DOI: 10.28919/ejma.2024.4.8

Eur. J. Math. Appl. (2024)4:8
URL: http://ejma.euap.org

RANDOM SOLUTIONS FOR IMPROVED CONFORMABLE FRACTIONAL DIFFERENTIAL EQUATIONS WITH RETARDATION AND ANTICIPATION

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Abstract. In this article, we present some results on existence, uniqueness, and Ulam-Hyers-Rassias stability for a class of nonlinear improved conformable fractional differential equations with Retardation and Anticipation. Our reasoning is based on some relevant fixed point theorems.

1. Introduction

The fractional calculus has long been an intriguing study topic in functional space theory due to its applications in the modeling and physical understanding of natural phenomenon. Indeed, various applications in viscoelasticity and electrochemistry have been explored. Noninteger derivatives of fractional order have been effectively applied to generalize the fundamental laws of nature. For more details, we recommend [1, 3–5, 16, 20], and its references. More details on differential equations and the different used methods to solve differential problems, see [6, 7, 18, 19].

Recently in [13], Khalil *et al.* gave a novel definition of fractional derivative which is a natural extension to the standard first derivative. The conformable fractional derivative is natural and it fulfills most of the properties that the classical integral derivative has such as product rule, quotient rule, linearity, chain rule, power rule and when used to modeling various physical problems, it brings us a lot of convenience. Indeed, since that time so many articles have been written and numerous equations have been solved with that concept [2, 8, 14, 15, 17].

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Key words and phrases. Random implicit differential equations; Riemann-Liouville fractional derivative; Caputo fractional derivative; fractional differential equations; improved conformable fractional derivative; existence; random solution; Ulam-Hyers-Rassias stability; retarded arguments; advanced arguments; fixed point.

Received 13/02/2024.

Very recently in [10], F. Gao and C. Chi claimed that there are still shortcomings for the conformable derivative and in order to overcome this difficulty, they proposed an improved conformable fractional derivative. The benefit of the improved conformable fractional derivative is that its physical behavior is closer than the conformable fractional derivative of Riemann-Liouville and Caputo fractional derivative. This improved conformable fractional derivative has a great potential to simulate various physical problems that typically employ the fractional derivative of Riemann-Liouville or Caputo type.

In this paper, we study the existence of random solutions for the initial value problem with nonlinear implicit fractional differential equation involving the improved Caputo-type conformable fractional derivative with retarded and advanced arguments:

$$ {}_0^C\tilde{\mathcal{T}}_\vartheta y(t,\alpha) = f\left(t, y^t(\cdot,\alpha), {}_0^C\tilde{\mathcal{T}}_\vartheta y(t,\alpha), \alpha\right), \ t \in [0,T], \tag{1} $$

$$ y(t,\alpha) = \xi(t,\alpha), \ t \in [-r,0], \ r > 0, \tag{2} $$

$$ y(t,\alpha) = \tilde{\xi}(t,\alpha), \ t \in [T, T+\delta], \ \delta > 0, \tag{3} $$

where $0 < \vartheta < 1$, ${}_0^C\tilde{\mathcal{T}}_\vartheta$ is the improved Caputo-type conformable fractional derivative defined in [10], $I := [0,T]$, $f : I \times C([-r,\delta],\mathbb{R}) \times \mathbb{R} \times \Psi \to \mathbb{R}$ is a given function, $\tilde{\xi} \in C([T,T+\delta],\mathbb{R})$ and $\xi \in C([-r,0],\mathbb{R})$ with $\xi(0) = 0$, Ψ is the sample space in a probability space and α is a random variable. By y^t, we denote the element of $C([-r,\delta])$ defined by:

$$ y^t(s,\alpha) = y(t+s,\alpha), \quad s \in [-r,\delta], \ \alpha \in \Psi. $$

This paper has the following structure: Section 2 presents certain notations and preliminaries about the improved conformable fractional derivatives used throughout this manuscript. In Section 3, we present two existence and uniqueness results for the problem (1)-(3) that are based on Schauder fixed point theorem and Banach contraction principle. Section 4 deals with the Ulam stability of our problem. In the last section, the results obtained are supported by illustrated examples.

2. Preliminaries

First, we give the definitions and the notations used in this paper. We denote by $C([-r,\delta],\mathbb{R})$ the Banach space of all continuous functions from $[-r,\delta]$ into $\mathbb{R}$ with the following norm

$$ \|y\|_{[-r,\delta]} = \sup_{-r \le t \le \delta} \{|y(t)|\}. $$

$AC(I,\mathbb{R})$ is the space of absolutely continuous functions on I, and

$$ AC^1(I) := \{y : I \longrightarrow \mathbb{R} : y' \in AC(I)\}, $$

where

$$ y'(t) = t\frac{d}{dt}y(t), \quad t \in I. $$

Also, define the following space:

$$ \mathcal{C} = \Big\{ u : [-r, T+\delta] \longmapsto \mathbb{R} : u\,|_{[-r,0]} \in C([-r,0]), u\,|_{[0,T]} \in AC^1([0,T]) $$

$$\text{and } u\,|_{[T,T+\delta]}\in C([T,T+\delta])\Bigg\},$$

with the norm

$$\|u\|_{\mathcal{C}} = \sup\{|u(t)| : -r \leq t \leq T+\delta\}.$$

Consider the space $X_b^p(0,T)$, $(b \in \mathbb{R},\ 1 \leq p \leq \infty)$ of those complex-valued Lebesgue measurable functions f on $[0,T]$ for which $\|f\|_{X_b^p} < \infty$, with:

$$\|f\|_{X_b^p} = \left(\int_0^T |t^b f(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}}, \quad (1 \leq p < \infty, b \in \mathbb{R}).$$

Definition 2.1. *([13]) The conformable fractional derivative of a given function $\psi : [0,+\infty) \longrightarrow \mathbb{R}$ of order ϑ is defined by*

$$\mathcal{T}_\vartheta(\psi)(t) = \lim_{\varepsilon\to 0} \frac{\psi\left(t+\varepsilon t^{1-\vartheta}\right) - \psi(t)}{\varepsilon},$$

for $t > 0$ and $\vartheta \in (0,1]$. If ψ is ϑ-differentiable in some $(0,a)$, $a > 0$, and $\lim_{t\to 0+} \mathcal{T}_a(\psi)(t)$ exists, then define $\mathcal{T}_\vartheta(\psi)(0) = \lim_{t\to 0+} \mathcal{T}_\vartheta(\psi)(t)$. If the conformable fractional derivative of ψ of order ϑ exists, then we simply say that ψ is ϑ-differentiable. It is easy to see that if ψ is differentiable, then $\mathcal{T}_\vartheta(\psi)(t) = t^{1-\vartheta}\psi'(t)$.

Definition 2.2. *(The improved Caputo-type conformable fractional derivative [10]) The improved Caputo-type conformable fractional derivative of a given function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ of order ϑ is defined by*

$${}_a^C\tilde{\mathcal{T}}_\vartheta(\psi)(t) = \lim_{\varepsilon\to 0}\left[(1-\vartheta)(\psi(t)-\psi(a)) + \vartheta\frac{\psi\left(t+\varepsilon(t-a)^{1-\vartheta}\right)-\psi(t)}{\varepsilon}\right],$$

where $-\infty < a < t < +\infty, a$ is a given number and $\vartheta \in [0,1]$.

Definition 2.3. *(The improved Riemann-Liouville-type conformable fractional derivative [10]) The improved Riemann-Liouville-type conformable fractional derivative of a given function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ of order ϑ is defined by*

$${}_a^{RL}\tilde{\mathcal{T}}_\vartheta(\psi)(t) = \lim_{\varepsilon\to 0}\left[(1-\vartheta)\psi(t) + \vartheta\frac{\psi\left(t+\varepsilon(t-a)^{1-\vartheta}\right)-\psi(t)}{\varepsilon}\right],$$

where $-\infty < a < t < +\infty, a$ is a given number and $\vartheta \in [0,1]$.

Lemma 2.4. *([10]) If $\vartheta \in [0,1]$, f and g are two ϑ-differentiable functions at a point t and m, n are two given numbers, then the improved conformable fractional derivatives satisfy the following properties:*

- ${}_a^C\tilde{\mathcal{T}}_\vartheta(mf+ng) = m\,{}_a^C\tilde{\mathcal{T}}_\vartheta(f) + n\,{}_a^C\tilde{\mathcal{T}}_\vartheta(g)$;
- ${}_a^{RL}\tilde{\mathcal{T}}_\vartheta(mf+ng) = m\,{}_a^{RL}\tilde{\mathcal{T}}_\vartheta(f) + n\,{}_a^{RL}\tilde{\mathcal{T}}_\vartheta(g)$;
- ${}_a^{RL}\tilde{\mathcal{T}}_\vartheta(fg) = (1-\vartheta)\,{}_a^{RL}\tilde{\mathcal{T}}_\vartheta(f)g + f\,{}_a^{RL}\tilde{\mathcal{T}}_\vartheta(g) - (1-\vartheta)fg$;
- ${}_a^{RL}\tilde{\mathcal{T}}_\vartheta(f(g(t))) = (1-\vartheta)f(g(t)) + \vartheta f'(g(t))\mathcal{T}_\vartheta(g(t))$.

Definition 2.5. *(The ϑ-fractional integral [10]) For $\vartheta \in (0,1]$ and a continuous function f, let*

$$\left(\mathcal{I}_\vartheta f\right)(t) = \frac{1}{\vartheta}\int_0^t \frac{f(s)}{s^{1-\vartheta}} e^{\left(1-\vartheta/\vartheta^2\right)\left(s^\vartheta - t^\vartheta\right)} ds.$$

When $\vartheta = 1, \mathcal{I}_1(f) = \int_0^t f(s)ds$, the usual Riemann integral.

Lemma 2.6. *([10]) If $\vartheta \in [0,1]$, ψ is ϑ-differentiable function at a point t and $\psi(0)=0$, then we have:*

- $\left(\mathcal{I}_{\vartheta}{}_0^C\tilde{\mathcal{T}}_{\vartheta}(\psi)\right)(t) = {}_0^C\tilde{\mathcal{T}}_{\vartheta}\left(\mathcal{I}_{\vartheta}\psi\right)(t) = \psi(t)$;
- $\left(\mathcal{I}_{\vartheta}{}_0^{RL}\tilde{\mathcal{T}}_{\vartheta}(\psi)\right)(t) = {}_0^{RL}\tilde{\mathcal{T}}_{\vartheta}\left(\mathcal{I}_{\vartheta}\psi\right)(t) = \psi(t)$.

By $B_{\mathbb{R}}$, we denote the σ-algebra of Borel subsets of $\mathbb{R}$. A mapping $\alpha : \Psi \to \mathbb{R}$ is said to be measurable if for any $D \in B_{\mathbb{R}^m}$, one has

$$\alpha^{-1}(D) = \{y \in \Psi : \alpha(y) \in D\} \subset \mathcal{A}.$$

Definition 2.7. *A mapping $N : \Psi \times \mathbb{R} \to \mathbb{R}$ is called jointly measurable if for any $D \in B_{\mathbb{R}^m}$, one has*

$$N^{-1}(D) = \{(y,x) \in \Psi \times \mathbb{R} : N(y,x) \in D\} \subset \mathcal{A} \times B_{\mathbb{R}},$$

where $\mathcal{A} \times B_{\mathbb{R}}$ is the product of the σ-algebras $\mathcal{A}$ defined in Ψ and $B_{\mathbb{R}}$.

Definition 2.8. *A function $N : \Psi \times \mathbb{R} \to \mathbb{R}$ is called jointly measurable if $N(\cdot, x)$ is measurable for all $x \in \mathbb{R}$ and $N(y, \cdot)$ is continuous for all $y \in \Psi$.*

N is called a random operator if $N(y,x)$ is measurable in y for all $x \in \mathbb{R}$, and it expressed as $N(y)x = N(y,x)$. We also say in this situation that $N(y)$ is a random operator on $\mathbb{R}$. $N(y)$ is called continuous (resp. completely continuous, compact and totally bounded) if $N(y,x)$ is continuous (resp. completely continuous, compact and totally bounded) in x for all $y \in \Psi$. The details and the properties of completely continuous random operators in Banach spaces are available in Itoh [12].

Definition 2.9. *([9]) Let $\mathcal{D}(X)$ be the family of all nonempty subsets of X and F be a mapping from Ψ into $\mathcal{D}(X)$. A mapping $N : \{(y,x) : y \in \Psi,\ x \in F(y)\} \to X$ is called random operator with stochastic domain F, if F is measurable (i.e., for all closed $B \subset X$, $\{y \in \Psi, F(y) \cap B \neq \emptyset\}$ is measurable) and for all open $D \subset X$ and all $x \in X$, $\{y \in \Psi : x \in F(y), N(y,x) \in D\}$ is measurable. N will be called continuous if every $N(y)$ is continuous. For a random operator N, a mapping $x : \Psi \to X$ is called a random (stochastic) fixed point of N if for $P-$almost all $y \in \Psi$, $x(y) \in F(y)$ and $N(y)x(y) = x(y)$, and for all open $D \subset X$, $\{y \in \Psi : x(y) \in D\}$ is measurable.*

Definition 2.10. *A function $g : I \times C([-r,\delta],\mathbb{R}) \times \mathbb{R} \times \Psi \to \mathbb{R}$ is called random Carathéodory if the following conditions are met:*

(i) *The map $(s,y) \to g(s,x,\bar{x},y)$ is jointly measurable for all $x \in C([-r,\delta],\mathbb{R})$ and $\bar{x} \in \mathbb{R}$;*
(ii) *The map $(x,\bar{x}) \to g(s,x,\bar{x},y)$ is continuous for all $s \in I$ and $y \in \Psi$.*

Theorem 2.11. *([12]) Let Y be a nonempty, closed convex bounded subset of the separable Banach space E and let $T : \Psi \times Y \longmapsto Y$ be a compact and continuous random operator. Then the random equation $T(y, x(y)) = x(y)$ has a random solution.*

Considering now the Ulam stability for problem (1)-(3). Let $x(\cdot,\alpha) \in \mathcal{C}$, $\epsilon > 0$ and $v : I \times \Psi \longmapsto [0,\infty)$ be a jointly measurable function. For $t \in I$, we have the following inequality:

$$\left|{}_0^C\tilde{\mathcal{T}}_{\vartheta}y(t,\alpha) - f\left(t, y^t(\cdot,\alpha), {}_0^C\tilde{\mathcal{T}}_{\vartheta}y(t,\alpha), \alpha\right)\right| \leq \epsilon v(t,\alpha). \tag{4}$$

Definition 2.12. *([4]) Problem (1)-(3) is Ulam-Hyers-Rassias (U-H-R) stable with respect to v if there exists a real number $a_{f,v} > 0$ such that for each $\epsilon > 0$ and for each solution $x(\cdot,\alpha) \in \mathcal{C}$ of inequality (4) there exists a solution $y(\cdot,\alpha) \in \mathcal{C}$ of (1)-(3) with*

$$\begin{cases} |x(t,\alpha) - y(t,\alpha)| \leq \epsilon a_{f,v} v(t,\alpha), \qquad t \in I, \ \alpha \in \Psi \\ |x(t,\alpha) - y(t,\alpha)| = 0, \qquad t \in [-r,0] \cup [T, T+\delta], \ \alpha \in \Psi. \end{cases}$$

Remark 2.13. *A function $x(\cdot,\alpha) \in \mathcal{C}$ is a solution of inequality (4) if and only if there exist $\sigma(\cdot,\alpha) \in \mathcal{C}$ such that*

(1) $|\sigma(t,\alpha)| \leq \epsilon v(t,\alpha)$, $t \in I$,
(2) ${}_0^C\tilde{\mathcal{T}}_\vartheta x(t,\alpha) = f\left(t, x^t(\cdot,\alpha), {}_0^C\tilde{\mathcal{T}}_\vartheta x(t,\alpha), \alpha\right) + \sigma(t,\alpha)$.

3. Existence of Solutions

Lemma 3.1. *Let $0 < \vartheta < 1$, $\tilde{\xi} \in C([T, T+\delta], \mathbb{R})$ and $\xi \in C([-r,0],\mathbb{R})$ with $\xi(0) = 0$, and $h : I \to \mathbb{R}$ be a continuous function. Then problem*

$$ {}_0^C\tilde{\mathcal{T}}_\vartheta y(t) = h(t), \quad t \in I := [0,T], \tag{5}$$

$$ y(t) = \xi(t), \ t \in [-r,0], \ r > 0, \tag{6}$$

$$ y(t) = \tilde{\xi}(t), \ t \in [T, T+\delta], \ \delta > 0, \tag{7}$$

has the following solution

$$ y(t) = \begin{cases} \xi(t), \ \text{if} \quad t \in [-r,0], \\ \dfrac{1}{\vartheta}\displaystyle\int_0^t \frac{h(s)}{s^{1-\vartheta}} e^{\frac{(1-\vartheta)(s^\vartheta - t^\vartheta)}{\vartheta^2}} ds, \ \text{if} \quad t \in I, \\ \tilde{\xi}(t), \ \text{if} \quad t \in [T, T+\delta]. \end{cases} \tag{8}$$

Proof. To obtain the integral equation (8), we apply the ϑ-fractional integral to both sides of (5), and by Lemma 2.6 we get

$$ y(t) = \frac{1}{\vartheta}\int_0^t \frac{h(s)}{s^{1-\vartheta}} e^{\frac{(1-\vartheta)(s^\vartheta - t^\vartheta)}{\vartheta^2}} ds. \tag{9}$$

Now, we apply the improved Caputo-type conformable fractional derivative of order ϑ to both sides of (9), for $t \in I$ we obtain

$$ {}_0^C\tilde{\mathcal{T}}_\vartheta y(t) = h(t).$$

Also, it is clear that $y(0) = 0$. □

Lemma 3.2. *A function $y(\cdot,\alpha) \in \mathcal{C}$ is random solution of problem (1)-(3) if and only if y satisfies the following integral equation*

$$y(t,\alpha) = \begin{cases} \xi(t,\alpha), & \text{if} \quad t \in [-r,0], \\ \dfrac{1}{\vartheta}\displaystyle\int_0^t s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^{\vartheta}-t^{\vartheta})}{\vartheta^2}} f\left(s, y^s(\cdot,\alpha), {}_0^C\tilde{\mathcal{T}}_{\vartheta} y(s,\alpha), \alpha\right) ds, & \text{if} \quad t \in I, \\ \tilde{\xi}(t,\alpha), & \text{if} \quad t \in [T, T+\delta]. \end{cases}$$

In the sequel, the following hypotheses are used:

(H_1): The function $f : I \times C([-r,\delta],\mathbb{R}) \times \mathbb{R} \times \Psi \longrightarrow \mathbb{R}$, is random Carathéodory.

(H_2): There exist continuous functions $p_1, p_2 : I \longrightarrow L^{\infty}(\Psi, \mathbb{R}_+)$,

$$|f(t,\beta_1,\bar{\beta}_1,\alpha) - f(t,\beta_2,\bar{\beta}_2,\alpha)| \le p_1(t,\alpha)\|\beta_1-\beta_2\|_{[-r,\delta]} + p_2(t,\alpha)|\bar{\beta}_1-\bar{\beta}_2|,$$

for $t \in I$ and $\beta_1, \beta_2 \in C([-r,\delta],\mathbb{R})$, and $\bar{\beta}_1, \bar{\beta}_2 \in \mathbb{R}$, with

$$p_1^*(\alpha) = \sup_{t\in I} p(t,\alpha) \quad \text{and} \quad p_2^*(\alpha) = \sup_{t\in I} p_2(t,\alpha) < 1, \quad \alpha \in \Psi.$$

(H_3): There exist continuous functions $k_1, k_2, k_3 : I \longrightarrow L^{\infty}(\Psi,\mathbb{R}_+)$, such that

$$|f(t,\beta,\bar{\beta},\alpha)| \le k_1(t,\alpha) + k_2(t,\alpha)\frac{\|\beta\|_{[-r,\delta]}}{1+\|\beta\|_{[-r,\delta]}} + k_3(t,\alpha)|\bar{\beta}|,$$

for $t \in I$, $\beta \in C([-r,\delta],\mathbb{R})$, $\bar{\beta} \in \mathbb{R}$ and $\alpha \in \Psi$.
Set

$$k_1^*(\alpha) = \sup_{t\in I} k_1(t,\alpha), \quad k_2^*(\alpha) = \sup_{t\in I} k_2(t,\alpha) \text{ and } k_3^*(\alpha) = \sup_{t\in I} k_3(t,\alpha) < 1.$$

Now we declare and demonstrate our first existence result for problem (1)-(3) based on the Banach contraction principle [11].

Theorem 3.3. *Assume that (H_1) and (H_2) hold. If*

$$\frac{p_1^*(\alpha)\left(1 - e^{\frac{(\vartheta-1)T^{\vartheta}}{\vartheta^2}}\right)}{(1-\vartheta)(1-p_2^*(\alpha))} < 1, \tag{10}$$

then the problem (1)-(3) has a unique solution.

Proof. Let $T : \mathcal{C} \times \Psi \longmapsto \mathcal{C}$ be the operator defined by

$$(Tx)(t,\alpha) = \begin{cases} \xi(t,\alpha), & \text{if} \quad t \in [-r,0], \\ \dfrac{1}{\vartheta}\displaystyle\int_0^t s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^{\vartheta}-t^{\vartheta})}{\vartheta^2}} \varrho(s,\alpha) ds, & \text{if} \quad t \in I, \\ \tilde{\xi}(t,\alpha), & \text{if} \quad t \in [T,T+\delta], \end{cases} \tag{11}$$

where ϱ is a function satisfying the following functional equation

$$\varrho(t,\alpha) = f\left(t, x^t(\cdot,\alpha), \varrho(t,\alpha), \alpha\right).$$

According to Lemma 3.2, the fixed points of T are solutions of problem (1)-(3).
Let $x_1, x_2 \in \mathcal{C}$. If $t \in [-r, 0]$ or $t \in [T, T+\delta]$ then,

$$|(Tx_1)(t,\alpha) - (Tx_2)(t,\alpha)| = 0.$$

For $t \in I$, we have

$$|(Tx_1)(t,\alpha) - (Tx_2)(t,\alpha)| \leq \frac{1}{\vartheta}\int_0^t s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^{\vartheta}-t^{\vartheta})}{\vartheta^2}} |\varrho_1(s,\alpha) - \varrho_2(s,\alpha)| ds, \tag{12}$$

where ϱ_1, ϱ_2 are the functions satisfying the following functional equations

$$\varrho_1(t,\alpha) = f\left(t, x_1^t(\cdot,\alpha), \varrho_1(t,\alpha), \alpha\right),$$
$$\varrho_2(t,\alpha) = f\left(t, x_2^t(\cdot,\alpha), \varrho_2(t,\alpha), \alpha\right).$$

By (H_2), we have

$$\begin{aligned} |\varrho_1(t,\alpha) - \varrho_2(t,\alpha)| &= \left|f\left(t, x_1^t(\cdot,\alpha), \varrho_1(t,\alpha), \alpha\right) - f\left(t, x_2^t(\cdot,\alpha), \varrho_2(t,\alpha), \alpha\right)\right| \\ &\leq p_1(t,\alpha)\|x_1 - x_2\|_{[-r,\delta]} + p_2(t,\alpha)|\varrho_1(t,\alpha) - \varrho_2(t,\alpha)| \\ &\leq p_1^*(\alpha)\|x_1 - x_2\|_{[-r,\delta]} + p_2^*(\alpha)|\varrho_1(t,\alpha) - \varrho_2(t,\alpha)|. \end{aligned}$$

Then,

$$|\varrho_1(t,\alpha) - \varrho_2(t,\alpha)| \leq \frac{p_1^*(\alpha)}{1 - p_2^*(\alpha)}\|x_1 - x_2\|_{[-r,\delta]}.$$

Therefore, for each $t \in I$, we get

$$\begin{aligned} |(Tx_1)(t,\alpha) - (Tx_2)(t,\alpha)| &\leq \frac{1}{\vartheta}\int_0^t s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^{\vartheta}-t^{\vartheta})}{\vartheta^2}} \frac{p_1^*(\alpha)}{1 - p_2^*(\alpha)}\|x_1 - x_2\|_{[-r,\delta]} ds \\ &\leq \left[\frac{1 - e^{\frac{(\vartheta-1)t^{\vartheta}}{\vartheta^2}}}{1-\vartheta}\right] \frac{p_1^*(\alpha)}{1 - p_2^*(\alpha)}\|x_1 - x_2\|_{[-r,\delta]}. \end{aligned}$$

Thus

$$\|Tx_1(\cdot,\alpha) - Tx_2(\cdot,\alpha)\|_{\mathcal{C}} \leq \frac{p_1^*(\alpha)\left(1 - e^{\frac{(\vartheta-1)T^{\vartheta}}{\vartheta^2}}\right)}{(1-\vartheta)(1 - p_2^*(\alpha))}\|x_1 - x_2\|_{\mathcal{C}}.$$

Hence, by the Banach contraction principle, T has a unique fixed point which is a unique random solution of the problem (1)-(3). □

Our second existence result for (1)-(3) is based on the fixed point theorem of Schauder [11].

Theorem 3.4. *Assume that (H_1) and (H_3) hold. Then problem (1)-(3) has at least one solution.*

Proof. We will establish the proof in various steps.
Step 1. T is continuous. Let $\{x_n\}$ be a sequence such that $x_n \longrightarrow x$ in $\mathcal{C}$. If $t \in [-r, 0]$ or $t \in [T, T+\delta]$ then

$$|(Tx_n)(t,\alpha) - (Tx)(t,\alpha)| = 0.$$

For $t \in I$, we have

$$|(Tx_n)(t,\alpha) - (Tx)(t,\alpha)| \leq \frac{1}{\vartheta}\int_0^t s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^{\vartheta}-t^{\vartheta})}{\vartheta^2}} |h_n(s,\alpha) - h(s,\alpha)| ds, \tag{13}$$

where

$$h_n(t,\alpha) = f\left(t, x_n^t(\cdot,\alpha), h_n(t,\alpha), \alpha\right),$$

and

$$h(t,\alpha)=f\left(t,x^{t}(\cdot,\alpha),h(t,\alpha),\alpha\right).$$

Since $x_n \longrightarrow x$, and by (H_1), we get $h_n(t,\alpha)\longrightarrow h(t,\alpha)$ as $n\longrightarrow\infty$ for each $t\in I$. Then by Lebesgue dominated convergence theorem and (H_1), equation (13) implies

$$|(Tx_n)(t,\alpha)-(Tx)(t,\alpha)|\longrightarrow 0 \; as \; n\longrightarrow\infty,$$

and hence

$$\|T(x_n)-T(x)\|_{\mathcal{C}}\longrightarrow 0 \; as \; n\longrightarrow\infty.$$

As a result, T is continuous.

Let the constant $R(\alpha)$ be such that:

$$R(\alpha)\geq\max\left\{\frac{k_1^*(\alpha)\eta}{k_2^*(\alpha)(1-\eta)},\|\xi(\cdot,\alpha)\|_{[-r,0]},\|\tilde{\xi}(\cdot,\alpha)\|_{[T,T+\delta]}\right\}, \tag{14}$$

with

$$\eta=\frac{k_2^*(\alpha)\left(1-e^{\frac{(\vartheta-1)T^{\vartheta}}{\vartheta^2}}\right)}{(1-k_3^*(\alpha))(1-\vartheta)}<1.$$

And, we define the following ball

$$B_{R(\alpha)}=\{y\in\Psi:\|y(\cdot,\alpha)\|_{\mathcal{C}}\leq R(\alpha)\}.$$

Then, $B_{R(\alpha)}$ is a convex, closed and bounded subset of $\mathcal{C}$.

Step 2. $T(B_{R(\alpha)})\subset B_{R(\alpha)}$.

Let $x\in B_{R(\alpha)}$ we show that $Tx\in B_{R(\alpha)}$.
If $t\in[-r,0]$, then

$$|T(x)(t,\alpha)|\leq\|\xi(\cdot,\alpha)\|_{[-r,0]}\leq R(\alpha),$$

and if $t\in[T,T+\delta]$, then

$$|T(x)(t,\alpha)|\leq\|\tilde{\xi}(\cdot,\alpha)\|_{[T,T+\delta]}\leq R(\alpha).$$

For $t\in I$, we have

$$|(Tx)(t,\alpha)|\leq\frac{1}{\vartheta}\int_0^t s^{\vartheta-1}e^{\frac{(1-\vartheta)(s^{\vartheta}-t^{\vartheta})}{\vartheta^2}}\left|f\left(s,y^{s}(\cdot,\alpha),{}_0^C\mathcal{T}_{\vartheta}y(s,\alpha),\alpha\right)\right|ds. \tag{15}$$

By the hypothesis (H_3), for $t\in I$, we have

$$\begin{aligned}|h(t,\alpha)|&=\left|f\left(t,x^{t}(\cdot,\alpha),h(t,\alpha),\alpha\right)\right|\\&\leq k_1(t,\alpha)+k_2(t,\alpha)\|x^{t}(\cdot,\alpha)\|_{[-r,\delta]}+k_3(t,\alpha)|h(t,\alpha)|,\end{aligned}$$

That means that

$$|h(t,\alpha)|\leq k_1^*(\alpha)+k_2^*(\alpha)\|x^{t}(\cdot,\alpha)\|_{[-r,\delta]}+k_3^*(\alpha)|h(t,\alpha)|,$$

then

$$|h(t,\alpha)|\leq\frac{k_1^*(\alpha)+k_2^*(\alpha)R(\alpha)}{1-k_3^*(\alpha)}:=\Lambda.$$

Thus for $t \in I$, from (15) we obtain

$$\begin{aligned} |(Tx)(t,\alpha)| &\leq \frac{\Lambda\left(1-e^{\frac{(\vartheta-1)T^{\vartheta}}{\vartheta^2}}\right)}{1-\vartheta} \\ &\leq R(\alpha), \end{aligned}$$

then, for $t \in [-r, T+\delta]$, we have $|Tx(t,\alpha)| \leq R(\alpha)$, which implies that $\|(Tx)(\cdot,\alpha)\|_{\mathcal{C}} \leq R(\alpha)$. Consequently,

$$T(B_{R(\alpha)}) \subset B_{R(\alpha)}.$$

Step 3: $T(B_{R(\alpha)})$ is equicontinuous and bounded.
By Step 2 we have $T(B_{R(\alpha)})$ is bounded.
Let $\gamma_1, \gamma_2 \in I = [0,T], \gamma_1 < \gamma_2,$ *and* $x(\cdot,\alpha) \in B_{R(\alpha)}$ then

$$\begin{aligned} & |(Tx)(\gamma_2,\alpha) - (Tx)(\gamma_1,\alpha)| \\ \leq & \left|\frac{1}{\vartheta}\int_0^{\gamma_2} s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^{\vartheta}-\gamma_2^{\vartheta})}{\vartheta^2}} h(s,\alpha)\,ds - \frac{1}{\vartheta}\int_0^{\gamma_1} s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^{\vartheta}-\gamma_1^{\vartheta})}{\vartheta^2}} h(s,\alpha)\,ds\right| \\ \leq & \frac{\Lambda}{1-\vartheta}\left[2 - 2e^{\frac{(1-\vartheta)(\gamma_1^{\vartheta}-\gamma_2^{\vartheta})}{\vartheta^2}} + e^{\frac{(\vartheta-1)\gamma_1^{\vartheta}}{\vartheta^2}} - e^{\frac{(\vartheta-1)\gamma_2^{\vartheta}}{\vartheta^2}}\right]. \end{aligned}$$

As $\gamma_1 \longrightarrow \gamma_2$ the right hand side of the above inequality tends to zero. As a result of Step 1 to Step 3, together with the Arzela-Ascoli theorem, We can say that T is continuous and completely continuous. From Schauder's theorem, we conclude that T has a fixed point with is a random solution of the problem (1)-(3). □

4. Ulam-Hyers-Rassias Stability

Theorem 4.1. *Assume that in addition to* (H_1)-(H_3)*, the following hypothesis hold.*

(H_4) *There exist a nondecreasing function* $v(\cdot,\alpha) \in \mathcal{C}$ *and* $\kappa_v > 0$*, such that for* $t \in I$*, we have*

$$\mathcal{I}_{\vartheta} v(t,\alpha) \leq \kappa_v v(t,\alpha).$$

(H_5) *There exists a continuous function* $q : I \longrightarrow L^{\infty}(\Psi, \mathbb{R}_+)$*, such that for* $t \in I$*, we have*

$$\frac{k_1(t,\alpha) + k_2(t,\alpha)}{1 - k_3(t,\alpha)} \leq q(t,\alpha)v(t,\alpha).$$

Then, problem (1)-(3) *is U-H-R stable.*

Set $q^* = \sup_{t\in I} q(t,\alpha).$

Proof. Let $x(\cdot,\alpha) \in \mathcal{C}$ be a solution if inequality (4), and assume that y is the unique solution of the problem

$$\begin{cases} {}^C_0\tilde{\mathcal{T}}_{\vartheta} y(t,\alpha) = f\left(t, y^t(\cdot,\alpha), {}^C_0\tilde{\mathcal{T}}_{\vartheta} y(t,\alpha), \alpha\right), \ t \in I, \\ y(t,\alpha) = x(t,\alpha), \quad t \in [-r,0] \cup [T, T+\delta]. \end{cases}$$

By Lemma 3.2, we obtain

$$y(t,\alpha)=\begin{cases}\xi(t,\alpha), \text{ if } \quad t\in[-r,0],\\[2ex] \dfrac{1}{\vartheta}\displaystyle\int_0^t s^{\vartheta-1}e^{\frac{(1-\vartheta)(s^\vartheta-t^\vartheta)}{\vartheta^2}} f\left(s, y^s(\cdot,\alpha), {}_0^C\tilde{\mathcal{T}}_\vartheta y(s,\alpha),\alpha\right)ds, \text{ if } \quad t\in I,\\[2ex] \tilde{\xi}(t,\alpha), \text{ if } \quad t\in[T,T+\delta].\end{cases}$$

Since x is a solution of the inequality (4), by Remark 2.13, for $t\in I$,we have

$$ {}_0^C\tilde{\mathcal{T}}_\vartheta x(t,\alpha)=f\left(t,x^t(\cdot,\alpha),{}_0^C\tilde{\mathcal{T}}_\vartheta x(t,\alpha),\alpha\right)+\sigma(t,\alpha). \tag{16}$$

Clearly, the solution of (16) is given by

$$x(t,\alpha)=\begin{cases}\xi(t,\alpha), \text{ if } \quad t\in[-r,0],\\[2ex] \dfrac{1}{\vartheta}\displaystyle\int_0^t s^{\vartheta-1}e^{\frac{(1-\vartheta)(s^\vartheta-t^\vartheta)}{\vartheta^2}}\left(f\left(s,x^s(\cdot,\alpha),{}_0^C\tilde{\mathcal{T}}_\vartheta x(s,\alpha),\alpha\right)+\sigma(s,\alpha)\right)ds, \text{ if } \quad t\in I,\\[2ex] \tilde{\xi}(t,\alpha), \text{ if } \quad t\in[T,T+\delta].\end{cases}$$

Hence, for $t\in[-r,0]\cup[T,T+\delta]$, we have

$$|x(t)-y(t)|=0.$$

And, for each $t\in I$, we have

$$\begin{aligned}|x(t)-y(t)| &\leq \frac{1}{\vartheta}\int_0^t s^{\vartheta-1}e^{\frac{(1-\vartheta)(s^\vartheta-t^\vartheta)}{\vartheta^2}}\Big|f\left(s,x^s(\cdot,\alpha),{}_0^C\tilde{\mathcal{T}}_\vartheta x(s,\alpha),\alpha\right)\\ &- f\left(s,y^s(\cdot,\alpha),{}_0^C\tilde{\mathcal{T}}_\vartheta y(s,\alpha),\alpha\right)\Big|ds+\frac{1}{\vartheta}\int_0^t s^{\vartheta-1}e^{\frac{(1-\vartheta)(s^\vartheta-t^\vartheta)}{\vartheta^2}}|\sigma(s,\alpha)|ds.\end{aligned}$$

By the hypothesis (H_3), for $t\in I$, we have

$$\left|f\left(t,x^t(\cdot,\alpha),h(t,\alpha),\alpha\right)\right|\leq k_1(t,\alpha)+k_2(t,\alpha)+k_3(t,\alpha)\left|f\left(t,x^t(\cdot,\alpha),h(t,\alpha),\alpha\right)\right|,$$

which implies that

$$\left|f\left(t,x^t(\cdot,\alpha),h(t,\alpha),\alpha\right)\right|\leq\frac{k_1(t,\alpha)+k_2(t,\alpha)}{1-k_3(t,\alpha)}.$$

Then, for each $t\in I$, we have

$$\begin{aligned}|x(t)-y(t)| &\leq \epsilon\kappa_v v(t,\alpha)+\frac{2}{\vartheta}\int_0^t\frac{k_1(t,\alpha)+k_2(t,\alpha)}{1-k_3(t,\alpha)}s^{\vartheta-1}e^{\frac{(1-\vartheta)(s^\vartheta-t^\vartheta)}{\vartheta^2}}ds\\ &\leq v(t,\alpha)\left(\epsilon\kappa_v+\frac{2q^*\left(1-e^{\frac{(\vartheta-1)T^\vartheta}{\vartheta^2}}\right)}{1-\vartheta}\right).\end{aligned}$$

Then for each $t\in[-r,T+\delta]$, we have

$$|x(t)-y(t)|\leq a_{f,v}\epsilon v(t,\alpha),$$

where

$$a_{f,v} = \kappa_v + \frac{2q^*\left(1 - e^{\frac{(\vartheta-1)T^{\vartheta}}{\vartheta^2}}\right)}{\epsilon(1-\vartheta)}.$$

Hence, problem (1)-(3) is U-H-R stable. □

5. Examples

Example 5.1. *We equip the space $\mathbb{R}_-^* := (-\infty, 0)$ with the standard σ-algebra, which consists of Lebesgue measurable subsets of $\mathbb{R}_-^*$. Now, we consider the following problem involving the improved Caputo-type conformable fractional derivative:*

$$(17)\qquad \begin{cases} x(t,\alpha) = \dfrac{e^t}{1+\alpha^2}, \quad t \in [1, \frac{3}{2}], \\ \\ {}_0^C\tilde{\mathcal{T}}_{\frac{1}{2}} x(t,\alpha) = \dfrac{\cos(t)}{64e^{t+3}(|\alpha|+1)(1+\|x\|_{[-r,\delta]} + |{}_0^C\tilde{\mathcal{T}}_{\frac{1}{2}} x(t,\alpha)|)}, \quad t \in [0,1], \\ \\ x(t,\alpha) = t^2(1+|\alpha|^3), \quad t \in [-\frac{1}{2}, 0]. \end{cases}$$

Set

$$f(t, x^t(\cdot,\alpha), {}_0^C\tilde{\mathcal{T}}_{\frac{1}{2}} x(t,\alpha), \alpha) = \frac{\cos(t)}{64e^{t+3}(|\alpha|+1)(1+\|x\|_{[-r,\delta]} + |{}_0^C\tilde{\mathcal{T}}_{\frac{1}{2}} x(t,\alpha)|)},$$

where $\vartheta = \frac{1}{2}$, $r = \delta = \frac{1}{2}$.
For each $\beta_1, \bar{\beta}_1 \in C([-r,\delta])$, $\beta_2, \bar{\beta}_2 \in \mathbb{R}$ and $t \in [0,1]$, we have

$$|f(t,\beta_1,\beta_2,\alpha) - f(t,\bar{\beta}_1,\bar{\beta}_2,\alpha)| \leq \frac{\cos(t)}{64e^{t+3}(|\alpha|+1)}\left[\|\beta_1 - \bar{\beta}_1\|_{[-r,\delta]} + |\beta_2 - \bar{\beta}_2|\right].$$

Therefore, (H_2) is verified with

$$p_1(t,\alpha) = p_2(t,\alpha) = \frac{\cos(t)}{64e^{t+3}(|\alpha|+1)},$$

and

$$p_1^*(\alpha) = p_2^*(\alpha) = \frac{1}{64e^3(|\alpha|+1)}.$$

Also, for $t \in I$ we have

$$\frac{p_1^*(\alpha)\left(1 - e^{\frac{(\vartheta-1)T^{\vartheta}}{\vartheta^2}}\right)}{(1-\vartheta)(1-p_2^*(\alpha))} = \frac{1-e^{-2}}{32e^3(|\alpha|+1) - \frac{1}{2}} < 1.$$

Then, the condition (10) is satisfied. Hence, as all conditions of Theorem 3.3 are met, therefore, the problem (17) admit a unique solution.

Example 5.2. *Consider the following problem:*

$$(18)\qquad \begin{cases} {}_0^C\tilde{\mathcal{T}}_{\frac{1}{4}} x(t,\alpha) = f(t, x^t(\cdot,\alpha), {}_0^C\tilde{\mathcal{T}}_{\frac{1}{4}} x(t,\alpha), \alpha), \quad t \in I = [0,2], \\ \\ x(t,\alpha) = 0, \quad t \in [-1,0] \cup [2,3], \end{cases}$$

where

$$f(t,x,\bar{x},\alpha)=\frac{1}{32+32e^{2-t}(|\alpha|^3+2)}\left[1+\frac{\|x\|_{[-r,\delta]}}{2+\|x\|_{[-r,\delta]}}+\frac{|\bar{x}|}{3+|\bar{x}|}\right],$$

for $t\in[0,2]$, $x\in C([-1,1])$, $\bar{x}\in\mathbb{R}$, $\vartheta=\frac{1}{4}$ *and* $r=\delta=1$.
All conditions of Theorem 3.4 are satisfied with

$$k_1(t,\alpha)=k_2(t,\alpha)=k_3(t,\alpha)=\frac{1}{32+32e^{2-t}(|\alpha|^3+2)},$$

$$k_1^*(\alpha)=k_2^*(\alpha)=k_3^*(\alpha)=\frac{1}{32+32(|\alpha|^3+2)},$$

and

$$\eta=\frac{k_2^*(\alpha)\left(1-e^{\frac{(\vartheta-1)T^{\vartheta}}{\vartheta^2}}\right)}{(1-k_3^*(\alpha))(1-\vartheta)}=\frac{4-4e^{\frac{-4(2)^{\frac{1}{4}}}{3}}}{93+96(|\alpha|^3+2)}<1.$$

Then, it follows that the problem (18) admit at least one random solution. Also, the hypothesis (H_4) *and* (H_5) *are satisfied with*

$$v(t,\alpha)=3\quad and\quad q(t,\alpha)=\frac{2}{93+96e^{2-t}(|\alpha|^3+2)}.$$

Hence, Theorem 4.1 implies that problem (18) is U-H-R stable.

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