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PSEUDO ALMOST PERIODIC SOLUTIONS OF INFINITE CLASS IN THE α-NORM UNDER THE LIGHT OF MEASURE THEORY

DJENDODE MBAINADJI[1,*], DJOKATA VOTSIA[2] AND ISSA ZABSONRE[3]

Abstract. The aim of this work is to study weighted pseudo almost periodic functions with infinite delay via measure theory. Using the Banach fixed point theorem and the techniques of fractional powers of an operator, we establish the existence and uniqueness of (μ,ν)-pseudo almost periodic solutions in the α-norm of the infinite class for some functional differential in Banach space when the delay is unbounded.

1. Introduction

In this paper we present a new approach to the study of weighted pseudo-almost periodic functions and their applications in evolution equations and partial functional differential equations. We use measure theory to define an ergodic function and study many interesting properties of such functions. Weighted pseudo-almost periodic functions. We study the existence and uniqueness of weighted pseudo almost periodic functions in the α norm for the following partial functional differential equation

$$u'(t) = -Au(t) + L(u_t) + f(t) \text{ for } t \in \mathbb{R}, \tag{1.1}$$

where $-A : D(A) \to X$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators on Banach space X, L is a bounded linear operator from $\mathscr{B}_\alpha$ into X and $f : \mathbb{R} \to X$ is a continuous function. The phase space $\mathcal{B}_\alpha$ defined by

$$\mathcal{B}_\alpha = \{\varphi \in \mathcal{B} : \varphi(\theta) \in D(A^\alpha) \text{ for } \theta \leq 0 \text{ and } A^\alpha\varphi \in \mathcal{B}\} \text{ with } \|\varphi\|_{\mathcal{B}_\alpha} = \|A^\alpha\varphi\|_{\mathcal{B}},$$

is a subset of $\mathcal{B}$, where $A^\alpha\varphi$ is defined by $A^\alpha\varphi(\theta) = A^\alpha(\varphi(\theta))$ for $\theta \in]-\infty,0]$ and $\mathcal{B}$ is a Banach space of function mappings from $]-\infty,0]$ into X and satisfying some axioms that will be introduced later. A^α is fractional α-power of A that will be describe later. For every $t \geq 0$, $u \in \mathcal{B}_\alpha$ the history function $u_t \in \mathcal{B}_\alpha$ defined by

$$u_t(\theta) = u(t+\theta) \text{ for } \theta \leq 0,$$

[1]Université Polytechnique de Mongo, Faculté des Sciences Fondamantales, Département de Mathématiques et Informatiques, B.P.4377, Mongo, Tchad

[2]École Nationale Supérieure des Travaux Publics, B.P. 60 , Ndjaména, Tchad

[3]Université Joseph KI-ZERBO, Unité de Recherche et de Formation en Sciences Exactes et Appliquées, Département de Mathématiques B.P.7021 Ouagadougou 03, Burkina Faso

*Corresponding author

E-mail addresses: mbainadjidjendode@gmail.com, vot.djo@mail.com, zabsonreissa@mail.com.

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L is a bounded linear operator from $\mathcal{B}_\alpha$ into X and $f : \mathbb{R}^+ \to X$ is a continuous function.

The study of weighted pseudo almost periodic functions started recently and became an interesting field in dynamical systems due both to its mathematical interest and applications in physics, mathematical biology, and control theory, among other areas. For this purpose some recent contributions concerning pseudo almost periodic solutions for abstract differential equations similar to equation (1.1) have been made. See for instance [3,5,6] and the references therein.
In [4], the authors presented a new approach to study weighted pseudo almost periodic functions using the measure theory. They presented a new concept of weighted periodic functions which is more general than the classical one. Then they establish many interesting results on the functional space of such functions like completeness and composition theorems.
In [8], the authors, investigated the existence and uniqueness of pseudo almost periodic solutions for some neutral partial functional differential equations in a Banach space when the delay is distributed.
Recently, [19], Issa Zabsonre and Djokata Votsia studied the existence and uniqueness of (μ, ν)-pseudo almost periodic solutions of infinite class for some neutral partial functional differential equations in a Banach space when the delay is distributed on $]-\infty, 0]$.
The organization of this work is as follows, in section 2 we recall some preliminary results about analytic semigroups and fractional power associated to its generator will be used throughout this work. In section 3, we recall some preliminary results variation constants formula and spectral decomposition. In section 4, we recall some preliminary results on (μ, ν)-pseudo almost automorphic functions and neutral partial functional differential equations that will be used in this work. In section 5, we give some properties of (μ, ν)-pseudo almost periodic functions of infinite class. In section 6, we discuss the main result of this paper. Using the strict contraction principle and we show the existence and uniqueness of (μ, ν)-pseudo almost periodic solution of infinite class for equation (1.1). The last section is devoted to some applications arising in population dynamics.

2. Analytic Semigroup

Let $(X, \|.\|)$ be a Banach space, let α be a constant such that $0 < \alpha < 1$ and let $-A$ be the infinitesimal generator of a bounded analytic semigroup of linear operator $(T(t))_{t\geq 0}$ on X. We assume without loss of generality that $0 \in \rho(A)$. Note that if the assumption $0 \in \rho(A)$ is not satisfied, one can substitute the operator A by the operator $(A - \sigma I)$ with σ large enough such that $0 \in \rho(A - \sigma I)$. This allows us to define the fractional power A^α for $0 < \alpha < 1$, as a closed linear invertible operator with domain $D(A^\alpha)$ dense in X. The closeness of A^α implies that $D(A^\alpha)$, endowed with the graph norm of A^α, $|x| = \|x\| + \|A^\alpha x\|$, is a Banach space. Since A^α is invertible, its graph norm $|.|$ is equivalent to the norm $|x|_\alpha = \|A^\alpha x\|$. Thus, $D(A^\alpha)$ equipped with the norm $|.|_\alpha$, is a Banach space, which we denote by X_α.

$(\mathbf{H}_0)$ The operator $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on the Banach space X. Moreover, we assume that $0 \in \rho(A)$.

Proposition 2.1. *[17] Let $0<\alpha<1$ and assume that $(\boldsymbol{H}_0)$ hold. Then we have*
i) $T(t): X \to D(A^\alpha)$ for every $t>0$,
ii) $T(t)A^\alpha x = A^\alpha T(t)x$ for every $x \in D(A^\alpha)$ and $t \geq 0$,
iii) for every $t>0$, $A^\alpha T(t)$ is bounded on X and there exist $M_\alpha > 0$ and $\omega > 0$ such that

$$\|A^\alpha T(t)\| \leq M_\alpha e^{-\omega t} t^{-\alpha} \text{ for } t>0,$$

iv) if $0<\alpha \leq \beta < 1$, then $D(A^\beta) \hookrightarrow D(A^\alpha)$.
v) There exists $N_\alpha > 0$ such that

$$\|(T(t)-I)A^{-\alpha}\| \leq N_\alpha t^\alpha \text{ for } t>0.$$

Recall that $A^{-\alpha}$ is given by the following formula

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} T(t) dt,$$

where the integral converges in the uniform operator topology for every $\alpha > 0$. Consequently, if $T(t)$ is compact for each $t>0$, then $A^{-\alpha}$ is compact.

3. Variation Constants Formula and Spectral Decomposition

In this work, we assume that the state space $(\mathcal{B}, |.|_{\mathcal{B}})$ is a normed linear space of functions mapping $]-\infty, 0]$ into X and satisfying the following fundamental axioms.

$(\mathbf{A}_1)$ There exist a positive constant H and functions $K(.), M(.) : \mathbb{R}^+ \to \mathbb{R}^+$, with K continuous and M locally bounded, such that for any $\sigma \in \mathbb{R}$ and $a>0$, if $u :]-\infty, a] \to X$, $u_\sigma \in \mathcal{B}$, and $u(.)$ is continuous on $[\sigma, \sigma+a]$, then for every $t \in [\sigma, a]$ the following conditions hold

(i) $u_t \in \mathcal{B}$,

(ii) $|u(t)| \leq H|u_t|$, which is equivalent to $|\varphi(0)| \leq H|\varphi|_{\mathcal{B}}$ for every $\varphi \in \mathcal{B}$,

(iii) $|u_t| \leq K(t-\sigma) \sup\limits_{\sigma \leq s \leq t} |u(s)| + M(t-\sigma)|u_\sigma|_{\mathcal{B}}$.

$(\mathbf{A}_2)$ For the function $u(.)$ in $(\mathbf{A}_1)$, $t \mapsto u_t$ is a $\mathcal{B}$-valued continuous function for $t \in [\sigma, \sigma+a]$.

$(\mathbf{B})$ The space $\mathcal{B}$ is a Banach space.

$(\mathbf{H}_1)$ $A^{-\alpha}\varphi \in \mathcal{B}$ for $\varphi \in \mathcal{B}$, the function $A^{-\alpha}\varphi$ is defined by

$$(A^{-\alpha}\varphi)(\theta) = A^{-\alpha}\varphi(\theta).$$

Consequently we get the following result.

Lemma 3.1. *[7] Assume that $(\boldsymbol{H}_0)$ and $(\boldsymbol{H}_1)$ hold. If $\mathcal{B}$ satisfies axioms $(\boldsymbol{A}_1)$, $(\boldsymbol{A}_2)$ and $(\boldsymbol{B})$. Then $\mathcal{B}_\alpha$ satisfies axioms $(\boldsymbol{A}_1)$, $(\boldsymbol{A}_2)$ and $(\boldsymbol{B})$.*
$(\boldsymbol{A}_1)$ if $u :]-\infty, a] \to X_\alpha$ is continuous on $[\sigma, a]$ with $x_\sigma \in \mathcal{B}_\alpha$ for some $\sigma < a$, then for all $t \in [\sigma, a]$,

(*i*) $u_t \in \mathcal{B}_\alpha$,
(ii) $|u(t)|_\alpha \leq H|u_t|_\alpha$, *which is equivalent to* $|\varphi(0)|_\alpha \leq H|\varphi|_\alpha$ *for every* $\varphi \in \mathcal{B}_\alpha$,
(iii) $|u_t|_\alpha \leq K(t-\sigma) \sup_{\sigma \leq s \leq t} |u(s)|_\alpha + M(t-\sigma)|u_\sigma|_\alpha$.
($\boldsymbol{A}_2$) *For the function* $u(.)$ *in* ($\boldsymbol{A}_1$), $t \to u_t$ *is* $\mathcal{B}_\alpha$*-valued continuous function for* $t \in [\sigma, \sigma + a]$.,

*(**B**) The space* $\mathcal{B}_\alpha$ *is a Banach space.*

Let

$$\mathcal{B}_\alpha = \{\varphi \in \mathcal{B} : \ \varphi(\theta) \in D(A^\alpha) \text{ for } \theta \leq 0 \text{ and } A^\alpha \varphi \in \mathcal{B}\} \ \text{ with } \ \|\varphi\|_{\mathcal{B}_\alpha} = \|A^\alpha \varphi\|_{\mathcal{B}},$$

We suppose that the phase space $\mathcal{B}$ satisfies the following axioms.

($\mathbf{C}_1$) If $(\varphi_n)_{n\geq 0}$ is the Cauchy sequence in $\mathcal{B}$ such that $\varphi_n \to 0$ in $\mathcal{B}$ as $n \to +\infty$, then, $(\varphi_n(\theta))_{n\geq 0}$ converges to 0 in X.

Let $C(]-\infty, 0], X)$ be the space of continuous functions from $]-\infty, 0]$ into X We suppose the following assumptions hold:

(C_2) $\mathcal{B} \subset C(]-\infty, 0], X)$.

(C_3) there exists $\lambda_0 \in \mathbb{R}$ such that for all $\lambda \in \mathbb{C}$ with $Re\ \lambda > \lambda_0$ and $x \in X$ we have $e^\lambda x \in \mathcal{B}$ and

$$K_0 = \sup_{Re\lambda > \lambda_0,\ x \in X,\ x \neq 0} \frac{|e^\lambda x|_{\mathcal{B}}}{|x|} < \infty,$$

where $(e^\lambda x)(\theta) = e^{\lambda\theta} x$ for $\theta \in]-\infty, 0]$ and $x \in X$.

To equation (1.1), we associate the following initial value problem

$$\begin{cases} \dfrac{d}{dt}u(t) = -Au(t) + L(u_t) + f(t) \text{ for } t \geq 0 \\ \\ u_0 = \varphi \in \mathcal{B}_\alpha, \end{cases} \tag{3.1}$$

where $f : \mathbb{R}^+ \to X$ is a continuous function.
For each $t \geq 0$, we define the linear operator $\mathcal{U}(t)$ on $\mathcal{B}_\alpha$ by

$$\mathcal{U}(t) = v_t(., \varphi),$$

where $v(., \varphi)$ is the solution of the following homogeneous equation

$$\begin{cases} \dfrac{d}{dt}v(t) = -Av(t) + L(v_t) \text{ for } t \geq 0 \\ \\ v_0 = \varphi \in \mathcal{B}_\alpha. \end{cases}$$

Proposition 3.2. *[16] Assume that $\mathcal{B}$ satisfies $(\boldsymbol{A}_1)$, $(\boldsymbol{A}_2)$, $(\boldsymbol{B})$, $(\boldsymbol{C}_1)$ and $(\boldsymbol{C}_2)$, then the generator $\mathcal{A}_{\mathcal{U}}$ of $(\mathcal{U}(t))_{t\geq 0}$ is defined on $\mathcal{B}_\alpha$ by*

$$\begin{cases} D(\mathcal{A}_{\mathcal{U}}) = \{\varphi \in \mathcal{B}_\alpha, \varphi' \in \mathcal{B}_\alpha, \varphi(0) \in (D(A), \varphi(0)' \in \overline{D(A)} \text{ and } \varphi(0)' = -A\varphi(0) + L(\varphi)\} \\ \\ \mathcal{A}_{\mathcal{U}}\varphi = \varphi' \in D(\mathcal{A}_{\mathcal{U}}). \end{cases}$$

Then $\mathcal{A}_{\mathcal{U}}$ is the infinitesimal generator of the semigroup $(\mathcal{U}(t))_t \geq 0$ on $\mathcal{B}_\alpha$.

Let $\langle X_0 \rangle$ be the space defined by $\langle X_0 \rangle$=$\{X_0 c : c \in X\}$, where the function $X_0 c$ is defined by

$$(X_0 c)(\theta) = \begin{cases} 0 \text{ if } \theta \in]-\infty, 0[\\ \\ c \text{ if } \theta = 0. \end{cases}$$

Consider the extension $\mathcal{A}_{\mathcal{U}}$ defined on $\mathcal{B}_\alpha \oplus \langle X_0 \rangle$ by

$$\begin{cases} D(\widetilde{\mathcal{A}_{\mathcal{U}}}) = \left\{\varphi \in C^1(]-\infty, 0], X_\alpha) : \varphi(0) \in D(A) \text{ and } \varphi(0)' \in \overline{D(A)}\right\} \\ \\ \widetilde{\mathcal{A}}_{\mathcal{U}}\varphi = X_0(A\varphi(0) + L(\varphi) - \varphi(0)'). \end{cases}$$

Lemma 3.3. *[1] Assume that $\mathcal{B}$ satisfies $(\boldsymbol{A}_1)$, $(\boldsymbol{A}_2)$, $(\boldsymbol{B})$, $(\boldsymbol{C}_1)$, $(\boldsymbol{C}_2)$ and $(\boldsymbol{C}_3)$ hold. Suppose that $(\boldsymbol{H}_0)$ holds. Then, $\mathcal{A}_{\mathcal{U}}$ satisfies the Hile-Yosida condition on $\mathcal{B}_\alpha \oplus \langle X_0 \rangle$ there exist $\widetilde{M} \geq 0$, $\widetilde{\omega} \in \mathbb{R}$ such that $]\widetilde{\omega}, +\infty[\subset \rho(\widetilde{\mathcal{A}_{\mathcal{U}}})$ and*

$$\left|(\lambda I - \widetilde{\mathcal{A}_{\mathcal{U}}})^{-n}\right| \leq \frac{\widetilde{M}}{(\lambda - \widetilde{\omega})^n} \text{ for } n \in \mathbb{N} \text{ and } \lambda > \widetilde{\omega}.$$

Let C_{00} be the space of X-valued continuous function on $]-\infty, 0]$ with compact support. Assume that:

(D) If φ is a Cauchy sequence in B and converges compactly to φ on $]-\infty, 0]$,then $\varphi \in \mathcal{B}$ and $|\varphi_n - \varphi| \to 0$.

Proposition 3.4. *[16] The family $(\mathcal{U}(t))_{t\geq 0}$ is a strongly semigroup on $\mathcal{B}_\alpha$, that is*

(***i***) $\mathcal{U}(0) = I$,
(***ii***) $\mathcal{U}(t+s) = \mathcal{U}(t)\mathcal{U}(s)$, *for* $t, s \geq 0$,
(***iii***) *for all* $\varphi \in \mathcal{B}_\alpha$, $\mathcal{U}(t)(\varphi)$ *is a continuous function of* $t \geq 0$ *with values in* $\mathcal{B}_\alpha$,
(***iv***) $\mathcal{U}(t)$ *satisfies the translation property, that's for* $t \geq 0$ *and* $\theta \leq 0$, *one has*

$$(\mathcal{U}(t)(\varphi))(\theta) = \begin{cases} (\mathcal{U}(t+\theta)(\varphi))(0), \text{ for } t+\theta \geq 0 \\ \\ \varphi(t+\theta) \text{ for } t+\theta \leq 0. \end{cases}$$

For $\varphi \in \mathcal{B}$ and $\theta \leq 0$, we define the linear operator W by

$$[W(t)\varphi](\theta) = \begin{cases} \varphi(0), \text{ if } t+\theta \geq 0 \\ \\ \varphi(t+\theta), \text{ if } t+\theta < 0. \end{cases}$$

$(W(t))_{t\geq 0}$ is exactly the solution semigroup associated to the following equation

$$\begin{cases} \dfrac{d}{dt}u(t) = 0 \\ \\ u_0 = 0. \end{cases}$$

Let $W_0(t) = W(t)_{/\widetilde{\mathcal{B}}}$, where $\widetilde{\mathcal{B}} := \{\varphi \in \mathcal{B} : \varphi(0) = 0\}$.

Definition 3.5. *[16] $\mathcal{B}$ is called a uniform fading memory space if it satisfies axioms $\mathcal{B}$ satisfies $(\boldsymbol{A}_1)$, $(\boldsymbol{A}_2)$, $(\boldsymbol{B})$, $(\boldsymbol{C}_1)$ and $(\boldsymbol{C}_2)$, $(\boldsymbol{D})$ and $\|W_0(t)\| \to 0$ as $t \to +\infty$.*

Lemma 3.6. *[14] If $\mathcal{B}$ is uniform fading memory space, then we can choose the function $K()$ and the function $M()$ such that $M(t) \to 0$ as $t \to +\infty$.*

Proposition 3.7. *[14] If the phase space $\mathcal{B}$ is a fading memory space, then the space $BC(]-\infty, 0], X]$ of bounded continuous X-valued functions on $]-\infty, 0]$ endowed with the uniform norm topology is continuous embedding in $\mathcal{B}$. In particular $\mathcal{B}$ satisfies $(\boldsymbol{C}_3)$, for $\lambda_0 > 0$.*

The following theorem gives a variation of constants formula for equation (3.1) in $\mathcal{B}_\alpha$.

Theorem 3.8. *[1] Assume that $(\boldsymbol{C}_1)$, $(\boldsymbol{C}_2)$ and $(\boldsymbol{C}_3)$ hold. Suppose that $(\boldsymbol{H}_0)$ holds. Then, for all $\varphi \in \mathcal{B}_\alpha$, the solution u of equation (3.1) is given by the following variation of constants formula*

$$u_t = \mathcal{U}(t)\varphi + \lim_{\lambda \to +\infty} \int_0^t \mathcal{U}(t-s)\widetilde{B}_\lambda(X_0 f(s))ds \text{ for } t \geq 0,$$

where $\widetilde{B}_\lambda = \lambda(\lambda I - \widetilde{\mathcal{A}_{\mathcal{U}}})^{-1}$ for $\lambda > \widetilde{\omega}$.

Definition 3.9. *We say a semigroup, $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic if*

$$\sigma(\mathcal{A}_{\mathcal{U}}) \cap i\mathbb{R} = \varnothing.$$

We make the following assumption.

$(\mathbf{H}_2)$ $T(t)$ is compact on X for each $t > 0$.

We have the following result on the spectral decomposition of the phase space $\mathcal{B}_\alpha$.

Theorem 3.10. *[7] Assume that $(\boldsymbol{H}_0)$, $(\boldsymbol{H}_1)$ and $(\boldsymbol{H}_2)$ hold. Suppose that $\mathcal{B}$ is a uniform fading memory space and the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic. Then $\mathcal{B}_\alpha$ is decomposed as a direct sum*

$$\mathcal{B}_\alpha = S \oplus U,$$

of two $\mathcal{U}(t)$ invariant closed subspaces S and U such that the restriction of $(\mathcal{U}(t))_{t\geq 0}$ on U is a group and there exist positive constants $\overline{M}$ and ω such that

$$|\mathcal{U}(t)\varphi|_\alpha \leq \overline{M}e^{-\omega t}|\varphi|_\alpha \ \textit{ for } t \geq 0 \ \textit{ and } \ \varphi \in S,$$

$$|\mathcal{U}(t)\varphi|_\alpha \leq \overline{M}e^{\omega t}|\varphi|_\alpha \ \textit{ for } \ t \leq 0 \textit{ and } \ \varphi \in U,$$

where S and U are called respectively the stable and unstable space, Π^s and Π^u denote respectively the projection operator on S and U.

4. (μ,ν)-Pseudo Almost Periodic Functions

In this section, we recall some properties about pseudo almost periodic functions. Let $BC(\mathbb{R}, X)$ be the space of all bounded and continuous function from $\mathbb{R}$ to X equipped with the uniform topology norm.
We denote by $\mathcal{N}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{N}$ satisfying $\nu(\mathbb{R}) = +\infty$ and $\mu([a,b]) < \infty$, for all $a, b \in \mathbb{R}$, $a \le b$.

Definition 4.1. *A bounded continuous function $\phi : \mathbb{R} \to X$ is called almost periodic, if for each $\varepsilon > 0$, there exists a relatively dense subset of $\mathbb{R}$ denote by $\mathcal{K}(\varepsilon, \phi, X)$ such that $|\phi(t+\tau) - \phi(t)| < \varepsilon$ for all $(t,\tau) \in \mathbb{R} \times \mathcal{K}(\varepsilon, \phi, X)$.*

We denote by $AP(\mathbb{R}, X)$, the space of all such functions.

Definition 4.2. *Let X_1 and X_2 be two Banach spaces. A bounded continuous function $\phi : \mathbb{R} \times X_1 \to X_2$ is called almost periodic in $t \in \mathbb{R}$ uniformly in $x \in X_1$, if for each $\varepsilon > 0$, there exists a relatively dense subset of $\mathbb{R}$ denote by $\mathcal{K}(\varepsilon, \phi, X)$ such that $|\phi(t+\tau, x) - \phi(t, x)| < \varepsilon$ for all $t \in \mathbb{R}$, $x \in X_1$, and $\tau \in \mathcal{K}(\varepsilon, \phi, X)$.*

We denote by $AP(\mathbb{R} \times X_1, X_2)$, the space of all such functions.
The next lemma gives a characterization of almost periodic functions.

Lemma 4.3. *A function $\phi \in C(\mathbb{R}; X)$ is almos periodic if and only if the spaces of functions $\{\phi_\tau : \tau \in \mathbb{R}\}$, where $(\phi_\tau)(t) = \phi(\tau + t)$ is relatively compact in $BC(\mathbb{R}; X)$.*

In the sequel, we recall some preliminary results concerning the (μ,ν)-pseudo almost periodic functions with infinite delay.

$$\mathscr{E}(\mathbb{R}; X_\alpha, \mu, \nu) = \Big\{ u \in BC(\mathbb{R}; X_\alpha) : \lim_{\tau \to +\infty} \frac{1}{\nu[-\tau,\tau]} \int_{-\tau}^{+\tau} |u(t)|_\alpha d\mu(t) = 0 \Big\}.$$

To study delayed differential equations for which the history belongs to $\mathcal{B}_\alpha$, we need to introduce the space

$$\mathscr{E}(\mathbb{R}; X_\alpha, \mu, \nu, \infty) = \Big\{ u \in BC(\mathbb{R}; X_\alpha) : \lim_{\tau \to +\infty} \frac{1}{\nu[-\tau,\tau]} \int_{-\tau}^{+\tau} \Big(\sup_{\theta \in]-\infty, t]} |u(t)|_\alpha \Big) d\mu(t) = 0 \Big\}.$$

In addition to above-mentioned spaces, we consider the following spaces

$$\mathscr{E}(\mathbb{R} \times X_\alpha, \mu, \nu) = \Big\{ u \in BC(\mathbb{R} \times X_\alpha; X_\alpha) : \lim_{\tau \to +\infty} \frac{1}{\nu[-\tau,\tau]} \int_{-\tau}^{+\tau} |u(t,x)|_\alpha d\mu(t) = 0 \Big\},$$

$$\mathscr{E}(\mathbb{R} \times X_\alpha, \mu, \nu, \infty) = \Big\{ u \in BC(\mathbb{R} \times X_\alpha; X_\alpha) : \lim_{\tau \to +\infty} \frac{1}{\nu[-\tau,\tau]} \int_{-\tau}^{+\tau} \Big(\sup_{\theta \in]-\infty, t]} |u(t,x)|_\alpha \Big) d\mu(t) = 0 \Big\},$$

where in both cases the limit(as $\tau \to +\infty$)is uniform in compact subset of X_α.
In view of previous definitions, it is clear that the space $\mathscr{E}(\mathbb{R}; X_\alpha, \mu, \nu, \infty)$ and $\mathscr{E}(\mathbb{R} \times X_\alpha, \mu, \nu, \infty)$ ara continuously embedded in $\mathscr{E}(\mathbb{R}; X_\alpha, \mu, \nu)$ and $\mathscr{E}(\mathbb{R} \times X_\alpha, \mu, \nu)$. On the other hand, one can observe that a ρ-weighted pseudo almost automorphic functions is μ-pseudo almost automorphic, where the measure μ is absolutely continuous with respect to the Lebesgue measure and its Radon-Nikodym derivative is ρ:

$$d\mu(t) = \rho(t)dt$$

and ν is the usual Lebesgue measure on $\mathbb{R}$, i.e $\nu[-\tau,\tau] = 2\tau$ for all $\tau \geq 0$.

Example 4.4. *[4] Let ρ be a nonnegative $\mathcal{N}$-measurable function. Denote by μ the positive measure defined by*

$$\mu(A) = \int_A \rho(t)dt \ \text{ for } \ A \in \mathcal{N}, \tag{4.1}$$

where dt denotes the Lebesgue measure on $\mathbb{R}$. The function ρ which occurs in equation (4.1) is called the Radon-Nikodym derivative of μ with respect to the Lebesgue measure on $\mathbb{R}$.

Definition 4.5. *A bounded continuous function $\phi : \mathbb{R} \to X$ is called $\alpha - (\mu,\nu)$-pseudo almost periodic if $\phi = \phi_1 + \phi_2$ where $\phi_1 \in AP(\mathbb{R}; X_\alpha)$ and $\phi_2 \in \mathscr{E}(\mathbb{R}; X_\alpha, \mu, \nu)$.*

We denote by $PAP(\mathbb{R}; X_\alpha, \mu, \nu)$, the space of all such functions.

Definition 4.6. *A bounded continuous function $\phi : \mathbb{R} \times X_\alpha \to X_\alpha$ is called uniformly $\alpha - (\mu,\nu)$-pseudo almost periodic if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AP(\mathbb{R} \times X_\alpha; X_\alpha)$ and $\phi_2 \in \mathscr{E}(\mathbb{R} \times X_\alpha; X_\alpha, \mu, \nu)$.*

We denote by $PAP(\mathbb{R} \times X_\alpha; X_\alpha, \mu, \nu)$, the space of all such functions.

Definition 4.7. *Let $\mu, \nu \in \mathcal{M}$. A bounded continuous function $\phi : \mathbb{R} \to X$ is called $\alpha - (\mu,\nu)$-pseudo almost periodic of infinite class if $\phi = \phi_1 + \phi_2$ where $\phi_1 \in AP(\mathbb{R}; X_\alpha)$ and $\phi_2 \in \mathscr{E}(\mathbb{R}; X_\alpha, \mu, \nu, \infty)$.*

We denote by $PAP(\mathbb{R}; X_\alpha, \mu, \nu, \infty)$, the space of all such functions.

Definition 4.8. *Let $\mu, \nu \in \mathcal{M}$. A bounded continuous function $\phi : \mathbb{R} \times X_\alpha \to X_\alpha$ is called uniformly $\alpha - (\mu,\nu)$-pseudo almost periodic of infinite class if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AP(\mathbb{R} \times X_\alpha; X_\alpha)$ and $\phi_2 \in \mathscr{E}(\mathbb{R} \times X_\alpha; X_\alpha, \mu, \nu, \infty)$.*

We denote by $PAP(\mathbb{R} \times X_\alpha; X_\alpha, \mu, \nu, \infty)$, the space of all such functions.

5. Properties of (μ,ν)-Pseudo Almost Periodic Functions of Infinite Class

From μ, $\nu \in \mathcal{M}$, we formulate the following hypotheses.
($\mathbf{H}_3$) Let μ, $\nu \in \mathcal{M}$ be such that

$$\limsup_{\tau \to +\infty} \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} = \delta < \infty.$$

We have the following result.

Lemma 5.1. *Assume ($\boldsymbol{H}_3$) holds and let $f \in BC(\mathbb{R}; X_\alpha)$. Then $f \in \mathscr{E}(\mathbb{R}; X_\alpha, \mu, \nu)$ if and only if for any $\varepsilon > 0$,*

$$\lim_{\tau \to +\infty} \frac{\mu(M_{\tau,\varepsilon}(f))}{\nu[-\tau,\tau]} = 0.$$

Where

$$M_{\tau,\varepsilon} = \Big\{ t \in [-\tau,\tau] : |f(t)|_\alpha \geq \varepsilon \Big\}$$

Proof. Suppose $f \in \mathscr{E}(\mathbb{R}; X_\alpha, \mu, \nu)$. Then

$$\begin{aligned}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}|f(t)|_\alpha d\mu(t) &= \frac{1}{\nu([-\tau,\tau])}\int_{\mathcal{M}_{\tau,\varepsilon}(f)}|f(t)|_\alpha d\mu(t)\\ &\quad+\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]\setminus\mathcal{M}_{\tau,\varepsilon}(f)}|f(t)|_\alpha d\mu(t)\\ &\geq \frac{1}{\nu([-\tau,\tau])}\int_{\mathcal{M}_{\tau,\varepsilon}(f)}|f(t)|_\alpha d\mu(t)\\ &\geq \frac{\varepsilon}{\nu([-\tau,\tau])}\int_{\mathcal{M}_{\tau,\varepsilon}(f)} d\mu(t)\\ &\geq \frac{\varepsilon\mu(M_{\tau,\varepsilon}(f))}{\nu([-\tau,\tau])}.\end{aligned}$$

Consequently

$$\lim_{\tau\to+\infty}\frac{\mu(M_{\tau,\varepsilon}(f))}{\nu([-\tau,\tau])}=0.$$

Suppose $f \in BC(\mathbb{R}; X_\alpha)$ such that for any $\varepsilon > 0$,

$$\lim_{\tau\to+\infty}\frac{\mu([\mathcal{M}_{\tau,\varepsilon}(f)}{\nu[-\tau,\tau])}=0.$$

We can assume $|f(t)|_\alpha \leq N$ for all $t \in \mathbb{R}$. Using $(\mathbf{H}_2)$, we have

$$\begin{aligned}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}|f(t)|_\alpha d\mu(t) &= \frac{1}{\nu([-\tau,\tau])}\int_{\mathcal{M}_{\tau,\varepsilon}(f)}|f(t)|_\alpha d\mu(t)\\ &\quad+\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]\setminus\mathcal{M}_{\tau,\varepsilon}(f)}|f(t)|_\alpha d\mu(t)\\ &\leq \frac{N}{\nu([-\tau,\tau])}\int_{\mathcal{M}_{\tau,\varepsilon}(f)} d\mu(t)\\ &\quad+\frac{\varepsilon}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]\setminus\mathcal{M}_{\tau,\varepsilon}(f)} d\mu(t)\\ &\leq \frac{N\mu(\mathcal{M}_{\tau,\varepsilon})}{\nu([-\tau,\tau])}+\frac{\varepsilon\mu([-\tau,\tau]}{\nu([-\tau,\tau])}.\end{aligned}$$

Which implies that

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}|f(t)|_\alpha d\mu(t)\leq\delta\varepsilon \text{ for any } \varepsilon>0.$$

Therefore $f \in \mathscr{E}(\mathbb{R}; X_\alpha, \mu, \nu)$. □

Lemma 5.2. *Assume that* $(\boldsymbol{H}_3)$ *holds. The space* $\mathscr{E}(\mathbb{R}; X_\alpha, \mu, \nu, \infty)$ *endowed with the uniform topology norm is a Banach space.*

Proof. We can see that $\mathscr{E}(\mathbb{R}; X_\alpha, \mu, \nu, \infty)$ is a vector subspace of $BC(\mathbb{R}; X_\alpha)$. To complete the proof, it is enough to prove that $\mathscr{E}(\mathbb{R}; X_\alpha, \mu, \nu, \infty)$ is closed in $BC(\mathbb{R}; X_\alpha)$. Let $(z_n)_n$ be a sequence in $\mathscr{E}(\mathbb{R}; X_\alpha, \mu, \nu, r)$ such that $\lim\limits_{n\to+\infty} z_n = z$ uniformly in $\mathbb{R}$. From $\nu(\mathbb{R}) = +\infty$, it follows $\nu([-\tau,\tau])>0$ for τ sufficiently large. Let $\|z\|_{\infty,\alpha}=\sup_{t\in\mathbb{R}}|z(t)|_\alpha$ and $n_0\in\mathbb{N}$ such that all $n\geq n_0$, we have

$$\begin{aligned}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]}|z(t)|_\alpha\Big)d\mu(t) &\leq \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]}|z_n(t)-z(t)|_\alpha\Big)d\mu(t)\\ &\quad+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]}|z_n(t)|_\alpha\Big)d\mu(t)\\ &\leq \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in\mathbb{R}}|z_n(t)-z(t)|_\alpha\Big)d\mu(t)\\ &\quad+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]}|z_n(t)|_\alpha\Big)d\mu(t)\\ &\leq \|z_n-z\|_{\infty,\alpha}\times\frac{\nu([-\tau,\tau])}{\mu([-\tau,\tau])}\\ &\quad+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]}|z_n(t)|_\alpha\Big)d\mu(t),\end{aligned}$$

which implies that

$$\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]}|z(t)|_\alpha\Big)d\mu(t)\leq\delta\varepsilon \text{ for any } \varepsilon>0.$$

From the definition of $PAP(\mathbb{R};X_\alpha,\mu,\nu,\infty)$, we deduce the following result. □

Proposition 5.3. *Let $\mu,\nu\in\mathcal{M}$ and assume that $(\boldsymbol{H}_3)$ holds. The space $PAP(\mathbb{R};X_\alpha,\mu,\nu,\infty)$ endowed with the uniform topology norm is a Banach space.*

Next result is a characterisation of $\alpha-(\mu,\nu)$-ergodic functions of infinite class.

Theorem 5.4. *Asume that $(\boldsymbol{H}_4)$ holds and let $\mu,\nu\in\mathcal{M}$ and I be a bounded interval (eventually $I=\varnothing$). Assume that $f\in BC(\mathbb{R};X_\alpha)$. The following assertions are equivalent*

i) $f\in\mathscr{E}(\mathbb{R};X_\alpha,\mu,\nu,\infty)$,

ii) $\displaystyle\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in]-\infty,t]}|f(\theta)|_\alpha\Big)d\mu(t)=0,$

iii) for any $\varepsilon>0$, $\displaystyle\lim_{\tau\to+\infty}\frac{\mu\Big(\{t\in[-\tau,\tau]\setminus I:\sup_{\theta\in]-\infty,t]}|f(\theta)|_\alpha>\varepsilon\}\Big)}{\nu([-\tau,\tau]\setminus I)}=0.$

Proof. *i)* ⇔ *ii)*. Denote By $A=\mu(I)$, $B=\displaystyle\int_I\Big(\sup_{\theta\in]-\infty,t]}|f(\theta)|_\alpha\Big)d\mu(t)$ and $C=\mu(I)$. We have A, B and $C\in\mathbb{R}$.

Since the interval I is bounded and the function f is bounded continuous. For $\tau>0$, such that $I\subset[-\tau,\tau]$ and $\nu([-\tau,\tau]\setminus I)>0$, we have

$$\frac{1}{\nu([-\tau,\tau])\setminus I}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in]-\infty,t]}|f(\theta|_\alpha\Big)=\frac{1}{\nu[-\tau,\tau]-A}\Big[\int_{[-\tau,\tau]}\Big(\sup_{\theta\in]-\infty,t]}|f(\theta)|_\alpha\Big)d\mu(t)-B\Big]$$

$$=\frac{\nu([-\tau,\tau])}{\nu[-\tau,\tau]-A}\Big[\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\Big(\sup_{\theta\in]-\infty,t]}|f(\theta)|_\alpha\Big)d\mu(t)-\frac{B}{\nu([-\tau,\tau])}\Big].$$

From above equalities and the fact $\nu(\mathbb{R}) = +\infty$, we deduce $ii)$ is equivalent to

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{[-\tau,\tau]} \Big(\sup_{\theta\in]-\infty,t]} |f(\theta)|_\alpha \Big) d\mu(t) = 0,$$

that $i)$. $iii) \Rightarrow ii)$ Denote by A_τ^ε and B_τ^ε the following sets:

$$A_\tau^\varepsilon = \Big\{ t \in [-\tau,\tau] \setminus I : \sup_{\theta\in]-\infty,t]} |f(\theta)|_\alpha > \varepsilon \Big\}$$

and

$$B_\tau^\varepsilon = \Big\{ t \in [-\tau,\tau] \setminus I : \sup_{\theta\in]-\infty,t]} |f(\theta)|_\alpha \le \varepsilon \Big\}.$$

Assume that $ii)$ holds, that is

$$\lim_{\tau\to+\infty} \frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]) \setminus I)} = 0. \tag{5.1}$$

From this equality

$$\begin{aligned} \int_{[-\tau,\tau]\setminus I} \Big(\sup_{\theta\in]-\infty,t]} |f(\theta)|_\alpha \Big) d\mu(t) &= \int_{A_\tau^\varepsilon} \Big(\sup_{\theta\in]-\infty,t]} |f(\theta)|_\alpha \Big) d\mu(t) \\ &+ \int_{B_\tau^\varepsilon} \Big(\sup_{\theta\in]-\infty,t]} |f(\theta)|_\alpha \Big) d\mu(t), \end{aligned}$$

we deduce that for τ sufficient large

$$\frac{1}{\nu([-\tau,\tau])} \int_{[-\tau,\tau]\setminus I} \Big(\sup_{\theta\in]-\infty,t]} |f(\theta)|_\alpha \Big) d\mu(t) \le \|f\|_{\infty,\alpha} \times \frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau] \setminus I)} + \varepsilon \frac{\mu(B_\tau^\varepsilon)}{\nu([-\tau,\tau] \setminus I)}.$$

Since $\nu(\mathbb{R}) = +\infty$ and by using $(\mathbf{H}_3)$ then for all $\varepsilon > 0$ we have

$$\frac{1}{\nu([-\tau,\tau] \setminus I)} \int_{[-\tau,\tau]\setminus I} \Big(\sup_{\theta\in]-\infty,t]} |f(\theta)|_\alpha \Big) d\mu(t) \le \delta\varepsilon$$

Consequently $ii)$ holds.

$ii) \Rightarrow iii)$

$$\begin{aligned} \int_{[-\tau,\tau]\setminus I} \Big(\sup_{\theta\in]-\infty,t]} |f(\theta)|_\alpha \Big) d\mu(t) &\ge \int_{A_\tau^\varepsilon} \Big(\sup_{\theta\in]-\infty,t]} |f(\theta)|_\alpha \Big) d\mu(t) \\ \frac{1}{\nu([-\tau,\tau] \setminus I)} \int_{[-\tau,\tau]\setminus I} \Big(\sup_{\theta\in]-\infty,t]} |f(\theta)|_\alpha \Big) d\mu(t) &\ge \varepsilon \frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau] \setminus I)} \\ \frac{1}{\varepsilon\nu([-\tau,\tau] \setminus I)} \int_{[-\tau,\tau]\setminus I} \Big(\sup_{\theta\in]-\infty,t]} |f(\theta)|_\alpha \Big) d\mu(t) &\ge \frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau] \setminus I)}, \end{aligned}$$

for τ sufficiently large, we obtain equation (5.1), that is $iii)$. □

From $\mu \in \mathcal{M}$, we formulate the following hypotheses.

$(\mathbf{H}_4)$ For all a, b and $c \in \mathbb{R}$ such that $0 \le a < b < c$, there exist δ_0 and $\alpha_0 > 0$ such that

$$|\delta| \ge \delta_0 \Rightarrow \mu(a+\delta, b+\delta) \ge \alpha_0 \mu(\delta, c+\delta).$$

$(\mathbf{H}_5)$ For all $\tau \in \mathbb{R}$ there exist $\beta > 0$ and a bounded interval Isuch that

$$\mu\Big(\{a+\tau : a \in A\} \Big) \le \beta\mu(A) \text{ when } A \in \mathcal{B} \text{ and satisfies } A \cap I = \varnothing.$$

We have the following result due to [4].

Lemma 5.5. *[4] Hypothesis* $(\boldsymbol{H}_5)$ *implies* $(\boldsymbol{H}_4)$.

Proposition 5.6. *[4, 6]* $\mu, \nu \in \mathcal{M}$ *satisfy* $(\boldsymbol{H}_4)$ *and* $f \in PAP(\mathbb{R}; X, \mu, \nu)$ *be such that*

$$f = g + h,$$

where $g \in AP(\mathbb{R}; X)$ *and* $h \in \mathscr{E}(\mathbb{R}; X, \mu, \nu)$. *Then*

$$\{g(t), t \in \mathbb{R}\} \subset \overline{\{f(t), t \in \mathbb{R}\}} (\text{the closure of the range of } f).$$

Corollary 5.7. *[6] Assume that* $(\boldsymbol{H}_4)$ *holds. Then the decomposition of* (μ, ν)*-pseudo almost periodic function in the form* $f = g + \phi$ *where* $g \in AP(\mathbb{R}; X)$ *and* $\phi \in \mathscr{E}(\mathbb{R}; X, \mu, \nu)$*, is unique.*

The following Proposition is a consequence of Proposition 5.6.

Proposition 5.8. *Let* $\mu, \nu \in \mathcal{M}$. *Assume that* $(\boldsymbol{H}_4)$ *holds. Then the decomposition of* $\alpha-(\mu, \nu)$*-pseudo-almost periodic function in the form* $\phi = \phi_1 + \phi_2$*, where* $\phi_1 \in AP(\mathbb{R}; X_\alpha)$ *and* $\phi_2 \in \mathscr{E}(\mathbb{R}; X_\alpha, \mu, \nu)$*, is unique.*

Proof. In fact, since as consequence of Corollary 5.7, the decomposition of a (μ, ν)-pseudo-almost periodic function $\phi_1 \in AP(\mathbb{R}; X_\alpha)$ and $\phi_2 \in \mathscr{E}(\mathbb{R}; X_\alpha, \mu, \nu)$, is unique.
Since $PAP(\mathbb{R}; X_\alpha, \mu, \nu, \infty) \subset PAP(\mathbb{R}; X, \mu, \nu)$, we get the desired result. $\square$

Definition 5.9. *Let* $\mu_1, \mu_2 \in \mathcal{M}$. *We say that* μ_1 *is equivalent to* μ_2*, denoting this as* $\mu_1 \sim \mu_2$ *if there exist constants* α *and* $\beta > 0$ *and a bounded interval* I *(eventually* $I = \varnothing$*) such that* $\alpha\mu_1(A) \leq \mu_2 \beta\mu_1(A)$*, when* $A \in \mathcal{B}$ *satisfies* $A \cap I = \varnothing$.

From [4]$\sim$ is binary equivalent relation on $\mathcal{M}$. The equivalence class of a given measure $\mu \in \mathbb{R}$ will then be denoted by

$$cl(\mu) = \Big\{\overline{\omega} \in \mathcal{M} : \mu \sim \overline{\omega}\}.$$

Theorem 5.10. *Let* $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathcal{M}$. *If* $\mu_1 \sim \mu_2$ *and* $\nu_1\nu_2$*, then* $PAP(\mathbb{R}; X_\alpha\mu_1, \nu_1, \infty) = PAP(\mathbb{R}; X_\alpha, \mu_2, \nu_2, \infty)$.

Proof. Since $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$ there exist constants α_1, α_2, β_1, $\beta_2 > 0$ and a bounded interval I (eventually $I = \varnothing$) such that $\alpha_1(A) \leq \mu_2 \leq \beta_1\mu(A)$ and $\alpha_2\nu_1(A) \leq \nu_2 \leq \beta_2\nu(A)$ for each $A \in \mathscr{B}$ satisfies $A \cap I = \varnothing$ i.e

$$\frac{1}{\beta_1\nu(A)} \leq \frac{1}{\nu_2(A)} \leq \frac{1}{\alpha_2\nu(A)}$$

Since $\mu_1 \sim \mu_2$ and β is the Lebesgue σ-field, we obtain for τ sufficiently Large that

$$\begin{aligned}
&\frac{\alpha_1\mu_1\Big(\{t \in [-\tau, \tau] \setminus I : \sup\limits_{\theta \in]-\infty, t]} |f(\theta)|_\alpha > \varepsilon\}\Big)}{\beta_2\mu_2([-\tau, \tau] \setminus I)} \\
&\leq \frac{\mu_2\Big(\{t \in [-\tau, \tau] \setminus I : \sup\limits_{\theta \in]-\infty, t]} |f(\theta)|_\alpha > \varepsilon\}\Big)}{\nu_2([-\tau, \tau] \setminus I)} \\
&\leq \frac{\beta_1\mu_1\Big(\{t \in [-\tau, \tau] \setminus I : \sup\limits_{\theta \in]-\infty, t]} |f(\theta)|_\alpha > \varepsilon\}\Big)}{\alpha_2\nu([-\tau, \tau] \setminus I)}.
\end{aligned}$$

By using Theorem 5.4, we deduce that $\mathscr{E}(\mathbb{R}; X_\alpha, \mu_1, \nu_1, \infty) = \mathscr{E}(\mathbb{R}; X_\alpha, \mu_2, \nu_2, \infty)$. From the definition of a (μ, ν)-pseudo almost periodic function, we deduce that
$PAP(\mathbb{R}; X_\alpha, \mu_1, \nu_1, \infty) = PAP(\mathbb{R}; X_\alpha, \mu_1, \nu_1, \infty)$. □

Let $\mu, \nu \in \mathcal{M}$, we denote by

$$cl(\mu, \nu) = \Big\{\overline{\omega}_1, \overline{\omega}_2 \in \mathcal{M} : \mu_1 \sim \mu_2, \nu_1 \sim \nu_2\Big\}.$$

Corollary 5.11. *Let $\mu, \nu \in \mathcal{M}$ satisfies $(\boldsymbol{H}_5)$. Then $PAP(\mathbb{R}; X_\alpha, \mu, \nu, \infty)$ is invariant by translation that is $f \in PAP(\mathbb{R}; X_\alpha, \mu, \nu, \infty)$ implies $f_\gamma \in PAP(\mathbb{R}; X_\alpha, \mu, \nu, \infty)$ for all $\gamma \in \mathbb{R}$.*

Proof. It suffices to prove that $\mathscr{E}(\mathbb{R}; X_\alpha, \mu, \nu, \infty)$ is invariant by translation. Let $f \in \mathscr{E}(\mathbb{R}; X, \mu, \nu)$ and $F^t(\theta) = \sup_{\theta \in]-\infty, t]} |f(\theta)|_\alpha$. Then $F^t \in \mathscr{E}(\mathbb{R}; \mathbb{R}, \mu, \nu)$ but since $\mathscr{E}(\mathbb{R}, \mathbb{R}, \mu, \nu)$ is invariant by translation, it follows that

$$\lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} F^t(\theta + \gamma) d\mu(t) = \lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in]-\infty, t]} |f(\theta + \gamma)|_\alpha d\mu(t),$$

which implies that $f(., \gamma) \in PAP(\mathbb{R}; X_\alpha, \mu, \nu, \infty)$. □

In what follows, we prove some preliminary results concerning the composition of (μ, ν)-pseudo almost periodic functions of infinite class.

Theorem 5.12. *Let $\mu, \nu, \phi, \mathcal{M}$, $\phi \in PAP(\mathbb{R} \times X_\alpha, \mu, \nu, \infty)$ and $h \in PAP(\mathbb{R}; X_\alpha, \mu, \nu, \infty)$. Assume that there exists a function $L_\phi : \mathbb{R} \to [0, \infty[$ such that*

$$|\phi(t, x_1) - \phi(t, x_2)|_\alpha \leq L_\phi |x_1 - x_2|_\alpha, \text{ for } t \in \mathbb{R} \text{ and for } x_1, x_2 \in X_\alpha. \tag{5.2}$$

If

$$\limsup_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \Big(\sup_{\theta \in]-\infty, t]} L_\phi(\theta) \Big) d\mu(t) < \infty \tag{5.3}$$

and

$$\lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \Big(\sup_{\theta \in]-\infty, t]} L_\phi(\theta) \Big) \xi(t) d\mu(t) = 0, \tag{5.4}$$

for each $\xi \in \mathscr{E}(\mathbb{R}, \mathbb{R}, \mu, \nu)$, then the function $t \to \phi(t, h(t))$ belongs to $PAP(\mathbb{R}; X_\alpha, \mu, \nu, \infty)$.

Proof. Assume that $\phi = \phi_1 + \phi_2$, $h = h_1 + h_2$, where $\phi \in AP(\mathbb{R} \times X_\alpha; X_\alpha)$, $\phi_2 \in \mathscr{E}(\mathbb{R} \times X_\alpha; \mu, \nu, \infty)$ and $h_1 \in AA(\mathbb{R}; X_\alpha)$, $h_2 \in \mathscr{E}(\mathbb{R}, X_\alpha, \mu, \nu, \infty)$. Consider the following decomposition

$$\phi(t, h(t)) = \phi_1(t, h_1(t)) + \Big[\phi(t, h(t)) - \phi(t, h_1(t))\Big] + \phi_2(t, h_1(t)).$$

From [5] $\phi_1(., h_1(.)) \in AP(\mathbb{R}; X_\alpha)$. It remains to prove that both ;$\phi(., h(.)) - \phi(., h_1(.))$ and $\phi_2(., h_1(.))$ belong to $\mathscr{E}(\mathbb{R}; X_\alpha, \mu, \nu, \infty)$. Consequently using (5.2), it follows that

$$\frac{\mu\Big(\Big\{t \in [-\tau, \tau] : \sup_{\theta \in]-\infty, t]} |\phi(\theta, h(\theta)) - \phi(\theta, h_1(\theta))|_\alpha > \varepsilon\Big\}\Big)}{\nu([-\tau, \tau])}$$

$$\leq \frac{\mu\Big(\Big\{t \in [-\tau, \tau] : \sup_{\theta \in]-\infty, t]} (L_\phi(\theta) |h_2(\theta)|_\alpha) > \varepsilon\Big\}\Big)}{\nu([-\tau, \tau])}$$

$$\leq \frac{\mu\Big(\Big\{t \in [-\tau,\tau] : (\sup_{\theta\in]-\infty,t]} L_\phi(\theta))(\sup_{\theta\in]-\infty,t]} |h_2(\theta)|_\alpha) > \varepsilon\Big\}\Big)}{\nu([-\tau,\tau])}.$$

Since h_2 is (μ,ν)-ergodic of infinite class , Theorem 5.4 and equation (5.7) yield that for above-mentioned ε, we have

$$\lim_{\tau\to+\infty} \frac{\mu\Big(\Big\{t \in [-\tau,\tau] : (\sup_{\theta\in]-\infty,t]} L_\phi(\theta))(\sup_{\theta\in]-\infty,t]} |h_2(\theta)|_\alpha) > \varepsilon\Big\}\Big)}{\nu([-\tau,\tau])} = 0,$$

and then, we obtain

$$\lim_{\tau\to+\infty} \frac{\mu\Big(\Big\{t \in [-\tau,\tau] : \sup_{\theta\in]-\infty,t]} |\phi(\theta,h(\theta)) - \phi(\theta,h_1(\theta))|_\alpha > \varepsilon\Big\}\Big)}{\nu([-\tau,\tau])} = 0. \tag{5.5}$$

By Theorem 5.4 and equation (5.5) follows that $\phi(t,h(t))-\phi(t,h_1(t))$ is (μ,ν)-ergodic of infinite class.

Now to complete the proof is enough to prove that $t \to \phi_2(t,h(t))$ is (μ,ν)-ergodic of infinite class. Since ϕ_2 is uniformly continuous on the compact set $K = \overline{\{h_1(t), t\in\mathbb{R}\}}$ with the respect of second variable x, we deduce that for given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $t \in \mathbb{R}$, ξ_1 and $\xi_2 \in K$ and one has

$$|\xi_1 - x_2| \leq \delta \Rightarrow |\phi_2(t,\xi_1) - \phi(\xi_2)|_\alpha \leq \varepsilon.$$

Therefore there exists $n(\varepsilon)$ and $\{z_i\}_{i=1}^{n(\varepsilon)} \subset K$ such that

$$K \subset \bigcup_{i=1}^{n(\varepsilon)} B_\delta(z_i,\delta)$$

and then

$$|\phi_2(t,h_1(t))|_\alpha \leq \varepsilon + \sum_{i=1}^{n(\varepsilon)} |\phi_2(t,z_i)|_\alpha.$$

Since

$$\forall i \in \Big\{1,\dots,n(\varepsilon)\Big\}, \lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]} |\phi_2(\theta,z_i)|_\alpha\Big)d\mu(t) = 0,$$

we deduce that

$$\forall \varepsilon > 0, \limsup_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]} |\phi_2(t,h_1(t))|_\alpha\Big)d\mu(t) \leq \varepsilon\delta,$$

that implies

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]} |\phi_2(t,h_1(t))|_\alpha\Big)d\mu(t) = 0.$$

Consequently $t \to \phi_2(t,h_1(t))$ is (μ,ν)-ergodic infinite class. □

For $\mu \in \mathcal{M}$ and $\delta \in \mathbb{R}$, we denote μ_δ the positive measure on $(\mathbb{R},\mathcal{N})$ defined by

$$\mu_\delta(A) = \mu([a+\delta : a \in A]) \tag{5.6}$$

Lemma 5.13. *[4] Let $\mu \in \mathcal{M}$ satisfy $(\boldsymbol{H}_5)$. Then the measures μ and μ_δ are equivalent for all $\delta \in \mathbb{R}$.*

Lemma 5.14. *[4] ($\boldsymbol{H}_5$) implies*

$$\text{for all } \sigma > 0,\ \limsup_{\tau\to+\infty} \frac{\mu([-\tau-\sigma,\tau+\sigma])}{\nu([-\tau,\tau])} < \infty.$$

We have the following result.

Theorem 5.15. *Assume that ($\boldsymbol{H}_5$) holds. Let $\mu,\nu \in \mathcal{M}$ and $\phi \in PAP(\mathbb{R}; X_\alpha,\mu,\nu,\infty)$, then the function $t \to \phi_t$ belongs to $PAP(\mathcal{B}_\alpha;\mu,\nu,\infty)$.*

Proof. Assume that $u = g + h$, where $g \in AP(\mathbb{R}; X_\alpha)$ and $h \in \mathscr{E}(\mathbb{R}; X_\alpha,\mu,\nu,\infty)$. We can see that $u_t = g_t + h_t$ and g_t is almost periodic. On the other hand for $\tau > 0$, we have

$$\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau} \sup_{\theta\in]-\infty,t]}\Big(\sup_{\theta\in]-\infty,0]}|h(\theta+\xi)|_\alpha\Big)d\mu(t) \le \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]}|h(\theta)|_\alpha\Big)d\mu(t),$$

which shows that ϕ_t belongs to $PAP(\mathcal{B}_\alpha;\mu,\nu,r)$. Thus we obtain the desired result. □

6. (μ,ν)-Pseudo Almost Periodic of Infinite Class

In what follows, we will be looking at the existence of bounded integral solutions of infinite class of equation (1.1).

Proposition 6.1. *[7] Assume that $\mathcal{B}$ satisfies ($\boldsymbol{A}_1$) ($\boldsymbol{A}_2$), ($\boldsymbol{B}$), ($\boldsymbol{C}_1$), ($\boldsymbol{C}_2$) Hold. Suppose that ($\boldsymbol{H}_0$), ($\boldsymbol{H}_1$) and ($\boldsymbol{H}_2$) hold and the semigroup $(U(t))_{t\ge0}$ is hyperbolic. If f is bounded on $\mathbb{R}$, then there exists a unique bounded solution u of equation (1.1) on $\mathbb{R}$, given by*

$$u_t = \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds,$$

where $\widetilde{B}_\lambda = \lambda(\lambda I - \mathcal{A}_{\mathcal{U}})^{-1}$ for$\lambda > \widetilde{\omega}$, Π^s and Π^u are projections of $\mathscr{B}_\alpha$ onto the stable and unstable subspaces respectively.

Proposition 6.2. *Let $h \in AP(\mathbb{R}, X)$ and Γ be the mapping defined for $t \in \mathbb{R}$ by*

$$\Gamma h(t) = \Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 h(s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 h(s))ds\Big](0).$$

Then $\Gamma h \in AP(\mathbb{R}; X_\alpha)$.

Proof. We can see that $\Gamma h \in BC(\mathbb{R}; X_\alpha)$. In fact

$$\begin{aligned}
|\Gamma h(t)|_\alpha &\le \Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}|\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 h(s))|_\alpha ds + \lim_{\lambda\to+\infty}\int_{t}^{+\infty}|\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 h(s))|_\alpha ds\Big](0)\\
&\le \Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\|\mathcal{A}_{\mathcal{U}}^\alpha\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 h(s))\|ds + \lim_{\lambda\to+\infty}\int_{t}^{+\infty}\|\mathcal{A}_{\mathcal{U}}^\alpha\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 h(s))\|ds\Big](0)\\
&\le \overline{M}\widetilde{M}\int_{-\infty}^{t}\frac{e^{-\omega(t-s)}}{(t-s)^\alpha}|\Pi^s|\|h(s)\|ds + \overline{M}\widetilde{M}\int_{t}^{+\infty}\frac{e^{\omega(t-s)}}{(s-t)^\alpha}|\Pi^u|\|h(s)\|ds\\
&\le \overline{M}\widetilde{M}\int_{-\infty}^{t}\frac{e^{-\omega(t-s)}}{(t-s)^\alpha}|\Pi^s|\|h(s)\|ds + \overline{M}\widetilde{M}\int_{t}^{+\infty}\frac{e^{-\omega(s-t)}}{(s-t)^\alpha}|\Pi^u|\|h(s)\|ds\\
&\le \frac{\overline{M}\widetilde{M}(|\Pi^s|+|\Pi^u|)\|h\|_\infty}{\omega^{1-\alpha}}\int_{0}^{+\infty}e^{-s}s^{-\alpha}ds = \frac{\overline{M}\widetilde{M}(|\Pi^s|+|\Pi^u|)\|h\|_\infty\Gamma(1-\alpha)}{\omega^{1-\alpha}} < \infty. \qquad (6.1)
\end{aligned}$$

Since h is almost periodic function, then the set of functions $\{h_\tau : \tau \in \mathbb{R}\}$, where $(h_\tau)(t) = h(\tau+t)$ is precompact in $BC(\mathbb{R}; X_\alpha)$. On the other hand, we have

$$
\begin{aligned}
(\Gamma h)_\tau(t) &= \Gamma h(\tau+t)\\
&= \Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{t+\tau}\mathcal{U}^s(t+\tau-s)\Pi^s(\widetilde{B}_\lambda X_0 h(s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^{t+\tau}\mathcal{U}^u(t+\tau-s)\Pi^u(\widetilde{B}_\lambda X_0 h(s))ds\Big](0)\\
&= \Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 h(s+\tau))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 h(s+\tau))ds\Big](0)\\
&= \Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 h_\tau(s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 h_\tau(s))ds\Big](0)\\
&= (\Gamma h_\tau)(t) \text{ for all } t\in\mathbb{R}.
\end{aligned}
$$

Consequently $(\Gamma h)_\tau = (\Gamma h_\tau)$, which implies that $\{(\Gamma h)_\delta : \delta\in\mathbb{R}\}$ is relatively compact int $BC(\mathbb{R};X_\alpha)$. Since Γ is continuous from $BC(\mathbb{R};X_\alpha)$ into $BC(\mathbb{R};X_\alpha)$. Thus $\Gamma h\in AP(\mathbb{R};X_\alpha)$. □

Theorem 6.3. *Let $\mu,\nu\in\mathcal{M}$ satisfy $(\boldsymbol{H}_4)$ and $g\in\mathscr{E}(\mathbb{R};X,\mu,\nu,\infty)$. Then $\Gamma g\in\mathscr{E}(\mathbb{R};X_\alpha,\mu,\nu,\infty)$*

Proof. In fact, for $\tau>0$ we get

$$
\begin{aligned}
\int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}|\Gamma g|_\alpha\Big)d\mu(t) &\le \int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}\Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{\theta}|\mathcal{U}^s(\theta-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s)|_\alpha ds\\
&\quad+\lim_{\lambda\to+\infty}\int_{\theta}^{+\infty}|\mathcal{U}^u(\theta-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s)|_\alpha ds\Big](0)\Big)d\mu(t)\\
&\le \int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}\Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{\theta}\|\mathcal{A}_{\mathcal{U}}^{\alpha}\mathcal{U}^s(\theta-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s)\|ds\\
&\quad+\lim_{\lambda\to+\infty}\int_{\theta}^{+\infty}\|\mathcal{U}_{\mathcal{U}}^{\alpha}\mathcal{U}^u(\theta-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s)\|ds\Big](0)\Big)d\mu(t)\\
&\le \overline{M}\widetilde{M}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}\int_{-\infty}^{\theta}\frac{e^{-\omega(\theta-s)}}{(\theta-s)^\alpha}|\Pi^s|\|g(s)\|ds\Big)d\mu(t)\\
&\quad+\overline{M}\widetilde{M}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}\int_{\theta}^{+\infty}\frac{e^{-\omega(\theta-s)}}{(s-\theta)^\alpha}|\Pi^u|\|g(s)\|ds\Big)d\mu(t)\\
&\le \overline{M}\widetilde{M}\Big[|\Pi^s|\int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}\int_{-\infty}^{\theta}\frac{e^{-\omega(\theta-s)}}{(\theta-s)^\alpha}\|g(s)\|ds\Big)d\mu(t)\\
&\quad+|\Pi^u|\int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}\int_{\theta}^{+\infty}\frac{e^{-\omega(\theta-s)}}{(s-\theta)^\alpha}\|g(s)\|ds\Big)d\mu(t)\Big].
\end{aligned}
$$

On the one hand using Fubini's Theorem, we have

$$
\int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}\int_{-\infty}^{\theta}\frac{e^{-\omega(t-s)}}{(\theta-s)^\alpha}\|g(s)\|d\mu(t)\Big)d\mu(t)
$$

$$
\le \int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}\int_{0}^{+\infty}\frac{e^{-\omega s}}{s^\alpha}\|g(\theta-s)\|ds\Big)d\mu(t)
$$

$$\leq \int_0^{+\infty} \frac{e^{-\omega s}}{s^\alpha}\Big(\sup_{\theta\in]-\infty,t]}\int_{-\tau}^{\tau}\|g(\theta-s)\|d\mu(t)\Big)ds.$$

By the Lebesgue dominated convergence Theorem and by using Corollary 5.11, it follows that

$$\lim_{\tau\to+\infty}\int_0^{+\infty}\frac{e^{-\omega s}}{s^\alpha}\frac{1}{\nu([-\tau,\tau])}\Big(\sup_{\theta\in]-\infty,t]}\int_{-\tau}^{\tau}\|g(\theta-s)\|d\mu(t)\Big)ds=0.$$

On the other hand by Fubini's theorem, we also have

$$\int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}\int_{-\infty}^{\theta}\frac{e^{-\omega(\theta-s)}}{(s-\theta)^\alpha}\|g(s)\|ds\Big)d\mu(t)$$

$$\leq \int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}\int_0^{+\infty}\frac{e^{-\omega s}}{s^\alpha}\|g(s+\theta)\|ds\Big)d\mu(t)$$

$$\leq \int_0^{+\infty}\frac{e^{-\omega s}}{s^\alpha}\Big(\sup_{\theta\in]-\infty,t]}\int_{-\tau}^{\tau}\|g(s+\theta)\|d\mu(t)\Big)ds$$

Resoning like above, it follows that

$$\lim_{\tau\to+\infty}\int_0^{+\infty}\frac{e^{-\omega s}}{s^\alpha}\frac{1}{\nu([-\tau,\tau])}\Big(\sup_{\theta\in]-\infty,t]}\int_{-\tau}^{\tau}|g(s+\theta)|d\mu(t)\Big)ds=0.$$

Consequently

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}|\Gamma g(\theta)|_\alpha)d\mu(t)=0.$$

Thus, we obtain the desired result. □

For the existence of (μ,ν)-pseudo almost periodic solution of infinite class, we make the following assumption.

$(\mathbf{H}_7)$ $f:\mathbb{R}\to X$ is compact $cl-(\mu,\nu)$-pseudo almost periodic of infinite class.

Proposition 6.4. *Assume that $\mathcal{B}_\alpha$ satisfies $(\boldsymbol{A}_1)$, $(\boldsymbol{A}_2)$ $(\boldsymbol{B})$, $(\boldsymbol{C}_1)$, $(\boldsymbol{C}_2)$ hold. Suppose that $(\boldsymbol{H}_0)$, $(\boldsymbol{H}_1)$, $(\boldsymbol{H}_2)$, $(\boldsymbol{H}_5)$ and $(\boldsymbol{H}_7)$ hold. Then (1.1) has a unique compact $\alpha-cl(\mu,\nu)$-pseudo almost periodic solution of infinite class.*

Proof. Since f is a (μ,ν)-pseudo almost periodic function, f has a decomposition $f=f_1+f_2$ where $f_1\in AP(\mathbb{R};X_\alpha)$ and $f_2\in\mathscr{E}(\mathbb{R};X_\alpha,\mu,\nu,\infty)$. Using Proposition 6.2 and Theorem 6.3, we get the desired result. □

Our next objective is to show the existence of (μ,ν)-pseudo almost automorphic solution of infinite class for the following problem

(6.2) $$u'(t)=-Au(t)+L(u_t)+f(t,u_t)\text{ for }t\in\mathbb{R},$$

where $f:\mathbb{R}\times\mathcal{B}_\alpha\to X$ is continuous.

We make the following assumptions:
$(\mathbf{H}_6)$ The unstable space $U\equiv\{0\}$.

($\mathbf{H}_7$) Let $\mu,\nu \in \mathcal{M}$ and $f:\mathbb{R}\times\mathcal{B}_\alpha \to X$ $cl(\mu,\nu)$-pseudo almost periodic infinite class such that there exists a positive constant L_f such that

$$\|f(t,\varphi_1)-f(t,\varphi_2)\| \le L_f\|\varphi_1-\varphi_2\|_{\mathcal{B}_\alpha} \text{ for all } t\in\mathbb{R}$$

$\varphi_1,\varphi_2\in\mathcal{B}_\alpha$ and L_f satisfies (5.3) and (5.4).

Theorem 6.5. *Assume that $\mathcal{B}_\alpha$ satisfies ($\boldsymbol{A}_1$), ($\boldsymbol{A}_2$), ($\boldsymbol{B}$), ($\boldsymbol{C}_1$), ($\boldsymbol{C}_2$) hold. Moreover suppose that ($\boldsymbol{H}_0$) ($\boldsymbol{H}_1$), ($\boldsymbol{H}_2$), ($\boldsymbol{H}_4$), ($\boldsymbol{H}_6$) and ($\boldsymbol{H}_7$) hold. If*

$$\frac{2C\overline{M}\widetilde{M}|\Pi^s\Gamma(1-\alpha)}{\omega^{1-\alpha}}<1,$$

then equation (6.2) has a unique compact $\alpha-cl(\mu,\nu)$-pseudo almost periodic solution of infinite class. Where $C=\max\big\{\sup_{t\in\mathbb{R}}|M(t)|,\sup_{t\in\mathbb{R}}|K(t)|\big\}$,

Proof. Let x be a function in $PAP(\mathbb{R};X,\mu,\nu,\infty)$, from Theorem 5.15, the function $t\to x_t$ belongs to $PAP(\mathcal{B}_\alpha;\mu,\nu,\infty)$. Hence Theorem 5.12 implies that the function $g(.):=f(.,x)$ is in $PAP(\mathbb{R};X,\mu,\nu,\infty)$. Since the unstable space $U\equiv\{0\}$, then $\Pi^u=0$. Consider the mapping

$$\mathcal{H}:PAP(\mathbb{R};X_\alpha,\mu,\nu,\infty)\to PAP(\mathbb{R};X_\alpha,\mu,\nu,\infty)$$

defined for $t\in\mathbb{R}$ by

$$(\mathcal{H}x)(t)=\Big[\lim_{\tau\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s,x_s))ds.$$

From the Proposition 6.1, Proposition 6.2 and taking into account Theorem 6.3, it suffices now to show that the the operator $\mathcal{H}$ has fixed point in $PAP(\mathbb{R};X_\alpha,\mu,\nu,\infty)$.
Let $C=\max\big\{\sup_{t\in\mathbb{R}}|M(t)|,\sup_{t\in\mathbb{R}}|K(t)|\big\}$ and $x_1,x_2\in PAP(\mathbb{R};X_\alpha,\mu,\nu,\infty)$. Then we have

$$\begin{aligned}
|(\mathcal{H}x_1)(t)-(\mathcal{H}x_2)(t)|_\alpha &\le \Big|\lim_{\tau\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s\big(\widetilde{B}X_0\big[f(s,x_{1s})-f(s,x_{2s})\big]\big)ds\Big|_\alpha\\
&\le \overline{M}\widetilde{M}|\Pi^s|\int_{-\infty}^{t}\frac{e^{-\omega(t-s)}}{(t-s)^\alpha}L_f\|x_{1s}-x_{2s}\|_{\mathcal{B}_\alpha}ds\\
&\le \overline{M}\widetilde{M}\Big[|\Pi^s|\int_{-\infty}^{t}\frac{e^{-\omega(t-s)}}{(t-s)^\alpha}L_f\Big(K(s)\sup_{0\le\xi\le s}|x_1(\xi)-x_2(\xi)|_\alpha\\
&\quad +M(s)\|x_{1_0}-x_{2_0}\|_{\mathcal{B}_\alpha}\Big)ds\\
&\le \frac{2C\overline{M}\widetilde{M}|\Pi^s|L_f}{\omega^{1-\alpha}}\Big(\int_0^{+\infty}e^{-s}s^{-\alpha}ds\Big)\|x_1-x_2\|_{\mathcal{B}_\alpha}\\
&\le \frac{2C\overline{M}\widetilde{M}|\Pi^s|L_f\Gamma(1-\alpha)}{\omega^{1-\alpha}}\|x_1-x_2\|_{\mathcal{B}_\alpha}.
\end{aligned}$$

This means that $\mathcal{H}$ is a strict contraction. Thus by Banach's fixed point theorem, $\mathcal{H}$ has a unique fixed point u in $PAP(\mathbb{R};X_\alpha,\mu,\nu,\infty)$. We conclude that equation (6.2), has one and only one α-$cl(\mu,\nu)$-pseudo almost periodic solution of infinite class. □

Proposition 6.6. *Assume that $\mathcal{B}_\alpha$ satisfies $(\boldsymbol{A}_1)$, $(\boldsymbol{A}_1)$,$(\boldsymbol{B})$, $(\boldsymbol{C}_1)$, $(\boldsymbol{C}_2)$ hold. Suppose that $(\boldsymbol{H}_0)$, $(\boldsymbol{H}_1)$, $(\boldsymbol{H}_2)$, $(\boldsymbol{H}_4)$, $(\boldsymbol{H}_6)$ and hold and f is lipschitz continuous with the respect of the second argument. If*

$$Lip(f) < \frac{\omega^{1-\alpha}}{2C\overline{M}\widetilde{M}|\Pi^s|L_f\Gamma(1-\alpha)},$$

then equation (6.2) has a unique compact α-$cl(\mu,\nu)$-pseudo almost automorphic solution of infinite class.

7. Application

For illustration, we propose to study the existence of solutions for the following model

$$(7.1)\qquad \begin{cases} \dfrac{\partial}{\partial t}z(t,x) = \dfrac{\partial^2}{\partial x^2}z(t,x) + \displaystyle\int_{-\infty}^{0} G(\theta)z(t+\theta,x)d\theta + x\big(\sin(t)+\sin\big(\sqrt{2}t\big)\big) + \arctan(t) \\ \qquad\qquad + \displaystyle\int_{-\infty}^{0} h\Big(t, \frac{\partial}{\partial x}z(t+\theta,x)\Big)d\theta \ \text{ for } t\in\mathbb{R}, \text{ and } x\in[0,\pi], \\ z(t,0) = z(t,\pi) = 0 \text{ for } t\in\mathbb{R}, \text{ and } x\in[0,\pi], \end{cases}$$

where $G :]-\infty,0] \to \mathbb{R}$ is continuous function and $h : \mathbb{R}^- \times \mathbb{R} \to$ is lipschitz continuous with the respect of the second argument. To rewrite (7.1) in abstract form, we introduce the space $X = L^2([0,\pi];\mathbb{R})$ vanishing at 0 and π, equipped with the L^2 norm that is to say for all $x \in X$,

$$\|x\|_{L^2} = \Big(\int_0^\pi |x(s)|^2 ds\Big)^{\frac{1}{2}}.$$

Let $A : X \to X$ be defined by

$$\begin{cases} D(A) = H^2(0,\pi)\cap H_0^1(0,\pi) \\ Ay = -y''. \end{cases}$$

Then the spectrum $\sigma(A)$ of A equals to the point spectrum $\sigma_p(A)$ and is given by

$$\sigma(A) = \sigma_p(A) = \{-n^2 :\ n \geq 1\}$$

and the associated eigenfunctions $(e_n)_{n\geq 1}$ are given by

$$e_n(s) = \sqrt{\frac{2}{\pi}}\sin(ns),\ s\in[0,\pi].$$

Then the operator is computed by

$$Ay = \sum_{n=1}^{+\infty} n^2(y,e_n)e_n,\ y\in D(A).$$

For each $y \in D(A^{\frac{1}{2}}) = \{y \in X : \sum_{n=1}^{+\infty} n(y,e_n)e_n \in X\}$, the operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}}y = \sum_{n=1}^{+\infty} n(y,e_n)e_n,\ y\in D(A).$$

Lemma 7.1. *[18] If $y \in D(A^{\frac{1}{2}})$, then y is absolutely continuous, $y' \in X$ and*

$$\|y\| = \|y'\| = \|A^{\frac{1}{2}}y\|.$$

It is well known that $-A$ is the generator of a compact analytic semigroup semigroup $(T(t))_{t\geq 0}$ on X which is given by

$$T(t)x = \sum_{n=1}^{+\infty} e^{-n^2t}(x, e_n)e_n, \ x \in X.$$

Then $(\mathbf{H}_0)$ and $(\mathbf{H}_2)$ are satisfies.

Let $\gamma > 0$, we define the phase space.

$$\mathcal{B} = C_\gamma = \{\varphi \in C(] - \infty, 0]; X) : \lim_{\theta \to -\infty} e^{\gamma\theta}\varphi(\theta) \text{ exist in } X\},$$

with the norm

$$\|\varphi\|_\gamma = e^{\gamma\theta} \sup_{\theta \leq 0} \|\varphi(\theta)\|, \text{ for } \varphi \in C_\gamma$$

From [14] this space satisfies axioms $(\mathbf{A}_1)$, $(\mathbf{A}_2)$ $(\mathbf{B})$, $(\mathbf{C}_1)$, $(\mathbf{C}_2)$ and $(\mathbf{C}_3)$. Moreover it is a uniform fading memory space, which implies that $(\mathbf{H}_3)$ is satisfied.

We choose $\alpha = \dfrac{1}{2}$. The norm in $\mathcal{B}_{\frac{1}{2}}$ is given by

$$\|\varphi\|_{\mathcal{B}_{\frac{1}{2}}} = e^{\gamma\theta} \sup_{\theta \leq 0} \|A^{\frac{1}{2}}\varphi(\theta)\| = e^{\gamma\theta} \sup_{\theta \leq 0} \sqrt{\int_0^\pi \Big(\frac{\partial}{\partial x}(\varphi)(\theta)(x)\Big)^2 dx}.$$

Moreover $A^{-\frac{1}{2}}\varphi \in \mathcal{B}$, for $\varphi \in \mathcal{B}$, then $(\mathbf{H}_1)$ is satisfies.

Let $h(t) = e^t$ for $t \in \mathbb{R}^-$ and define $|\varphi|_{\mathcal{B}_{h_\alpha}}$ by

$$|\varphi|_{\mathcal{B}_{h_\alpha}} = \int_{-\infty}^0 h(s) \sup_{s \leq \theta \leq 0} |\varphi(\theta)| ds.$$

We define $f : \mathbb{R} \times \mathcal{B}_{\frac{1}{2}} \to X$ and $L : \mathcal{B}_{\frac{1}{2}} \to X$ as follows

$$f(t, \varphi)(x) = x\big(\sin(t) + \sin\big(\sqrt{2}t\big)\big) + \arctan(t) + \int_{-\infty}^0 h(t, \frac{\partial}{\partial x} z(t+\theta, x)\Big) d\theta, \quad \text{for } t \in \mathbb{R}, \ x \in [0, \pi]$$

$$L(\varphi)(x) = \int_{-\infty}^0 G(\theta)\varphi(\theta)(x) d\theta, \text{ for } \theta \leq 0.$$

$(\mathbf{H}_8)$ $e^{-2\gamma}h \in L^2(\mathbb{R}^-)$.

Lemma 7.2. *[7] Assume that $(\boldsymbol{H}_8)$ holds. Then L is a bounded operator from $\mathcal{B}_{\frac{1}{2}}$ to X.*

Let us pose $v(t) = z(t, x)$. Then equation (7.1) takes the following abstract form

$$v'(t) = -Av(t) + L(v_t) + f(t, v_t) \text{ for } t \in \mathbb{R}. \tag{7.2}$$

Consider the measure μ and ν where its Randon-Nikodym derivates are respectively ρ_1 and ρ_2

$$\rho_1(t) = \begin{cases} 1 & \text{for } t > 0 \\ e^t & \text{for } t \leq 0 \end{cases}$$

and

$$\rho_2(t) = |t| \text{ for } t \in \mathbb{R}$$

i.e $d\mu(t) = \rho_1(t)dt$ and $d\mu(t) = \rho_2(t)dt$, where dt denotes the Lebesgue measure on $\mathbb{R}$ and

$$\mu(A) = \int_A \rho_1(t)dt \text{ for } \nu(A) = \rho_2(t)dt \text{ for } A \in \mathcal{N}.$$

From [3] $\mu, \nu \in \mathcal{M}$ satisfies Hypothesis $(\mathbf{H}_4)$.

$$\lim_{\tau\to+\infty} \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} = \limsup_{\tau\to+\infty} \frac{\displaystyle\int_{-r}^{0} e^t dt + \int_0^\tau dt}{\displaystyle 2\int_0^\tau tdt} = \limsup_{\tau\to+\infty} \frac{1+e^{-\tau}+\tau}{\tau^2} = 0 < \infty,$$

which implies that $(\mathbf{H}_3)$ is satisfied.

Since $A^{\frac{1}{2}}\Big(x\big(\sin(t)+\sin\big(\sqrt{2}t\big)\Big) = \sin(t)+\sin\big(\sqrt{2}t\big)$ and the function $t \mapsto \sin(t)+\sin\big(\sqrt{2}t\big)$ belongs to $AP(\mathbb{R};X)$, it follows that the function $t \mapsto x\Big(\sin(t)+\sin\big(\sqrt{2}t\big)\Big)$ belongs to $AP(\mathbb{R};X_{\frac{1}{2}})$.

On other hand, we have the following

$$\begin{aligned}
\frac{1}{\nu([-r,r])}\int_{-\tau}^{\tau} \sup_{\theta\in]-\infty,0]} |\arctan(\theta)|_{\frac{1}{2}} dt &= \frac{1}{\nu([-r,r])}\int_{-\tau}^{\tau} \sup_{\theta\in]-\infty,0]} |A^{\frac{1}{2}}\arctan(\theta)|dt \\
&= \frac{1}{\nu([-r,r])}\int_{-\tau}^{\tau} \sup_{\theta\in]-\infty,0]} |\frac{1}{1+\theta^2}|dt \\
&\le \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} \to 0 \text{ as } \tau\to\infty
\end{aligned}$$

It follows that $t \to \arctan t$ is (μ,ν)-ergodic of infinite class consequently, f is uniformly (μ,ν)-pseudo almost periodic of infinite class.

For every $\varphi_1, \varphi_2 \in \mathcal{B}_{\frac{1}{2}}$ and $t \ge 0$, we have

$$\begin{aligned}
\|f(t,\phi_1)(x) - f(t,\phi_2)(x)\|^2 &= \int_0^\pi \Big(\int_{-\infty}^{\theta} |h(\theta)\Big[\frac{\partial}{\partial x}\varphi_1(\theta)(x) - \frac{\partial}{\partial x}\varphi_2(\theta)(x)|^2 d\theta\Big)dx \\
&\le \int_0^\pi \Big(\int_{-\infty}^{\theta} |e^{-2\gamma\theta}e^{2\gamma\theta}h(\theta)\Big[\frac{\partial}{\partial x}\varphi_1(\theta)(x) - \frac{\partial}{\partial x}\varphi_2(\theta)(x)|^2 d\theta\Big]\Big)dx \\
&\le \int_0^\pi \Big[\Big(\int_{-\infty}^{0} e^{-4\gamma\theta}h^2(\theta)d\theta\Big)\Big(\int_{-\infty}^{0} e^{4\gamma\theta}\Big|\frac{\partial}{\partial x}\varphi_1(\theta)(x) - \frac{\partial}{\partial x}\varphi_2(\theta)(x)|^2 d\theta\Big|\Big)dx \\
&\le \Big(\int_{-\infty}^{0} e^{-4\gamma\theta}h^2(\theta)d\theta\Big)\sup_{\theta\le 0}\Big(\int_0^\pi e^{2\gamma\theta}\Big|\frac{\partial}{\partial x}\varphi_1(\theta)(x) - \frac{\partial}{\partial x}\varphi_2(\theta)(x)|^2 dx\Big)\Big(\int_{-\infty}^{0} e^{2\gamma\theta}d\theta\Big) \\
&\le \frac{1}{2\mu}\Big(\int_{-\infty}^{0} e^{-4\gamma\theta}h^2(\theta)dx\Big)\sup_{\theta\le 0}\Big(\int_0^\pi e^{2\gamma\theta}\Big|\frac{\partial}{\partial x}\varphi_1(\theta)(x) - \frac{\partial}{\partial x}\varphi_2(\theta)(x)|^2 dx\Big).
\end{aligned}$$

By Lemma (7.1), we have

$$\begin{aligned}
\|f(t,\phi_1)(x) - f(t,\phi_2)(x)\|^2 &\le \frac{1}{2\mu}\Big(\int_{-\infty}^{0} e^{-4\gamma\theta}h^2(\theta)d\theta\Big)\sup_{\theta\le 0}\Big(\int_0^\pi e^{2\gamma\theta}\Big\|A^{\frac{1}{2}}(\varphi_1(\theta)(x) - \varphi_2(\theta)(x))\Big\|^2 dx\Big) \\
&\le \frac{1}{2\mu}\Big(\int_{-\infty}^{0} e^{-4\gamma\theta}h^2(\theta)d\theta\Big)\|\varphi_1 - \varphi_2\|^2_{\mathcal{B}_{\frac{1}{2}}}.
\end{aligned}$$

Consequently, we conclude that f is Lipschitz continuous and $cl(\mu,\nu)$-pseudo almost periodic of infinite class. . Then $(\mathbf{H}_7)$ is satisfied.

Lemma 7.3. *[13] If* $\int_{-\infty}^{0} |G(\theta)|d\theta < 1$, *then the seimigroup* $(\mathcal{U}(t))_{t\geq 0}$ *is hyperbolic.*

Proposition 7.4. *Under the above assumptions equation* (7.2) *has a unique* α-$cl(\mu,\nu)$*-pseudo almost periodic solution* v *of infinite class.*

Conclusion

In this work we have studied the existence of compact α-$cl(\mu,\nu)$-pseudo almost periodic solution of infinite class. In next work, we will present α-$cl(\mu,\nu)$-S^p pseudo almost periodic of infinite class.

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