $\phi\in C_b^n(\mathbb{R}\times X_1;X_2)$ is said to be C^n-almost periodic functions in $t\in\mathbb{R}$ uniformly in $x\in X_1$ if for each $\varepsilon>0$ and all compact $K\subset X_1$, there exists a relatively dense subset of $\mathbb{R}$ denote by $\mathcal{K}(\varepsilon,\phi,K)$ such that $|\phi_\tau(.,x)-\phi(.,x)|_n<\varepsilon$ for all $t\in\mathbb{R}$, $x\in K$, $\tau\in\mathcal{K}(\varepsilon,\phi,K)$, $\phi_\tau(.,x)(t)=\phi(t+\tau,x)$. Here

$$|\phi(.,x)|_n=\sup_{t\in\mathbb{R}}\sum_{i=0}^{n}\Big|\frac{\partial^i\phi}{\partial t^i}(t,x)\Big|.$$

We denote by $AP^{(n)}(\mathbb{R}\times X_1;X_2)$, the space of all such functions.

Since it is well known that for any almost periodic functions h_1 and h_2 and $\varepsilon>0$, there exists a relatively dense set of their common ε almost periodic. Consequently, we get the following result.

Proposition 3.7. *[4] $h\in AP^{(n)}(\mathbb{R};X)$ if and only if $h^{(i)}\in AP(\mathbb{R};X)$ for $i=1,...,n$.*

Since $AP(\mathbb{R};X)$ equipped with uniform norm topology is a Banach space, then we get the following result.

Proposition 3.8. *[4] $AP^{(n)}(\mathbb{R};X)$ provided with the norm $|\,.\,|_n$ is a Banach space.*

Example 3.9. The following example of C^n-almost periodic function has been given in [7]. Let

$$g(t)=\sin(\alpha t)+\sin(\beta t)$$

where $(\alpha/\beta)\notin\mathbb{Q}$. Then the function $h(t)=e^{g(t)}$ is C^n-almost periodic function for any $n\geq 1$. In [7], one can find example of function which is C^n-almost periodic but not C^{n+1}-almost periodic.

In the sequel, we use some preliminary results concerning the (μ,ν)-Pseudo almost periodic functions which have been established recently in [5].

$\mathcal{E}(\mathbb{R};X,\mu,\nu)$ stands for the space of functions

$$\mathcal{E}(\mathbb{R};X,\mu)=\Big\{u\in BC(\mathbb{R};X):\ \lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}|u(t)|d\mu(t)=0\Big\}.$$

To study delayed differential equations for which the history belong to $C([-r,0];X)$, we need to introduce the space

$$\mathcal{E}(\mathbb{R};X,\mu,\nu,r)=\Big\{u\in BC(\mathbb{R};X):\ \lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|u(\theta)|\Big)d\mu(t)=0\Big\}.$$

In addition to above-mentioned space, we consider the following spaces

$$\mathcal{E}(\mathbb{R}\times X_1,X_2,\mu,\nu)=\Big\{u\in BC(\mathbb{R}\times X_1;X_2):\ \lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}|u(t,x)|_{X_2}d\mu(t)=0\Big\},$$

$$\mathcal{E}(\mathbb{R}\times X_1;X_2,\mu,\nu,r)=\Big\{u\in BC(\mathbb{R}\times X_1;X_2):\ \lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|u(\theta,x)|_{X_2}\Big)d\mu(t)=0\Big\}.$$

where in both cases the limit (as $\tau \to +\infty$) is uniform in compact subset of X_1.

In view of previous definitions, it is clear that the spaces $\mathcal{E}(\mathbb{R}; X, \mu, \nu, r)$ and $\mathcal{E}(\mathbb{R}\times X_1; X_2, \mu, \nu, r)$ are continuously embedded in $\mathcal{E}(\mathbb{R}; X, \mu, \nu)$ and $\mathcal{E}(\mathbb{R} \times X_1, X_2, \mu, \nu)$, respectively.
On the other hand, one can observe that a ρ-weighted pseudo almost periodic functions is μ-pseudo almost periodic, where the measure μ is absolutely continuous with respect to the Lebesgue measure and its Radon-Nikodym derivative is ρ:

$$d\mu(t) = \rho(t)dt.$$

Example 3.10. [5] Let ρ be a nonnegative $\mathcal{B}$-measurable function. Denote by μ the positive measure defined by

$$\mu(A) = \int_A \rho(t)dt, \text{ for } A \in \mathcal{B}, \tag{3.1}$$

where dt denotes the Lebesgue measure on $\mathbb{R}$. The function ρ which occurs in equation (3.1) is called the Radon-Nikodym derivative of μ with respect to the Lebesgue measure on $\mathbb{R}$.

Definition 3.11. A function $h \in C_b^n(\mathbb{R}; X)$ is said to be C^n-ergodic functions if $h^{(i)} \in \mathcal{E}(\mathbb{R}; X, \mu)$ for $i = 1, ..., n$. We denote by $\mathcal{E}^{(n)}(\mathbb{R}; X, \mu)$ the space of the C^n-ergodic functions.

Definition 3.12. A function $h \in C_b^n(\mathbb{R}; X)$ is said to be C^n-ergodic functions of class r if $h^{(i)} \in \mathcal{E}(\mathbb{R}; X, \mu)$ for $i = 1, ..., n$. We denote by $\mathcal{E}^{(n)}(\mathbb{R}; X, \mu, r)$ the space of the C^n-ergodic functions.

From $\mu, \nu \in \mathcal{M}$, we formulate the following hypothese.

($\mathbf{H_2}$) Let $\mu, \nu \in \mathcal{M}$ be such that $\limsup\limits_{\tau\to+\infty} \dfrac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} = \alpha < \infty$.

We have the following result.

Lemma 3.13. *Asumme ($\boldsymbol{H_2}$) holds and let $f \in C_b^n(\mathbb{R}; X)$. Then $f \in \mathcal{E}^{(n)}(\mathbb{R}; X, \mu, \nu)$ if and only if for any $\varepsilon > 0$ and for $i = 1, ..., n$,*

$$\lim_{\tau\to+\infty} \frac{\mu(M_{\tau,\varepsilon}(f^{(i)}))}{\nu([-\tau,\tau]} = 0$$

where

$$M_{\tau,\varepsilon}(f^{(i)}) = \{t \in [-\tau, \tau] : \ |f^{(i)}(t)| \geq \varepsilon\}.$$

Proof. Suppose that $f \in \mathcal{E}^{(n)}(\mathbb{R}; X, \mu, \nu)$, then by Definition 3.11 $f^{(i)} \in \mathcal{E}(\mathbb{R}; X, \mu, \nu)$. We have

$$\begin{aligned}
\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau} |f^{(i)}(t)|d\mu(t) &= \frac{1}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(f^{(i)})} |f^{(i)}(t)|d\mu(t) \\
&+ \frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]\setminus M_{\tau,\varepsilon}(f^{(i)})} |f^{(i)}(t)|d\mu(t) \\
&\geq \frac{1}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(f^{(i)})} |f^{(i)}(t)|d\mu(t)
\end{aligned}$$

$$\geq \frac{\varepsilon}{\nu([-\tau,\tau])} M_{\tau,\varepsilon}(f^{(i)}).$$

Consequently

$$\lim_{\tau\to+\infty} \frac{\mu(M_{\tau,\varepsilon}(f^{(i)}))}{\nu([-\tau,\tau]} = 0.$$

Suppose that $f \in C_b^n(\mathbb{R};X)$ such that for any $\varepsilon > 0$ and for $i = 1, ..., n$,

$$\lim_{\tau\to+\infty} \frac{\mu(M_{\tau,\varepsilon}(f^{(i)}))}{\nu([-\tau,\tau]} = 0$$

We can assume $|f^{(i)}(t)| \leq N$ for all $t \in \mathbb{R}$. Using $(\mathbf{H_2})$, we have

$$\begin{aligned}
& \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} |f^{(i)}(t)| d\mu(t) \\
= & \frac{1}{\nu([-\tau,\tau])} \int_{M_{\tau,\varepsilon}(f^{(i)})} |f^{(i)}(t)| d\mu(t) \\
+ & \frac{1}{\nu([-\tau,\tau])} \int_{[-\tau,\tau]\setminus M_{\tau,\varepsilon}(f^{(i)})} |f^{(i)}(t)| d\mu(t) \\
\leq & \frac{N}{\nu([-\tau,\tau])} \int_{M_{\tau,\varepsilon}(f^{(i)})} d\mu(t) + \frac{1}{\nu([-\tau,\tau])} \int_{[-\tau,\tau]\setminus M_{\tau,\varepsilon}(f^{(i)})} |f^{(i)}(t)| d\mu(t) \\
\leq & \frac{N}{\nu([-\tau,\tau])} \int_{M_{\tau,\varepsilon}(f^{(i)})} d\mu(t) + \frac{\varepsilon}{\nu([-\tau,\tau])} \int_{[-\tau,\tau]} d\mu(t) \\
\leq & \frac{N}{\nu([-\tau,\tau])} M_{\tau,\varepsilon}(f^{(i)}) + \frac{\varepsilon\mu([-\tau,\tau])}{\nu([-\tau,\tau])}.
\end{aligned}$$

Which implies that

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} |f^{(i)}(t)| d\mu(t) \leq \alpha\varepsilon \text{ for any } \varepsilon > 0.$$

Therefore $f^{(i)} \in \mathcal{E}(\mathbb{R};X,\mu,\nu)$ for $i = 1, ..., n$, which implies that $f \in \mathcal{E}^{(n)}(\mathbb{R};X,\mu,\nu)$.☐

Definition 3.14. Let $\mu \in \mathcal{M}$. A bounded continuous function $\phi \in C_b^n(\mathbb{R};X)$ is called C^n-(μ,ν)-pseudo almost periodic if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AP^{(n)}(\mathbb{R},X)$ and $\phi_2 \in \mathcal{E}^{(n)}(\mathbb{R};X,\mu,\nu)$.

We denote by $PAP^{(n)}(\mathbb{R};X,\mu,\nu)$ the space of all such functions.

Definition 3.15. Let $\mu \in \mathcal{M}$ and X_1 and X_2 be two Banach spaces. A bounded continuous function $\phi \in C_b^n(\mathbb{R};X_1 \to X_2)$ is called uniformly C^n-(μ,ν)-pseudo almost periodic if $\phi = \phi_1 + \phi_2$, where
$\phi_1 \in AP^{(n)}(\mathbb{R}\times X_1;X_2)$ and $\phi_2 \in \mathcal{E}^{(n)}(\mathbb{R}\times X_1, X_2, \mu)$.

We denote by $PAP^{(n)}(\mathbb{R}\times X_1;X_2,\mu,\nu)$, the space of all such functions.

Definition 3.16. $\mu \in \mathcal{M}$. A bounded continuous function $\phi \in C_b^n(\mathbb{R};X)$ is C^n-called (μ,ν)-pseudo almost periodic of class r if $\phi = \phi_1+\phi_2$, where $\phi_1 \in AP^{(n)}(\mathbb{R};X)$ and $\phi_2 \in \mathcal{E}^{(n)}(\mathbb{R};X,\mu,\nu,r)$.

We denote by $PAP^{(n)}(\mathbb{R};X,\mu,r)$, the space of all such functions.

Definition 3.17. $\mu \in \mathcal{M}$. Let X_1 and X_2 be two Banach spaces. A bounded continuous function $\phi \in C_b^n(\mathbb{R}; X_1 \to X_2)$ is called uniformly C^n-(μ,ν)-pseudo almost periodic of class r if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AP^{(n)}(\mathbb{R}\times X_1; X_2)$ and $\phi_2 \in \mathcal{E}^{(n)}(\mathbb{R}\times X_1; X_2, \mu, r)$.

We denote by $PAP^{(n)}(\mathbb{R}\times X_1; X_2, \mu, \nu, r)$, the space of all such functions.

4. Properties of C^n-(μ,ν)-pseudo almost periodic functions of class r

Lemma 4.1. $\mu \in \mathcal{M}$. *The space $PAP^{(n)}(\mathbb{R}; X, \mu, \nu, r)$ endowed with the $|.|_n$ norm is a Banach space.*

Proof. Let $(x_m)_m$ be a sequence in $PAP^{(n)}(\mathbb{R}; X, \mu, \nu, r)$ such that $\lim\limits_{m\to\infty} x_m = x$ in $BC^n(\mathbb{R}; X)$. For each m, let $x_m = y_n + z_m$ with $y_m \in AP^{(n)}(\mathbb{R}; X)$ and $z_m \in \mathcal{E}^{(n)}(\mathbb{R}; X, \mu, \nu, r)$. Since $y_m \in AP^{(n)}(\mathbb{R}; X)$, then from Proposition 3.7 $y_m^{(i)} \in AP(\mathbb{R}; X)$ and by [12, Lemma 1.2], $(y_m^{(i)})_m$ converges to some $y^{(i)} \in AP(\mathbb{R}; X)$ for $i = 0, 1, ..., n$. Consequently by Proposition 3.7 $y \in AP^{(n)}(\mathbb{R}; X)$.

Since $z_m \in \mathcal{E}^{(n)}(\mathbb{R}; X, \mu, \nu, r)$, Definition 3.12 implies that $z_m^{(i)} \in \mathcal{E}(\mathbb{R}; X, \mu, \nu, r)$ and $(z_m^{(i)})_m$ converges to some $z^{(i)} \in BC(\mathbb{R}; X)$ for $i = 0, 1, ..., n$. We have

$$\begin{aligned}
\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|z^{(i)}(\theta)|\Big)d\mu(t) &\le \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|z_m^{(i)}(\theta)-z^{(i)}(\theta)|\Big)d\mu(t)\\
&\quad+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|z_m^{(i)}(\theta)|\Big)d\mu(t)\\
&\le \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{t\in\mathbb{R}}|z_m^{(i)}(t)-z^{(i)}(t)|\Big)d\mu(t)\\
&\quad+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|z_m^{(i)}(\theta)|\Big)d\mu(t)\\
&\le \|z_m^{(i)}-z^{(i)}\| + \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|z_m^{(i)}(\theta)|\Big)d\mu(t).
\end{aligned}$$

Then we get $z^{(i)} \in \mathcal{E}(\mathbb{R}; X, \mu, \nu, r)$ for $i = 0, 1, ..., n$, i.e $z \in \mathcal{E}^{(n)}(\mathbb{R}; X, \mu, \nu, r)$. It follows that $x \in PAP^{(n)}(\mathbb{R}; X, \mu, \nu, r)$.$\square$

Next result is a characterization of (μ,ν)-ergodic functions of class r.

Theorem 4.2. *Assume that $(\boldsymbol{H_2})$ holds and let $\mu, \nu \in \mathcal{M}$ and I be a bounded interval (eventually $I = $). Assume that $f \in C_b^n(\mathbb{R}; X)$. Then the following assertions are equivalent:*

i) $f \in \mathcal{E}^{(n)}(\mathbb{R}, X, \mu, \nu, r)$.

ii) $\lim\limits_{\tau\to+\infty}\dfrac{1}{\nu([-\tau,\tau]\setminus I)}\displaystyle\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t) = 0$ *for* $i = 0, 1, ..., n$.

iii) For any $\varepsilon > 0$, $\lim\limits_{\tau\to+\infty}\dfrac{\mu\Big(\Big\{t\in[-\tau,\tau]\setminus I : \sup\limits_{\theta\in[t-r,t]}|f^{(i)}(\theta)| > \varepsilon\Big\}\Big)}{\nu([-\tau,\tau]\setminus I)} = 0$ *for* $i = 0, 1, ..., n$.

Proof. $i) \Leftrightarrow ii)$ Denote by $A = \nu(I)$, $B = \displaystyle\int_I\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t)$. We have A and $B \in \mathbb{R}$, since the interval I is bounded and the function $f^{(i)}$ is bounded and continuous for $i = 0, 1, ..., n$. For $\tau > 0$ such that $I \subset [-\tau, \tau]$ and $\nu([-\tau, \tau] \setminus I) > 0$, we have

$$\begin{aligned}\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t) &= \frac{1}{\nu([-\tau,\tau])-A}\Big[\int_{[-\tau,\tau]}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t)-B\Big]\\ &= \frac{\nu([-\tau,\tau])}{\nu([-\tau,\tau])-A}\Big[\frac{1}{\nu([-r,r])}\int_{[-\tau,\tau]}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t)-\frac{B}{\nu([-\tau,\tau])}\Big].\end{aligned}$$

From above equalities and the fact that $\nu(\mathbb{R}) = +\infty$, we deduce that $ii)$ is equivalent to

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t) = 0, \quad \text{for } i = 0,1,...,n$$

and by Definition 3.11 that is $i)$.

$iii) \Rightarrow ii)$ Denote by A^{ε}_{τ} and B^{ε}_{τ} the following sets

$$A^{\varepsilon}_{\tau} = \Big\{t\in[-\tau,\tau]\setminus I : \sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)| > \varepsilon\Big\} \text{ and } B^{\varepsilon}_{\tau} = \Big\{t\in[-\tau,\tau]\setminus I) : \sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)| \le \varepsilon\Big\}.$$

Assume that $iii)$ holds, that is

$$\lim_{\tau\to+\infty}\frac{\mu(A^{\varepsilon}_{\tau})}{\nu([-\tau,\tau]\setminus I)} = 0. \tag{4.1}$$

From the equality

$$\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t) = \int_{A^{\varepsilon}_{\tau}}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t)+\int_{B^{\varepsilon}_{\tau}}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t),$$

we deduce that for τ sufficiently large

$$\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t) \le \|f^{(i)}\|_{\infty}\frac{\mu(A^{\varepsilon}_{\tau})}{\nu([-\tau,\tau]\setminus I)}+\varepsilon\frac{\mu(B^{\varepsilon}_{\tau})}{\nu([-\tau,\tau]\setminus I)}.$$

By using $(\mathbf{H_2})$, it follows that

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t) \le \alpha\varepsilon, \quad \text{for any } \varepsilon>0, \quad \text{for } i=0,1,...,n,$$

consequently (ii) holds.

$ii) \Rightarrow iii)$ Assume that $ii)$ holds. From the following inequality

$$\begin{aligned}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t) &\ge \int_{A^{\varepsilon}_{\tau}}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t)\\ \frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t) &\ge \varepsilon\frac{\mu(A^{\varepsilon}_{\tau})}{\nu([-\tau,\tau]\setminus I)}\\ \frac{1}{\varepsilon\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t) &\ge \frac{\mu(A^{\varepsilon}_{\tau})}{\nu([-\tau,\tau]\setminus I)}, \quad \text{for } i=0,1,...,n\end{aligned}$$

For τ sufficiently large, we obtain equation (4.1), that is $iii)$.□

From $\mu \in \mathcal{M}$, we formulate the following hypotheses.

$(\mathbf{H_3})$ For all a, b and $c \in \mathbb{R}$, such that $0 \le a < b \le c$, there exist δ_0 and $\alpha_0 > 0$ such that

$$|\delta| \ge \delta_0 \Rightarrow \mu(a+\delta, b+\delta) \le \alpha_0\mu(\delta, c+\delta).$$

($\mathbf{H_4}$) For all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that

$$\mu(\{a+\tau : \ a \in A\} \leq \beta\mu(A) \ \text{ when } A \in \mathcal{B} \text{ satisfies } \ A \cap I = .$$

We have the following results due to [5]

Lemma 4.3. *[5] Hypothesis* ($\boldsymbol{H_4}$) *implies* ($\boldsymbol{H_3}$).

Proposition 4.4. *[5, 8] $\mu, \nu \in \mathcal{M}$ satisfy ($\boldsymbol{H_3}$) and $f \in PAP(\mathbb{R}; X, \mu, \nu)$ be such that*

$$f = g + h$$

where $g \in AP(\mathbb{R}, X)$ and $h \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$. Then

$$\{g(t), \ t \in \mathbb{R}\} \subset \overline{\{f(t), \ t \in \mathbb{R}\}} \text{ (the closure of the range of } f).$$

Corollary 4.5. *[8] Assume that ($\boldsymbol{H_3}$) holds. Then the decomposition of a (μ, ν)-pseudo almost periodic function in the form $f = g + \phi$ where $g \in AP(\mathbb{R}; X)$ and $\phi \in \mathcal{E}(\mathbb{R}; X, \mu, \nu)$, is unique.*

The following corollary is a consequence of Corollary 4.5.

Proposition 4.6. *Let $\mu, \nu \in \mathcal{M}$. Assume ($\boldsymbol{H_3}$) holds. Then the decomposition of a (μ, ν)-pseudo-almost periodic function $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AP^{(n)}(\mathbb{R}; X)$ and $\phi_2 \in \mathcal{E}^{(n)}(\mathbb{R}; X, \mu, \nu, r)$, is unique.*

Proof. Let In fact $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AP^{(n)}(\mathbb{R}; X)$ and $\phi_2 \in \mathcal{E}^{(n)}(\mathbb{R}; X, \mu, \nu, r)$, then $\phi_1^{(i)} \in AP(\mathbb{R}; X)$ and $\phi_2^{(i)} \in \mathcal{E}(\mathbb{R}; X, \mu, \nu, r)$ for $i = 0, 1, ..., n$. Since as a consequence of Corollary 4.5, the decomposition of a (μ, ν)-pseudo-almost periodic function $\phi^{(i)} = \phi_1^{(i)} + \phi_2^{(i)}$, where $\phi_1^{(i)} \in AP(\mathbb{R}; X)$ and $\phi_2^{(i)} \in \mathcal{E}(\mathbb{R}; X, \mu, \nu)$, is unique and $PAP(\mathbb{R}; X, \mu, \nu, r) \subset PAP(\mathbb{R}; X, \mu, \nu)$, then the decomposition of a (μ, ν)-pseudo-almost periodic function of class r, $\phi^{(i)} = \phi_1^{(i)} + \phi_2^{(i)}$, where $\phi_1^{(i)} \in AP(\mathbb{R}; X)$ and $\phi_2 \in \mathcal{E}(\mathbb{R}; X, \mu, \nu, r)$, is unique. Consequently, we get the desired result.□

Definition 4.7. Let $\mu_1, \mu_2 \in \mathcal{M}$. We say that μ_1 is equivalent to μ_2, denoting this as $\mu_1 \sim \mu_2$ if there exist constants α and $\beta > 0$ and a bounded interval I (eventually $I =$) such that

$$\alpha\mu_1(A) \leq \mu_2(A) \leq \beta\mu_1(A), \ \text{ when } A \in \mathcal{B} \text{ satisfies } A \cap I = .$$

From [5] $\sim$ is a binary equivalence relation on $\mathcal{M}$. the equivalence class of a given measure $\mu \in \mathcal{M}$ will then be denoted by

$$cl(\mu) = \{\varpi \in \mathcal{M} : \ \mu \sim \varpi\}.$$

Theorem 4.8. *Let $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathcal{M}$. If $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$, then $PAP^{(n)}(\mathbb{R}; X, \mu_1, \nu_1, r) = PAP^{(n)}(\mathbb{R}; X, \mu_2, \nu_2, r)$.*

Proof. Since $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$ there exist some constants $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ and a bounded interval I (eventually $I =$) such that $\alpha_1\mu_1(A) \leq \mu_2(A) \leq \beta_1\mu_1(A)$ and $\alpha_2\nu_1(A) \leq \nu_2(A) \leq \beta_2\nu_1(A)$ for each $A \in \mathcal{B}$ satisfies $A \cap I =$ i.e

$$\frac{1}{\beta_2\nu_1(A)} \leq \frac{1}{\nu_2(A)} \leq \frac{1}{\alpha_2\nu_1(A)}.$$

Let $f \in C_b^n(\mathbb{R}, X)$, since $\mu_1 \sim \mu_2$ and $\mathcal{B}$ is the Lebesgue σ-field, we obtain for τ sufficiently large, it follows that

$$\frac{\alpha_1\mu_1\Big(\Big\{t \in [-\tau,\tau] \setminus I : \sup_{\theta\in[t-r,t]} |f^{(i)}(\theta)| > \varepsilon\Big\}\Big)}{\beta_2\nu_1([-\tau,\tau] \setminus I)} \leq \frac{\mu_2\Big(\Big\{t \in [-\tau,\tau] \setminus I : \sup_{\theta\in[t-r,t]} |f^{(i)}(\theta)| > \varepsilon\Big\}\Big)}{\nu_2([-\tau,\tau] \setminus I)}$$
$$\leq \frac{\beta_1\mu_1\Big(\Big\{t \in [-\tau,\tau] \setminus I : \sup_{\theta\in[t-r,t]} |f^{(i)}(\theta)| > \varepsilon\Big\}\Big)}{\alpha_2\nu_1([-\tau,\tau] \setminus I)}$$

By using Theorem 4.2 we deduce that $\mathcal{E}^{(n)}(\mathbb{R}, X, \mu_1, \nu_1, r) = \mathcal{E}^{(n)}(\mathbb{R}, X, \mu_2, \nu_2, r)$. From the definition of a (μ,ν)-pseudo almost periodic function, we deduce that $PAP^{(n)}(\mathbb{R}; X, \mu_1, \nu_1, r) = PAP^{(n)}(\mathbb{R}; X, \mu_2, \nu_2, r)$.□

Let $\mu, \nu \in \mathcal{M}$ we denote by

$$cl(\mu,\nu) = \{\varpi_1, \varpi_2 \in \mathcal{M} :\ \mu \sim \varpi_2 \text{ and } \ \nu \sim \varpi_2\}.$$

Proposition 4.9. *[8] Let $\mu, \nu \in \mathcal{M}$ satisfy ($\boldsymbol{H_4}$). Then $PAP(\mathbb{R}, X, \mu, \nu)$ is invariant by translation, that is $f \in PAP(\mathbb{R}, X, \mu, \nu)$ implies $f_\alpha \in PAP(\mathbb{R}, X, \mu, \nu)$ for all $\alpha \in \mathbb{R}$.*

We can deduce the following result.

Corollary 4.10. *Let $\mu, \nu \in \mathcal{M}$ satisfy ($\boldsymbol{H_4}$). Then $PAP^{(n)}(\mathbb{R}, X, \mu, \nu)$ is invariant by translation, that is $f \in PAP^{(n)}(\mathbb{R}, X, \mu, \nu)$ implies $f_\alpha \in PAP^{(n)}(\mathbb{R}, X, \mu, \nu)$ for all $\alpha \in \mathbb{R}$.*

In what follows, we prove some preliminary results concerning the composition of (μ,ν)-pseudo almost periodic functions of class r.

Theorem 4.11. *Let $\mu, \nu \in \mathcal{M}$, $\phi \in PAP^{(n)}(\mathbb{R}\times X_1; X_2, \mu, \nu, r)$ and $h \in PAP^{(n)}(\mathbb{R}; X_1, \mu, \nu, r)$. Assume that there exists a function $L_\phi : \mathbb{R} \to [0, +\infty[$ sastisfies*

$$|\phi(t,x_1) - \phi(t,x_2)| \leq l_\phi(t)|x_1 - x_2| \quad \text{for} \quad t \in \mathbb{R} \quad \text{and for} \quad x_1, x_2 \in X_1. \tag{4.2}$$

If

$$\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]} L_\phi(\theta)\Big)d\mu(t) < \infty \quad \text{and} \quad \lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]} L_\phi(\theta)\Big)\xi(t)d\mu(t) = 0 \tag{4.3}$$

for each $\xi \in \mathcal{E}^{(n)}(\mathbb{R}, \mu, \nu)$ and for almost $\tau > 0$, then the function $t \to \phi(t, h(t))$ belongs to $PAP^{(n)}(\mathbb{R}; X_2, \mu, \nu, r)$.

Proof. Assume that $\phi = \phi_1 + \phi_2$, $h = h_1 + h_2$ where $\phi_1 \in AP^{(n)}(\mathbb{R}\times X_1; X_2)$, $\phi_2 \in \mathcal{E}^{(n)}(\mathbb{R}\times X_1; X_2, \mu, \nu, r)$ and $h_1 \in AP^{(n)}(\mathbb{R}; X_1)$, $h_2 \in \mathcal{E}^{(n)}(\mathbb{R}; X_1, \mu, \nu, r)$, then $\phi_1^{(i)} \in AP(\mathbb{R}\times X_1; X_2)$, $\phi_2^{(i)} \in \mathcal{E}(\mathbb{R}\times X_1; X_2, \mu, \nu, r)$ and $h_1^{(i)} \in AP(\mathbb{R}; X_1)$, $h_2^{(i)} \in \mathcal{E}(\mathbb{R}; X_1, \mu, \nu, r)$ for $i = 0, 1, ..., n$. Consider the following decomposition

$$\phi^{(i)}(t, h(t)) = \phi_1^{(i)}(t, h_1^{(i)}(t)) + [\phi^{(i)}(t, h^{(i)}(t)) - \phi^{(i)}(t, h_1^{(i)}(t))] + \phi_2^{(i)}(t, h_1^{(i)}(t)).$$

From [6, 13], $\phi_1^{(i)}(., h_1^{(i)}(.)) \in AP(\mathbb{R}; X_2)$ for $i = 0, 1, ..., n$. It remains to prove that both $\phi^{(i)}(., h^{(i)}(.)) - \phi^{(i)}(., h_1^{(i)}(.))$ and $\phi_2^{(i)}(., h_1^{(i)}(.))$ belong to $\mathcal{E}(\mathbb{R}; X_2, \mu, \nu, r)$ for $i = 0, 1, ..., n$.

Using equation (4.2), it follows that

$$\begin{aligned}
&\frac{\mu\Big(\Big\{t\in[-\tau,\tau]:\ \sup_{\theta\in[t-r,t]}|\phi^{(i)}(\theta,h^{(i)}(\theta))-\phi^{(i)}(\theta,h_1^{(i)}(\theta))|>\varepsilon\Big\}\Big)}{\nu([-\tau,\tau])}\\
\leq\ &\frac{\mu\Big(\Big\{t\in[-\tau,\tau]:\ \sup_{\theta\in[t-r,t]}(L_\phi(\theta)|h_2^{(i)}(\theta)|)>\varepsilon\Big\}\Big)}{\nu([-\tau,\tau])}\\
\leq\ &\frac{\mu\Big(\Big\{t\in[-\tau,\tau]:\Big(\sup_{\theta\in[t-r,t]}L_\phi(\theta)\Big)\Big(\sup_{\theta\in[t-r,t]}|h_2^{(i)}(\theta)|\Big)>\varepsilon\Big\}\Big)}{\nu([-\tau,\tau])}.
\end{aligned}$$

Since $h_2^{(i)}$ is (μ,ν)-ergodic of class r, Theorem 4.2 and equation (4.3) yield that for the above-mentioned ε, we have

$$\lim_{\tau\to+\infty}\frac{\mu\Big(\Big\{t\in[-\tau,\tau]:\Big(\sup_{\theta\in[t-r,t]}L_\phi(\theta)\Big)\Big(\sup_{\theta\in[t-r,t]}|h_2^{(i)}(\theta)|\Big)>\varepsilon\Big\}\Big)}{\nu([-\tau,\tau])}=0,$$

and then we obtain

$$\lim_{\tau\to+\infty}\frac{\mu\Big(\Big\{t\in[-\tau,\tau]:\ \sup_{\theta\in[t-r,t]}|\phi^{(i)}(\theta,h^{(i)}(\theta))-\phi(\theta,h_1(\theta))|>\varepsilon\Big\}\Big)}{\nu([-\tau,\tau])}=0, \tag{4.4}$$

By Theorem 4.2, equation (4.4) shows that $t\mapsto\phi^{(i)}(t,h^{(i)}(t))-\phi^{(i)}(t,h_1^{(i)}(t))$ is (μ,ν)-ergodic of class r for $i=0,1,...,n$.

Now to complete the proof, it is enough to prove that $t\mapsto\phi_2(t,h(t))$ is (μ,ν)-ergodic of class r. Since $\phi_2^{(i)}$ is uniformly continuous on the compact set $K_i=\overline{\{h_1^{(i)}(t):\ t\in\mathbb{R}\}}$ with respect to the second variable x, we deduce that for given $\varepsilon>0$, there exists $\delta>0$ such that, for all $t\in\mathbb{R}$, ξ_1 and $\xi_2\in K_i$, one has

$$\|\xi_1-\xi_2\|\leq\delta\Rightarrow\|\phi_2^{(i)}(t,\xi_1^{(i)}(t))-\phi_2^{(i)}(t,\xi_2^{(i)}(t))\|\leq\varepsilon.$$

Therefore, there exist $n(\varepsilon)$ and $\{z_k^{(i)}\}_{k=1}^{m(\varepsilon)}\subset K$, such that

$$K_i\subset\bigcup_{k=1}^{m(\varepsilon)}B_\delta(z_k^{(i)},\delta)$$

and then

$$\|\phi_2^{(i)}(t,h_1^{(i)}(t))\|\leq\varepsilon+\sum_{k=1}^{m(\varepsilon)}\|\phi_2^{(i)}(t,z_i)\|$$

Since

$$\forall k\in\{1,...,m(\varepsilon)\},\quad\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}|\phi_2^{(i)}(\theta,z_k^{(i)})|\Big)d\mu(t)=0,$$

we deduce that

$$\forall\varepsilon>0,\quad\limsup_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}|\phi_2^{(i)}(\theta,h_1^{(i)}(t))|\Big)d\mu(t)\leq\varepsilon,$$

that implies

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}|\phi_2^{(i)}(\theta,h_1^{(i)}(t))|\Big)d\mu(t)=0.$$

Consequently $t \mapsto \phi_2^{(i)}(t, h^{(i)}(t))$ is (μ,ν)-ergodic of class r for $i = 0, 1, ..., n$.
Using Proposition 3.7 and Definition 3.11 it follows that the function $t \to \phi(t, h(t))$ belongs to $PAP^{(n)}(\mathbb{R}; X_2, \mu, \nu, r)$.$\square$

For $\mu \in \mathcal{M}$ and $\alpha \in \mathbb{R}$, we denote μ_α the positive measure on $(\mathbb{R}, \mathcal{B})$ defined by

$$\mu_\alpha(A) = \mu([a + \alpha : \ a \in A]) \tag{4.5}$$

Lemma 4.12. *[5] Let $\mu \in \mathcal{M}$ satisfy $(\boldsymbol{H_3})$. Then the measures μ and μ_α are equivalent for all $\alpha \in \mathbb{R}$.*

Lemma 4.13. *[5] $(\boldsymbol{H_3})$ implies*

$$\textit{for all } \ \sigma > 0 \ \ \limsup_{\tau\to+\infty} \frac{\mu([-\tau-\sigma, \tau+\sigma])}{\mu([-\tau, \tau])} < +\infty.$$

We have the following result.

Theorem 4.14. *Assume that $(\boldsymbol{H_3})$ holds. Let $\mu, \nu \in \mathcal{M}$ and $\phi \in PAP^{(n)}(\mathbb{R}; X, \mu, \nu, r)$, then the function $t \to \phi_t$ belongs to $PAP^{(n)}(C([-r, 0]; X), \mu, \nu, r)$.*

Proof. Assume that $\phi = g + h$ where $g \in AP^{(n)}(\mathbb{R}; X)$ and $h^{(n)} \in \mathcal{E}(\mathbb{R}; X, \mu, \nu, r)$. Then we can see that, $\phi_t = g_t + h_t$ and for $i = 0, 1, ..., n$ $g_t^{(i)}$ is almost periodic, which implies that $g_t \in AP^{(n)}(C([-r, 0]; X), \mu, \nu, r)$. For $i = 0, 1, ..., n$, tet us denote by

$$M_\alpha(\tau) = \frac{1}{\nu_\alpha([-\tau, \tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta\in[t-r,t]} |h^{(i)}(\theta)|\Big) d\mu_\alpha(t),$$

where μ_α and ν_α are the positive measures defined by equation (4.5). By using Lemma 4.12, it follows that μ_α and μ are equivalent and ν_α and ν are also equivalent. hen by using Theorem 4.8 we have $\mathcal{E}^{(n)}(\mathbb{R}; X, \mu_\alpha, \nu_\alpha, r) = \mathcal{E}^{(n)}(\mathbb{R}; X, \mu, \nu, r)$, therefore $h^{(i)} \in \mathcal{E}(\mathbb{R}; X, \mu_\alpha, \nu_\alpha, r)$, that is

$$\lim_{\tau\to+\infty} M_\alpha(\tau) = 0, \ \text{ for all } \ \alpha \in \mathbb{R}.$$

On the other hand, for $r > 0$ we have

$$\begin{aligned}
&\frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta\in[t-r,t]} \Big[\sup_{\xi\in[-r,0]} |h^{(i)}(\theta+\xi)|\Big]\Big) d\mu(t) \le \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta\in[t-2r,t]} |h^{(i)}(\theta)|\Big) d\mu(t) \\
\le\ & \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta\in[t-2r,t-r]} |h^{(i)}(\theta)| + \sup_{\theta\in[t-r,t]} |h^{(i)}(\theta)|\Big) d\mu(t) \\
\le\ & \frac{1}{\nu([-\tau, \tau])} \int_{-\tau-r}^{+\tau-r} \Big(\sup_{\theta\in[t-r,t]} |h^{(i)}(\theta)|\Big) d\mu(t+r) + \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta\in[t-r,t]} |h^{(i)}(\theta)|\Big) d\mu(t) \\
\le\ & \frac{1}{\nu([-\tau, \tau])} \int_{-\tau-r}^{+\tau+r} \Big(\sup_{\theta\in[t-r,t]} |h^{(i)}(\theta)|\Big) d\mu(t+r) + \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta\in[t-r,t]} |h^{(i)}(\theta)|\Big) d\mu(t) \\
\le\ & \Big[\frac{\nu([-\tau-r, \tau+r])}{\nu([-\tau, \tau])}\Big] \times \frac{1}{\nu([-\tau-r, \tau+r]} \int_{-\tau-r}^{+\tau+r} \Big(\sup_{\theta\in[t-r,t]} |h^{(i)}(\theta)|\Big) d\mu(t+r) \\
&+ \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta\in[t-r,t]} |h^{(i)}(\theta)|\Big) d\mu(t).
\end{aligned}$$

Consequently

$$\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\Big[\sup_{\xi\in[-r,0]}|h^{(i)}(\theta+\xi)|\Big]\Big)d\mu(t) \leq \Big[\frac{\nu([-\tau-r,\tau+r])}{\nu([-\tau,\tau])}\Big]\times M_r(\tau+r)$$
$$+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|h^{(i)}(\theta)|\Big)d\mu(t),$$

which shows using Lemma 4.12 and Lemma 4.13 that $\phi_t^{(i)}$ belongs to $PAP(C([-r,0];X),\mu,\nu,r)$ for $i=0,1,...,n$. Thus, we obtain the desired result.□

5. C^n-(μ,ν)-Pseudo almost periodic solutions of class r

In what follows, we will be looking at the existence of bounded integral solutions of class r of equation (1.1).

Proposition 5.1. *[10] Assume that ($\boldsymbol{H_0}$) and ($\boldsymbol{H_1}$) hold and the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic. If $f\in BC(\mathbb{R};X)$, then there exists a unique bounded solution u of equation (1.1) on $\mathbb{R}$, given by*

$$u_t=\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds \quad \textit{for } t\in\mathbb{R},$$

where Π^s and Π^u are the projections of C_0 onto the stable and unstable subspaces, respectively.

Proposition 5.2. *[?] Let $h\in AP(\mathbb{R};X)$ and Γ be the mapping defined for $t\in\mathbb{R}$ by*

$$\Gamma h(t)=\Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 h(s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 h(s))ds\Big](0).$$

Then $\Gamma h\in AP(\mathbb{R},X)$.

Corollary 5.3. *Let $h\in AP^{(n)}(\mathbb{R};X)$ and Γ be the mapping defined for $t\in\mathbb{R}$ by*

$$\Gamma h(t)=\Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 h(s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 h(s))ds\Big](0).$$

Then $\Gamma h\in AP^{(n)}(\mathbb{R},X)$.

Proof. In fact, since $h\in AP^{(n)}(\mathbb{R};X)$ then $h^{(i)}\in AP(\mathbb{R};X)$ and $\Gamma h^{(i)}\in AP(\mathbb{R},X)$ for $i=0,1,...,n$.□

Theorem 5.4. *Let $\mu,\nu\in\mathcal{M}$ satisfy ($\boldsymbol{H_3}$) and $g\in\mathcal{E}^{(n)}(\mathbb{R};X,\mu,\nu,r)$. Then $\Gamma g\in\mathcal{E}^{(n)}(\mathbb{R};X,\mu,\nu,r)$.*

Proof. In fact, since $g\in\mathcal{E}^{(n)}(\mathbb{R};X,\mu,\nu,r)$ then $g^{(i)}\in\mathcal{E}(\mathbb{R};X,\mu,\nu,r)$ for $i=0,1,...,n$. For $\tau>0$ we get

$$\begin{aligned}
\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}|\Gamma g^{(i)}(\theta)|ds\Big)d\mu(t) &\leq \overline{M}\widetilde{M}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}\int_{-\infty}^{\theta}e^{-\omega(\theta-s)}|\Pi^s|\ |g^{(i)}(s)|ds\Big)d\mu(t)\\
&+\overline{M}\widetilde{M}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}\int_{\theta}^{+\infty}e^{\omega(\theta-s)}|\Pi^u|\ |g^{(i)}(s)|ds\Big)d\mu(t)\\
&\leq \overline{M}\widetilde{M}|\Pi^s|\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}e^{\omega r}\int_{-\infty}^{\theta}e^{-\omega(t-s)}|g^{(i)}(s)|ds\Big)d\mu(t)\\
&+\overline{M}\widetilde{M}|\Pi^u|\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}\int_{\theta}^{+\infty}e^{\omega(t-s)}|g^{(i)}(s)|ds\Big)d\mu(t).
\end{aligned}$$

On the one hand using Fubini's theorem, we have

$$\begin{aligned}
\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}e^{\omega r}\int_{-\infty}^{\theta}e^{-\omega(t-s)}|g^{(i)}(s)|ds\Big)d\mu(t) &\leq \int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}e^{\omega r}\int_{-\infty}^{t}e^{-\omega(t-s)}|g^{(i)}(s)|ds\Big)d\mu(t)\\
&\leq e^{\omega r}\int_{-\tau}^{\tau}\int_{-\infty}^{t}e^{-\omega(t-s)}|g^{(i)}(s)|dsd\mu(t)\\
&\leq e^{\omega r}\int_{-\tau}^{\tau}\int_{0}^{+\infty}e^{-\omega s}|g^{(i)}(t-s)|dsd\mu(t)\\
&\leq e^{\omega r}\int_{0}^{+\infty}e^{-\omega s}\int_{-\tau}^{\tau}|g^{(i)}(t-s)|d\mu(t)ds.
\end{aligned}$$

By using Corollary 4.10, we deduce that

$$\begin{aligned}
&\lim_{\tau\to+\infty}\frac{e^{-\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}|g^{(i)}(t-s)|d\mu(t)\to 0 \ \text{ for all } \ s\in\mathbb{R}^+\\
&\text{and } \frac{e^{-\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}|g^{(i)}(t-s)|d\mu(t)\leq e^{-\omega s}|g|_{\infty}.
\end{aligned}$$

Since $g^{(i)}$ is a bounded function, then the function $s\mapsto e^{-\omega s}|g^{(i)}|_{\infty}$ belongs to $L^1([0,+\infty[)$, in view of the Lebesgue dominated convergence theorem, it follows that

$$e^{\omega r}\lim_{\tau\to+\infty}\int_{0}^{+\infty}e^{-\omega s}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}|g^{(i)}(t-s)|d\mu(t)ds=0.$$

On the other hand by Fubini's theorem, we also have

$$\begin{aligned}
\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}\int_{\theta}^{+\infty}e^{\omega(t-s)}|g^{(i)}(s)|ds\Big)d\mu(t) &\leq \int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}\int_{t-r}^{+\infty}e^{\omega(t-s)}|g^{(i)}(s)|ds\Big)d\mu(t)\\
&\leq \int_{-\tau}^{\tau}\int_{t-r}^{+\infty}e^{\omega(t-s)}|g^{(i)}(s)|dsd\mu(t)\\
&\leq \int_{-\tau}^{\tau}\int_{-\infty}^{r}e^{\omega s}|g^{(i)}(t-s)|dsd\mu(t)\\
&\leq \int_{-\infty}^{r}e^{\omega s}\int_{-\tau}^{\tau}|g^{(i)}(t-s)|d\mu(t)ds.
\end{aligned}$$

Since the function $s \mapsto e^{\omega s}|g^{(i)}|_\infty$ belongs to $L^1(]-\infty, r])$, resoning like above, it follows that

$$\lim_{\tau\to+\infty}\int_{-\infty}^{r} e^{\omega s}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}|g^{(i)}(t-s)|d\mu(t)ds = 0.$$

Consequently

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}|(\Gamma g^{(i)})(\theta)|\Big)d\mu(t) = 0,$$

which implies that $\Gamma g^{(i)} \in \mathcal{E}^{(n)}(\mathbb{R}; X, \mu, \nu, r)$ for $i = 0, 1, ..., n$.. Thus, we obtain the desired result.□

For the existence of C^n-(μ, ν)-pseudo almost periodic solution of class r, we make the following assumption.

$(\mathbf{H_5})$ $f : \mathbb{R} \to X$ is in C^n-(μ, ν)-pseudo almost periodic of class r.

Proposition 5.5. *Assume $(\boldsymbol{H_0})$, $(\boldsymbol{H_1})$, $(\boldsymbol{H_3})$ and $(\boldsymbol{H_5})$ hold. Then equation (1.1) has a unique $C^n - cl(\mu, \nu)$-pseudo almost periodic solution of class r.*

Proof. Since f is a C^n-(μ, ν)-pseudo almost periodic function, f has a decomposition $f = f_1 + f_2$ where $f_1 \in AP^{(n)}(\mathbb{R}; X)$ and $f_2 \in \mathcal{E}^{(n)}(\mathbb{R}; X, \mu, \nu, r)$. Using Proposition 5.1, Corollary 5.3 and Theorem 5.4, we get the desired result.□

Our next objective is to show the existence of C^n-(μ, ν)-pseudo almost periodic solutions of class r for the following problem

$$u'(t) = Au(t) + L(u_t) + f(t, u_t) \text{ for } t \in \mathbb{R} \tag{5.1}$$

where $f : \mathbb{R} \times C \to X$ is continuous.

For the sequel, we make the following assumption.

$(\mathbf{H_6})$ Let $\mu, \nu \in \mathcal{M}$ and $f : \mathbb{R} \times C([-r, 0]; X)) \to X$ C^n-$cl(\mu, \nu)$-pseudo almost periodic of class r such that there exists a continuous function $L_f : \mathbb{R} \to [0, +\infty[$ such that

$$|f(t, \varphi_1) - f(t, \varphi_2)| \le L_f(t)|\varphi_1 - \varphi_2| \quad \text{for all } \ t \in \mathbb{R} \ \text{ and } \ \varphi_1, \varphi_2 \in C([-r, 0]; X)$$

and L_f satisfies (4.3).

Theorem 5.6. *Assume $(\boldsymbol{H_0})$, $(\boldsymbol{H_1})$, $(\boldsymbol{H_2})$, $(\boldsymbol{H_4})$ and $(\boldsymbol{H_6})$ hold. If*

$$\overline{M}\widetilde{M}\sup_{t\in\mathbb{R}}\Big(|\Pi^s|\int_{-\infty}^{t} e^{-\omega(t-s)}L_f(s)ds + |\Pi^u|\int_{t}^{+\infty} e^{\omega(t-s)}L_f(s)ds\Big) < 1.$$

Then equation (5.1) has a unique C^n-$cl(\mu, \nu)$-pseudo almost periodic solution of class r.

Proof. Let x be a function in $PAP^{(n)}(\mathbb{R}; X, \mu, \nu, r)$, from Theorem 4.14 the function $t \to x_t$ belongs to $PAP(C([-r, 0]; X), \mu, r)$. Hence Theorem 4.11 implies that the function $g(.) := f(., x_.)$ is in $PAP^{(n)}(\mathbb{R}; X, \mu, r)$. Consider the mapping

$$\mathcal{H} : PAP^{(n)}(\mathbb{R}; X, \mu, \nu, r) \to PAP^{(n)}(\mathbb{R}; X, \mu, \nu, r)$$

defined for $t \in \mathbb{R}$ by

$$(\mathcal{H}x)(t) = \Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s,x_s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s,x_s))ds\Big](0).$$

From Proposition 5.1, Corollary 5.3 and Theorem 5.4, it suffices now to show that the operator $\mathcal{H}$ has a unique fixed point in $PAP^{(n)}(\mathbb{R};X,\mu,\nu,r)$. Let $x_1,x_2 \in PAP^{(n)}(\mathbb{R};X,\mu,\nu,r)$. Then for $i=0,1,...,n$, we have

$$\begin{aligned}
& |\ \mathcal{H}x_1^{(i)}(t) - \mathcal{H}x_2^{(i)}(t)| \\
\leq\ & \Big|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0[f((s,x_{1s}^{(i)})) - f((s,x_{1s}^{(i)}))]ds\Big| \\
& +\Big|\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^s(t-s)\Pi^u(\widetilde{B}_\lambda X_0[f((s,x_{2s}^{(i)})) - f((s,x_{2s}^{(i)}))]ds\Big| \\
\leq\ & \overline{M}\widetilde{M}\Big(|\Pi^s|\int_{-\infty}^{t}e^{-\omega(t-s)}L_f(s)|x_{1s}^{(i)} - x_{2s}^{(i)}|ds + |\Pi^u|\int_{t}^{+\infty}e^{\omega(t-s)}L_f(s)|x_{1s}^{(i)} - x_{2s}^{(i)}|ds\Big) \\
\leq\ & \overline{M}\widetilde{M}\sup_{t\in\mathbb{R}}\Big(|\Pi^s|\int_{-\infty}^{t}e^{-\omega(t-s)}L_f(s)ds + |\Pi^u|\int_{t}^{+\infty}e^{\omega(t-s)}L_f(s)ds\Big)|x_1^{(i)} - x_2^{(i)}|_\infty,
\end{aligned}$$

which implies that

$$\begin{aligned}
& \sum_{i=0}^{n}|\ \mathcal{H}x_1^{(i)}(t) - \mathcal{H}x_2^{(i)}(t)| \\
& \leq \overline{M}\widetilde{M}\sup_{t\in\mathbb{R}}\Big(|\Pi^s|\int_{-\infty}^{t}e^{-\omega(t-s)}L_f(s)ds + |\Pi^u|\int_{t}^{+\infty}e^{\omega(t-s)}L_f(s)ds\Big)|x_1 - x_2|_n.
\end{aligned}$$

This means that $\mathcal{H}$ is a strict contraction. Thus by Banach's fixed point theorem, $\mathcal{H}$ has a unique fixed point u in $PAP^{(n)}(\mathbb{R};X,\mu,\nu,r)$. We conclude that equation (5.1), has one and only one C^n-$cl(\mu,\nu)$-pseudo almost periodic solution of class r.□

Proposition 5.7. *Assume ($\boldsymbol{H_0}$), ($\boldsymbol{H_1}$), ($\boldsymbol{H_2}$) and, ($\boldsymbol{H_4}$) and f is lipschitz continuous with respect the second argument. If*

$$Lip(f) < \frac{\omega}{\overline{M}\widetilde{M}(|\Pi^s| + |\Pi^u|)}$$

then equation (5.1) has a unique C^n-$cl(\mu,\nu)$-pseudo almost periodic solution of class r, where $Lip(f)$ is the lipschitz constant of f.

Proof. Let us pose $k = Lip(f)$, for $i = 0,1,...,n$, we have

$$\begin{aligned}
|\ \mathcal{H}x_1^{(i)}(t) - \mathcal{H}x_2^{(i)}(t)| \ &\leq\ \overline{M}\widetilde{M}\Big(|\Pi^s|\int_{-\infty}^{t}e^{-\omega(t-s)}k|x_{1s}^{(i)} - x_{2s}^{(i)}|ds + |\Pi^u|\int_{t}^{+\infty}e^{\omega(t-s)}k|x_{1s}^{(i)} - x_{2s}^{(i)}|ds\Big) \\
&\leq\ \frac{k\overline{M}\widetilde{M}(|\Pi^s| + |\Pi^u|)}{\omega}|x_1^{(i)} - x_2^{(i)}|_\infty,
\end{aligned}$$

which implies that

$$\sum_{i=0}^{n}|\ \mathcal{H}x_1^{(i)}(t) - \mathcal{H}x_2^{(i)}(t)| \ \leq\ \frac{k\overline{M}\widetilde{M}(|\Pi^s| + |\Pi^u|)}{\omega}|x_1 - x_2|_n.$$

Consequently $\mathcal{H}$ is a strict contraction if $k < \dfrac{\omega}{\overline{M}\widetilde{M}(|\Pi^s| + |\Pi^u|)}$.$\square$

6. Application

For illustration, we propose to study the existence of solutions for the following model

$$\begin{cases} \dfrac{\partial}{\partial t}z(t,x) = \dfrac{\partial^2}{\partial x^2}z(t,x) + \displaystyle\int_{-r}^{0} G(\theta)z(t+\theta,x))d\theta + \exp(\sin t + \sin(\sqrt{2}t)) + \cos(t) \\ \qquad\qquad + \displaystyle\int_{-r}^{0} h(\theta, z(t+\theta,x))d\theta \text{ for } t\in\mathbb{R} \text{ and } x\in[0,\pi] \\ z(t,0) = z(t,\pi) = 0 \text{ for } t\in\mathbb{R}, \end{cases} \tag{6.1}$$

where $G : [-r, 0] \to \mathbb{R}$ is a continuous function and $h : [-r, 0] \times \mathbb{R} \to \mathbb{R}$ is continuous and lipschitzian with respect to the second argument. To rewrite equation (6.1) in the abstract form, we introduce the space $X = C_0([0, \pi]; \mathbb{R})$ of continuous function from $[0, \pi]$ to $\mathbb{R}^+$ equipped with the uniform norm topology. Let $A : D(A) \to X$ be defined by

$$\begin{cases} D(A) = \{y \in X \cap C^2([0,\pi],\mathbb{R}) : \ y'' \in X\} \\ Ay = y''. \end{cases}$$

Then A satisfied the Hille-Yosida condition in X. Moreover the part A_0 of A in $\overline{D(A)}$ is the generator of strongly continuous compact semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$. It follows that $(\mathbf{H_0})$ and $(\mathbf{H_1})$ are satisfied.

We define $f : \mathbb{R} \times C \to X$ and $L : C \to X$ as follows

$$\begin{aligned} f(t,\varphi)(x) &= \exp(\sin t + \sin(\sqrt{2}t)) + \cos(t) + \int_{-r}^{0} h(\theta, \varphi(\theta)(x))d\theta \ \text{ for } x\in[0,\pi] \ \text{ and } \ t\in\mathbb{R}, \\ L(\varphi)(x) &= \int_{-r}^{0} G(\theta)\varphi(\theta)(x))d\theta \ \text{ for } \ -r\leq\theta\leq 0 \ \text{ and } \ x\in[0,\pi]. \end{aligned}$$

Let us pose $v(t) = z(t, x)$. Then equation (6.1) takes the following abstract form

$$v'(t) = Av(t) + L(v_t) + f(t, v_t) \text{ for } t \in \mathbb{R}. \tag{6.2}$$

Consider the measures μ and ν where its Radon-Nikodym derivative are respectively $\rho_1, \rho_2 : \mathbb{R} \to \mathbb{R}$ defined by

$$\rho_1(t) = \begin{cases} 1 \ \text{ for } \ t > 0 \\ e^t \ \text{ for } \ t \leq 0. \end{cases}$$

and

$$\rho_2(t) = |t| \ \text{ for } \ t \in \mathbb{R}$$

i.e $d\mu(t) = \rho_1(t)dt$ and $d\nu(t) = \rho_2(t)dt$ where dt denotes the Lebesgue measure on $\mathbb{R}$ and

$$\mu(A) = \int_A \rho_1(t)dt \ \text{ for } \ \nu(A) = \int_A \rho_2(t)dt \ \text{ for } \ A \in \mathcal{B}.$$

From [5] $\mu, \nu \in \mathcal{M}$, μ, ν satisfy Hypothesis $(\mathbf{H_4})$ and $\exp(\sin t + \sin(\sqrt{2}t))$ is C^n-almost periodic. We have

$$\limsup_{\tau\to+\infty} \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} = \limsup_{\tau\to+\infty} \frac{\displaystyle\int_{-\tau}^{0} e^t dt + \int_0^\tau dt}{\displaystyle 2\int_0^\tau t dt} = \limsup_{\tau\to+\infty} \frac{1-e^{-\tau}+\tau}{\tau^2} = 0 < \infty,$$

which implies that $(\mathbf{H_2})$ is satisfied.

For all $t \in \mathbb{R}$, $|\cos^{(i)} t| \le 1$ for $i = 0, 1, ..., n$, which implies that:

$$\begin{aligned} \lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \sup_{\theta\in[t-r,t]} |\cos^{(i)}(\theta)| d\mu(t) &= \lim_{\tau\to+\infty} \Big(\frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{0} \sup_{\theta\in[t-r,t]} |\cos^{(i)}(\theta)| e^t dt \\ &\quad + \frac{1}{\nu([-\tau,\tau])} \int_0^\tau \sup_{\theta\in[t-r,t]} |\cos^{(i)}(\theta)| dt \Big) \\ &\le \lim_{\tau\to+\infty} \Big(\frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{0} e^t dt + \frac{1}{\nu([-\tau,\tau])} \int_0^\tau dt \Big) \\ &\le \lim_{\tau\to+\infty} \frac{1-e^{-\tau}+\tau}{\tau^2} = 0. \end{aligned}$$

It follows that $t \mapsto \cos^{(i)} t$ belongs to $\mathcal{E}^{(n)}(\mathbb{R}; X, \mu, \nu, r)$, consequently, $f \in PAP^{(n)}(\mathbb{R}; X, \mu, \nu, r)$. Moreover, L is a bounded linear operator from C to X.
Let k be the lipschiz constant of h, then for every $\varphi_1, \varphi_2 \in C$ and $t \ge 0$, we have

$$\begin{aligned} |f(t,\varphi_1) - f(t,\varphi_2)| &= r \sup_{0\le x\le\pi} |h(t,\varphi_1)(x) - h(t,\varphi_2)(x)| \\ &\le kr \sup_{\substack{-r<\theta\le 0 \\ 0\le x\le\pi}} |\varphi_1(\theta)(x) - \varphi_2(\theta)(x)|. \end{aligned}$$

Consequently, we conclude that f is Lipschitz continuous.
For the hyberbolicity, we suppose that

$(\mathbf{H_7})$ $\displaystyle\int_{-r}^{0} |G(\theta)| d\theta < 1.$

Lemma 6.1. *[10] Assume that $(\boldsymbol{H_7})$ holds. Then the semigroup $(\mathcal{U}(t))_{t\ge0}$ is hyperbolic.*

Then by Proposition 5.7 we deduce the following result.

Theorem 6.2. *Under the above assumptions, if $Lip(h)$ is small enough, then equation (6.2) has a unique C^n-$cl(\mu,\nu)$-pseudo almost periodic solution v of class r.*

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