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ON A DISCRETE WIENER–HOPF EQUATION

M.S. SGIBNEV

ABSTRACT. We prove the existence of a solution to the discrete inhomogeneous Wiener–Hopf equation whose kernel is an arithmetic probability distribution which generates a random walk drifting to $-\infty$. We establish some asymptotic properties of the solution, depending on the corresponding properties of both the inhomogeneous term of the equation and its kernel.

1. INTRODUCTION

Let $\mathbb{Z}$ be the set of all integers, $\mathbb{Z}_+$ be the set of all nonnegative integers and $\mathbb{Z}_- := \mathbb{Z} \setminus \mathbb{Z}_+$. The classical Wiener–Hopf equation is

$$z(x) = \int_0^\infty k(x-y)z(y)\,dy + g(x), \qquad x \geq 0. \tag{1}$$

Its discrete analog is the following infinite systems of equations

$$\sum_{k=0}^\infty a_{j-k}\, z_k = g_j, \qquad j \in \mathbb{Z}_+, \tag{2}$$

where $\{a_j\}_{j=-\infty}^\infty$ and $\{g_j\}_{j=0}^\infty$ are known number sequences, and $\{z_j\}_{j=0}^\infty$ is the sequence sought. Moreover, the coefficients a_j satisfy the condition $\sum_{j=-\infty}^\infty |a_j| < \infty$ (see [6, § 13, Subsection 1]). The system (2) can be written in the form similar to (1):

$$z_j = \sum_{k=0}^\infty f_{j-k}\, z_k + g_j, \qquad j \in \mathbb{Z}_+,$$

where $f_0 = 1 - a_0$, $f_j = -a_j$, $j = \pm 1, \pm 2, \ldots$; or, equivalently, in the form

$$z_j = \sum_{k=-\infty}^{j} z_{j-k}\, f_k + g_j, \qquad j \in \mathbb{Z}_+, \tag{3}$$

The sequence $\{f_j\}_{j=-\infty}^\infty$ is a discrete analog of the kernel $k(x)$ in the classical Wiener–Hopf equation (1). We call the sequence $\{f_j\}_{j=-\infty}^\infty$ the *kernel* of equation (3). We shall consider equation (3) whose kernel is an arithmetic probability distribution $F = \{f_j\}_{j=-\infty}^\infty$ with span one, which generates a random walk drifting to $+\infty$. Recall (see [1, Chapter V, § 2, Definition 3]) that a probability destribution F on $\mathbb{R}$ is called *arithmetic* if it is concentrated on a set of points of the form $0, \pm\lambda, \pm 2\lambda, \ldots$ The greatest such λ is called the *span* of F. Let X_k, $k \geq 1$, be independent random variables with the same distribution F not concentrated at zero: $\mathsf{P}(X_k = j) = f_j$,

SOBOLEV INSTITUTE OF MATHEMATICS, NOVOSIBIRSK, 630090, RUSSIAN FEDERATION
E-mail address: `sgibnev@math.nsc.ru`.

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$j \in \mathbb{Z}$. These variables generate the random walk $S_0 = 0$, $S_n = X_1 + \ldots + X_n$, $n \geq 1$. According to Theorem 1 in [1, Chapter XII, § 2], there exist only two types of random walks: 1) the oscillating type (S_n oscillates with probability one between $-\infty$ and $+\infty$); 2) the drifting type (with probability one S_n tends either to $-\infty$ or to $+\infty$). The random walk $\{S_n\}$ drifts to $-\infty$ if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathsf{P}(S_n > 0) < \infty$$

(see [1, Chapter XII, § 7, Theorem 2]). If there exists $\mathsf{E}X_1 := \sum_{j=-\infty}^{\infty} j f_j < 0$ (E stands for "expectation"), then the random walk drifts to $-\infty$ according to the law of large numbers: $\lim_{n\to\infty} S_n/n = \mathsf{E}X_1$ almost surely [9, Proposition IV.7.1].

Denote by l_1 the space of summable sequences $a = \{a_n\}_{n=-\infty}^{\infty} \subset \mathbb{C}$. Define the *convolution* $a * b$ of sequences $a = \{a_n\}_{n=-\infty}^{\infty}$ and $b = \{b_n\}_{n=-\infty}^{\infty}$ from l_1 as the sequence $c = \{c_n\}_{n=-\infty}^{\infty} \in l_1$, where

$$c_n := (a * b)_n := \sum_{j=-\infty}^{\infty} a_{n-j} b_j, \qquad n \in \mathbb{Z}.$$

This definition will also be used even when the sequences a and b do not belong to l_1, provided the sum makes sense for all $n \in \mathbb{Z}$. Since complex-valued measures concentrated on $\mathbb{Z}$, may be regarded as sequences, their convolutions will be understood as convolutions of the corresponding sequences. We shall not distinguish such measures and the corresponding sequences. Is is sometimes convenient to denote a sequence as a function of an integer argument. To each sequence $a = \{a_n\}_{n=-\infty}^{\infty} \in l_1$, let correspond the series $\widehat{a}(s) := \sum_{n=-\infty}^{\infty} a_n s^n$ which absolutely converges on the circle $\{s \in \mathbb{C} : |s| = 1\}$. In particular, $\widehat{F}(s) = \sum_{n=-\infty}^{\infty} f_n s^n$. Denote by δ_0 the atomic measure of unit mass concentrated at zero. We have $\delta_0(n) = 0$ for $n \neq 0$ and $\delta_0(0) = 1$. The space l_1 is a Banach algebra with norm $\|a\|_1 = \sum_{n=-\infty}^{\infty} |a_n|$ and with convolution $a * b$ as its multiplication. The addition, subtraction and multiplication of sequences by numbers are carried out term by term: $a \pm b = \{a_n \pm b_n\}$, $\alpha a = \{\alpha a_n\}$, $\alpha \in \mathbb{C}$. The measure δ_0 is the unit of the algebra l_1 (see [5, Supplement, § 1, Subsection 1, Example 4]). For $c \in \mathbb{C}$ we set $c/\infty = 0$.

2. Existence of solution

Denote by F^{n*} the nth fold convolution of the distribution F:

$$F^{1*} := F, \qquad F^{(n+1)*} := F^{n*} * F, \quad n \geq 1, \qquad F^{0*} := \delta_0.$$

Definition 1. Let F be an arithmetic probability ddistribution with span 1. The discrete measure

$$U := \sum_{k=0}^{\infty} F^{k*} = \{U(\{n\})\}_{n=-\infty}^{\infty} =: \{u_n\}_{n=-\infty}^{\infty}$$

is called the *renewal measure* generated by F.

Definition 2. A probability distribution F (and the generated random walk) is called *transient* if $U(I) < \infty$ for all finite intervals I, and *recurring* otherwise.

If the distribution F is transient, then the number of times the random walk $\{S_n\}$ hits each finite interval I is finite with probability one and the expectation of the number of such hits is equal to $U(I)$. If F is a recurring arithmetic distribution with span 1, then the number of times the random walk $\{S_n\}$ hits every point k, $k \in \mathbb{Z}$, is infinite with probability one [1, Chapter VI,

§ 10, Theorem 3]). Therefore, the random walk $\{S_n\}$ of drifting type is transient. Hence $u_n < \infty$ for all $n \in \mathbb{Z}$.

Consider equation (3). Define the sequence $g = \{g_j\}_{j=0}^{\infty}$ for negative values of j by putting $g_j = 0$, $j \in \mathbb{Z}_-$.

Theorem 1. *Let F be an arithmetic probability distribution with span 1 of drifting to $-\infty$ type, U be the corresponding renewal measure and let $g \in l_1$. Then there exists a solution $z = \{z_j\}_{j=0}^{\infty}$ to equaiton (3) such that*

$$\sup_{n\geq 0} |z_n| \leq \sqrt{2}(u_{-1} + u_0 + u_1)\|g\|_1.$$

Proof. It suffices to establish existence for nonnegative sequences g; in case of real sequences use the representation $g_j = g_j^+ - g_j^-$, where $g_j^+ := \max(g_j, 0)$, $g_j^- := -\min(g_j, 0)$, and for complex g use $g_j = \Re g_j + i\,\Im g_j$. Then for real sequences, $z = z^+ - z^-$, where $z^{\pm}$ are the solutions of (3) for $g^{\pm}$ instead of g. Parallel with (3) consider the discrete *renewal equation*

$$\zeta_j = \sum_{k=-\infty}^{\infty} \zeta_{j-k}\, f_k + g_j, \qquad j \in \mathbb{Z}. \tag{4}$$

Let us construct solutions to equations (3) and (4) by the successive approximations:

$$z_j^{(0)} = g_j, \quad z_j^{(n)} = \sum_{k=-\infty}^{j} z_{j-k}^{(n-1)}\, f_k + g_j, \qquad j \in \mathbb{Z}_+, \quad n = 1, 2, \dots; \tag{5}$$

$$\zeta_j^{(0)} = g_j, \quad \zeta_j^{(n)} = \sum_{k=-\infty}^{\infty} \zeta_{j-k}^{(n-1)}\, f_k + g_j, \qquad j \in \mathbb{Z}, \quad n = 1, 2, \dots \tag{6}$$

Clearly, $z_j^{(n)} \uparrow$ and $\zeta_j^{(n)} \uparrow$ as $n \uparrow$. Thus, the limits

$$z_j := \lim_{n\to\infty} z_j^{(n)}, \quad j \in \mathbb{Z}_+, \qquad \zeta_j := \lim_{n\to\infty} \zeta_j^{(n)}, \quad j \in \mathbb{Z},$$

exist with $z_j \leq \zeta_j$, $j \in \mathbb{Z}_+$. Let us show that the elements of the sequence ζ are finite. The inequality

$$\sup_{n\in\mathbb{Z}} u_n \leq u_{-1} + u_0 + u_1$$

holds true (see [1, Chapter VI, § 10, Theorem 1]). Since $\zeta^{(n)} = \sum_{k=0}^{n} F^{k*} * g$, the sequence $\zeta = \{\zeta_j\}_{j=-\infty}^{\infty}$ can be represented in the form $\zeta = U * g$. Therefore,

$$\zeta_n = \sum_{j=-\infty}^{\infty} u_{n-j} g_j \leq (u_{-1} + u_0 + u_1) \sum_{j=0}^{\infty} g_j = (u_{-1} + u_0 + u_1)\|g\|_1.$$

By Beppo Levi's theorem [5, Chapter 5, § 5, Subsection 5], we can pass to the limit under the summation sign in equalities (5) and (6), and the sequences z and ζ are solutions to equations (3) and (4), respectively. In addition, the following estimate holds true:

$$\sup_{n\geq 0} |z_n| \leq (u_{-1} + u_0 + u_1)\|g\|_1.$$

The inequality is also valid when the inhomogeneous term of the equation is a real sequence. This follows from the relations $z = z^+ - z^-$ and $\|g\|_1 = \|g^+\|_1 + \|g^-\|_1$. For complex sequences g, the supremum is estimated by

$$\sqrt{2}(u_{-1} + u_0 + u_1)\|g\|_1. \qquad \square$$

3. Wiener–Hopf factorization and explicit form of solution

Put $\overline{\mathscr{T}}_+ := \min\{n \geq 1 : S_n \geq 0\}$. The random variable $\overline{\mathscr{H}}_+ := S_{\overline{\mathscr{T}}_+}$ is called the *first weak ascending ladder height*. Similarly, $\overline{\mathscr{T}}_- := \min\{n \geq 1 : S_n < 0\}$ and $\overline{\mathscr{H}}_- := S_{\overline{\mathscr{T}}_-}$ is the *first strong descending ladder height*. We have the following factorization identity: for $|\xi| \leq 1$, $\Re s = 0$,

$$1 - \xi \mathsf{E}(e^{sX_1}) = \left[1 - \mathsf{E}\left(\xi^{\overline{\mathscr{T}}_-} e^{s\overline{\mathscr{H}}_-}\right)\right]\left[1 - \mathsf{E}\left(\xi^{\overline{\mathscr{T}}_+} e^{s\overline{\mathscr{H}}_+}\right)\right]. \tag{7}$$

Note that (4) was deduced in [11, Section 2] from an analogous identity in [1, § XVIII.3] for another collection of ladder variables. Denote by $F_+ := \{f_j^+\}_{j=0}^{\infty}$ and $F_- := \{f_j^-\}_{j=-\infty}^{-1}$ the distributions of the random variables $\overline{\mathscr{H}}_+$ and $\overline{\mathscr{H}}_-$, respectively. It is convenient to define these sequences on the whole set $\mathbb{Z}$ as follows: $f_j^+ = 0$ for $j < 0$ and $f_j^- = 0$ for $j \geq 0$. Then the distributions $F_+ := \{f_j^+\}_{j=-\infty}^{\infty}$ and $F_- := \{f_j^-\}_{j=-\infty}^{\infty}$ are elements of the Banach algebra l_1. It follows from (7) that

$$\delta_0 - F = (\delta_0 - F_-) * (\delta_0 - F_+). \tag{8}$$

Let

$$U_\pm := \sum_{k=0}^{\infty} F_\pm^{k*} = \{U_\pm(\{n\})\}_{n=-\infty}^{\infty} =: \{u_n^\pm\}_{n=-\infty}^{\infty}$$

be the renewal measures generated by the distributions $F_\pm$, respectively. Denote by $\mathbf{1}_{\mathbb{Z}_-}$ the indicator of the subset $\mathbb{Z}_-$ in $\mathbb{Z}$: $\mathbf{1}_{\mathbb{Z}_-}(j) = 1$ for $j \in \mathbb{Z}_-$ and $\mathbf{1}_{\mathbb{Z}_-}(j) = 0$ for $j \in \mathbb{Z}_+$. A similar meaning has the notation $\mathbf{1}_{\mathbb{Z}_+}$.

The following result was proved in [11, Section 3, Theorem 1]. Let F be an arithmetic probability distribution with span 1 and let $g \in l_1$. Then the sequence

$$z_j = \left\{U_+ * \left[(U_- * g)\mathbf{1}_{\mathbb{Z}_+}\right]\right\}_j, \qquad j \in \mathbb{Z}_+, \tag{9}$$

is the solution to equation (3) coinciding with the solution obtained by successive approximations.

4. An asymptotic property of the solution

First, mention the following auxiliary result. Let $F = \{f_j\}_{j=-\infty}^{\infty}$ be an arithmetic probability distribution with span 1 such that

$$\mu := \mathsf{E}X_1 = \sum_{j=-\infty}^{\infty} j f_j \in [-\infty, 0),$$

and let $\{a_n\}_{n=-\infty}^{\infty} \in l_1$. Then $(U * a)_j \to 0$ as $j \to +\infty$. In fact, according to the discrete renewal theorem, $u_n \to 1/|\mu|$ as $n \to -\infty$ and $u_n \to 0$ as $n \to +\infty$. (a symmetrical variant of Theorem 1 in [1, Chapter XI, § 1]). Therefore, the sequence $\{u_n\}$ is bounded by some number $C > 0$, and the expression under the summation sign in the equality

$$(U * a)_j = \sum_{n=-\infty}^{\infty} u_{j-n} a_n$$

is majorized by the corresponding elements of the summable sequence $\{C|a_j|\}_{j=-\infty}^{\infty}$. So we may pass to the limit under the summation sign as $j \to +\infty$, see [3, Chapter V, Section 26, Theorem D], where the space with measure $(X, \mathbf{S}, \nu)$ has the following form: $X = \mathbb{Z}$, $\mathbf{S}$ is the collection of all subsets in $\mathbb{Z}$, ν is the counting measure, that is, $\nu(A)$ is the number of of the elements of the subset $A \subset \mathbb{Z}$ if A is a finite set and $\nu(A) = \infty$ if A is an infinite set.

We go to an asymptotic property of the solution.

Theorem 2. *Let be an arithmetic probability distribution with span* 1 *which generates a random walk drifting to* $-\infty$*, and let* $g \in l_1$*. Then the solution* $z = \{z_j\}_{j=0}^{\infty}$ *to equation* (3)*, obtained by successive approximations* (5)*, satisfies the following relation:*

$$z_j \to 0 \qquad \text{as } j \to \infty.$$

Proof. Turn to equality (9). As shown in the proof of Lemma 1 below, the span of F_- is equal to 1. Hence, by the discrete renewal theorem, $u_n^- \to 1/|\mu_-|$ as $n \to -\infty$, where $\mu_- := \mathsf{E}\mathscr{H}_- \in [-\infty, 0)$. Applying the above auxiliary result, we get $U_- * g(j) \to 0$ as $j \to \infty$, whence $\max_{j\geq 0} |U_- * g(j)| < \infty$. As mentioned in [1, Chapter XVIII, § 4, Example a)], drifting of $\{S_n\}$ to $+\infty$ takes place when the distribution of the random variable $\mathscr{H}_- := S_{\mathscr{T}_-}$ is defect; here $\mathscr{T}_- := \min\{n \geq 1 : S_n \leq 0\}$. In our case, i.e., when $\{S_n\}$ drifts to $-\infty$, this means — by symmetry reasons — that the distribution F_+ is defect: $F_+(\mathbb{Z}) < 1$. Hence U_+ is a finite measure and we have by the majorized convergence theorem

$$z_j = \left\{U_+ * \left[(U_- * g)\mathbf{1}_{\mathbb{Z}_+}\right]\right\}_j = \sum_{k=0}^{\infty} \left[(U_- * g)\mathbf{1}_{\mathbb{Z}_+}\right]_{j-k} u_k^+$$

$$= \sum_{k=0}^{j} (U_- * g)_{j-k} u_k^+ \to 0 \qquad \text{as } j \to \infty. \qquad \square$$

5. Rate of convergence

Definition 3. A sequence $\{\alpha_n\}_{n=-\infty}^{\infty}$ of positive numbers is called *submultiplicative* if

$$\alpha_{k+l} \leq \alpha_k \alpha_l. \tag{10}$$

The following relations are valid [2, Chapter III, § 19]:

$$\rho_1 = \sup_{n>0} \frac{1}{\sqrt[n]{\alpha_{-n}}} = \lim_{n\to+\infty} \frac{1}{\sqrt[n]{\alpha_{-n}}};$$
$$\rho_2 = \inf_{n>0} \sqrt[n]{\alpha_n} = \lim_{n\to+\infty} \sqrt[n]{\alpha_n};$$
$$0 < \rho_1 \leq \rho_2 < +\infty.$$

Denote by $W^{<\alpha>}$ the collection of all formal series $x = \sum_{n=-\infty}^{\infty} a_n X^n$ for which

$$\|x\| := \sum_{n=-\infty}^{\infty} |a_n|\alpha_n < \infty.$$

Inequality (10) implies that together with all pairs $x = \sum_{n=-\infty}^{\infty} a_n X^n$ and $y = \sum_{n=-\infty}^{\infty} b_n X^n$ in $W^{<\alpha>}$, their formal product

$$\sum_{n=-\infty}^{\infty} c_n X^n = \sum_{n=-\infty}^{\infty} \Big(\sum_{m=-\infty}^{\infty} a_{n-m} b_m \Big) X^n$$

also belongs to $W^{<\alpha>}$ and $\|xy\| \leq \|x\|\|y\|$. Thus, $W^{<\alpha>}$ is a normed ring (Banach algebra) with respect to the introduced norm and multiplication, and to the usual operations of addition and multiplication by numbers for formal series. With each element $x = \sum_{n=-\infty}^{\infty} a_n X^n \in W^{<\alpha>}$, associate the series $\widehat{a}(w) := \sum_{n=-\infty}^{\infty} a_n w^n$, which is absolutely convergent in the annulus

$$\mathsf{R} := \{w \in \mathbb{C} : \rho_1 \leq |w| \leq \rho_2\}.$$

It is convenient to identify the formal series $x = \sum_{n=-\infty}^{\infty} a_n X^n$ with the sequence $\{a_n\}_{n=-\infty}^{\infty}$ which may be regarded as the measure on the set of all integers $\mathbb{Z}$ taking the value a_n on the one-point set $\{n\}$. If $a_n = 0$ for all $n < 0$, then $\widehat{a}(w)$, $|w| \leq \rho_2$, is the generating function of the sequence $\{a_n\}_{n=0}^{\infty}$. Under this interpretation of the elements of the algebra $W^{<\alpha>}$, its unity is the measure δ_0 of unit mass concentrated at zero.

Recall the invertibility condition for elements in the Banach algebra $W^{<\alpha>}$. Let $\mathscr{M}$ be the space of maximal ideals of the algebra $W^{<\alpha>}$. The following facts are well known [2, Chapter I, § 4]. Each maximal ideal $M \in \mathscr{M}$ generates some homomorphism $h : W^{<\alpha>} \to \mathbb{C}$ and M is the kernel of this homomorphism. Denote by $\nu(M)$ the value of h at $\nu \in W^{<\alpha>}$. An element $\nu \in W^{<\alpha>}$ if invertible if and only if ν does not belong to any maximal ideal $M \in \mathscr{M}$. In other words, ν is invertible if and only if $\nu(M) \neq 0$ for each $M \in \mathscr{M}$. Furhter, let M be a maximal ideal of the algebra $W^{<\alpha>}$. Then there exists $w \in \mathsf{R}$ such that $x(M) = \sum_{n=-\infty}^{\infty} a_n w^n$ for all $x = \sum_{n=-\infty}^{\infty} a_n X^n \in W^{<\alpha>}$. Thus, the element $x = \sum_{n=-\infty}^{\infty} a_n X^n \in W^{<\alpha>}$ is invertible if $\sum_{n=-\infty}^{\infty} a_n w^n \neq 0$ for each $w \in \mathsf{R}$ (see [2, Chapter III, § 19]).

We shall need the following renewal theorem for arithmetic distributions (see, for example, the symmetrical variant of Lemma 1 in [10], where the case $\mu > 0$ was considered).

Theorem 3. *Let $F = \{f_j\}_{j=-\infty}^{\infty}$ be an arithmetic probability distribution with span 1 such that*

$$\mu = \sum_{j=-\infty}^{\infty} j f_j < 0, \qquad \sum_{j=-\infty}^{\infty} j^2 f_j < \infty.$$

Then the renewal measure U generated by the distribution F can be represented in the following form:

$$U = \frac{1}{|\mu|} \mathbf{1}_{\mathbb{Z}_- \cup \{\mathbf{0}\}} + R,$$

where $R \in l_1$.

We now formulate a result about rate of convergence of the solution.

Theorem 4. *Let $\{\alpha_n\}_{n=-\infty}^{\infty}$ be a nondecreasing submultiplicative sequence such that $\alpha_n \equiv 1$ for $n < 0$. Let $F = \{f_j\}_{j=-\infty}^{\infty}$ be an arithmetic probability distribution with span 1. Suppose that*

$$\mu = \sum_{j=-\infty}^{\infty} j f_j < 0, \qquad \sum_{j=-\infty}^{\infty} j^2 f_j < \infty, \qquad \sum_{j=1}^{\infty} j \alpha_j f_j < \infty.$$

If $\rho_2 > 1$, then suppose also that $\widehat{F}(\rho_2) < 1$. Moreover, let

$$\alpha_n \sum_{j=n}^{\infty} g_j \to 0, \qquad \text{as } n \to \infty.$$

Then the solution z of equation (3) satisfies the following asymptotic relation:

$$z_j = o\left(\frac{1}{\alpha_j}\right) \qquad \text{as } j \to \infty.$$

The proofs of the following lemmas will be given after the proof of Theorem 4.

Lemma 1. *Let the hypotheses of Theorem 4 be fulfilled. Then the distribution F_- satisfies the conditions of Theorem 3.*

Lemma 2. *Let the hypotheses of Theorem 4 be fulfilled. Then $F_+ \in W^{<\alpha>}$.*

Lemma 3. *Let the hypotheses of Theorem 4 be fulfilled. Then $U_+ \in W^{<\alpha>}$.*

Proof of Theorem 4 . First, we prove that

$$(U_- * g)_j = o\left(\frac{1}{\alpha_j}\right) \qquad \text{as } j \to \infty. \tag{11}$$

We have

$$\alpha_n g_n = \alpha_n \sum_{j=n}^{\infty} g_j - \alpha_n \sum_{j=n+1}^{\infty} g_j = \alpha_n \sum_{j=n}^{\infty} g_j - \frac{\alpha_n}{\alpha_{n+1}} \alpha_{n+1} \sum_{j=n+1}^{\infty} g_j \to 0$$

as $n \to \infty$. By Lemma 1 and Theorem 3

$$U_- = \frac{1}{|\mu_-|}\mathbf{1}_{\mathbb{Z}_- \cup \{\mathbf{0}\}} + R_-, \qquad R_- = \{r_n^-\}_{n=-\infty}^{0} \in l_1,$$

where $\mu_- = \mathsf{E}\overline{\mathscr{H}}_-$. For $j > 0$, we have

$$\begin{aligned}|\alpha_j (U_- * g)_j| &= \alpha_j \left| \sum_{n=-\infty}^{0} u_n^- g_{j-n} \right| = \alpha_j \left| \frac{1}{|\mu_-|} \sum_{n=-\infty}^{0} g_{j-n} + \sum_{n=-\infty}^{0} r_n^- g_{j-n} \right| \\ &\le \frac{\alpha_j}{|\mu_-|} \left| \sum_{k=j}^{\infty} g_k \right| + \sum_{n=-\infty}^{0} \alpha_n |r_n^-| \alpha_{j-n} |g_{j-n}| \to 0 \qquad \text{as } j \to \infty,\end{aligned}$$

since the first sum on the right tends to zero by assumption and the second one tends to zero by the majorized convergence theorem. Relation (11) is proved. The assertion of the theorem now follows from the relation

$$\begin{aligned}|\alpha_j z_j| &= \alpha_j \left| \left\{ U_+ * \left[(U_- * g) \mathbf{1}_{\mathbb{Z}_+} \right] \right\}_j \right| = \alpha_j \left| \sum_{k=0}^{j} u_k^+ \left[(U_- * g)_{j-k} \right. \right| \\ &\le \sum_{k=0}^{j} \alpha_k u_k^+ \alpha_{j-k} |(U_- * g)_{j-k}| \to 0 \qquad \text{as } j \to \infty,\end{aligned}$$

by the majorized convergence theorem (see (11) and Lemma 3). □

Proof of Lemma 1. A discrete probability distribution is calle *lattice* if its points of discontinuity form an arithmetic progression, that is, a set of points of the form $a + jd$, where a, d are fixed numbers and j takes integer values [8, Section 1.2]. It is well known [8, Theorem 2.1.4] that F is a lattice distribution if and anly if there exists a real number $t_0 \ne 0$ such that $|\mathscr{F}(it_0)| = 1$; here $\mathscr{F}(s) := \int_{\mathbb{R}} e^{sx}\, F(dx)$ is the Laplace transform (moment generating function) of F. Perusing the proof of Theorem 2.1.4 in [8], we see that F is an arithmetic distribution if and only if there exist a real number $t_0 \ne 0$ such that $\mathscr{F}(it_0) = 1$. The span λ of a l attice distribution F is determined by the following relations [7, Chapter IV, Exercise 7]: $|\mathscr{F}(it)| < 1$ for $0 < |t| < 2\pi/\lambda$ (t real) and $|\mathscr{F}(2\pi i/\lambda)| = 1$. If F is an arithmetic distribution, then its span λ is determined by the following relations: $\mathscr{F}(it) \ne 1$ for $0 < |t| < 2\pi/\lambda$ (t real) and $\mathscr{F}(2\pi i/\lambda) = 1$. Thus, the distribution F_- of the random variable $\overline{\mathscr{H}}_-$ is an arithmetic distribution with span 1. This follows from the above criterion of arithmeticity and factorization identity (8) for $\xi = 1$. Further, $\mu_- = \mathsf{E}\overline{\mathscr{H}}_- \le \mathsf{E}X_1 < 0$, since $\overline{\mathscr{H}}_- \le X_1$. Finally, let us prove that $\mathsf{E}\overline{\mathscr{H}}_-^2 < \infty$.

Let $\{\beta_n\}_{n=-\infty}^{\infty}$ be a submultiplicative sequence such that $\beta_n = 1 + n^2$ for $n < 0$ and $\beta_n \equiv 1$ for $n \geq 0$. Then $F \in W^{<\beta>}$ by assumption. The distribution F_+ is defective. Therefore, $\|F_+\|_1 = F_+(\mathbb{Z}_+) < 1$. It follows that the element $\delta_0 - F_+ \in l_1$ is invertible in the Banach algebra l_1: $U_+ = \{u_n^+\}_{n=-\infty}^{\infty} = (\delta_0 - F_+)^{-1} \in l_1$, where $u_n^+ \equiv 0$ for $n < 0$. Obviously, $U_+ \in W^{<\beta>}$. Equality (9) implies

$$\delta_0 - F_- = (\delta_0 - F) * U_+ \in W^{<\beta>}.$$

Consequently, $F_- \in W^{<\beta>}$, that is, $\sum_{n=-\infty}^{-1} n^2 f_n^- < \infty$, as was to be proved. □

Proof of Lemma 2. Put $w = e^s$ and $\xi \in (0, 1)$ in (7). We obtain

$$1 - \xi\widehat{F}(w) = [1 - \widehat{F}_{\xi-}(w)][1 - \widehat{F}_{\xi+}(w)], \qquad 0 < \xi < 1, \quad |w| = 1,$$

where $F_{\xi\pm}$ are positive measures concentrated on the sets $\mathbb{Z}_\pm$ respectively; moreover, $F_{\xi+}(\mathbb{Z}_+) \leq F_+(\mathbb{Z}_+) < 1$ and $F_{\xi-}(\mathbb{Z}_-) \leq \xi < 1$. We have

$$\delta_0 - F_{\xi+} = U_{\xi-} * (\delta_0 - \xi F),$$

where $U_{\xi-} := \sum_{n=0}^{\infty} F_{\xi-}^{n*}$ is a finite measure. Therefore,

$$F_{\xi+}(\{j\}) = \xi(U_{\xi-} * F)(\{j\})$$

for integer $j > 0$. Letting $\xi \uparrow 1$, we get $f_j^+ = (U_- * F)(\{j\})$ for positive $j \in \mathbb{Z}_+$. Show that the distribution F_- satisfies the hypotheses of Theorem 3. Put $\beta_n = (1 + |n|)^2$ for $n \in \mathbb{Z}_-$ and $\beta_n = 1$ for $n \in \mathbb{Z}_+$. Clearly, $\{\beta_n\}$ is a submultiplicative sequence. The underlying distribution F certainly belongs to the algebra $W^{<\beta>}$. The defect distribution F_+ on $\mathbb{Z}_+$, being a summable sequence, also belongs to $W^{<\beta>}$, and so is the finite renewal measure U_+. Equality (8) implies that

$$\delta_0 - F_- = (\delta_0 - F) * (\delta_0 - F_+)^{-1} = (\delta_0 - F) * U_+ \in W^{<\beta>},$$

whence $F_- \in W^{<\beta>}$. Obviously, $\mu_- \leq \mu < 0$, so that Theorem 3 is applicable to F_-. By Theorem 3 for F_-,

$$\begin{aligned}
\sum_{n=1}^{\infty} \alpha_n f_n^+ &= \sum_{n=1}^{\infty} \alpha_n \sum_{k=-\infty}^{0} f_{n-k} u_k^- \\
&\leq \sum_{k=-\infty}^{0} u_k^- \sum_{n=1}^{\infty} \alpha_{n-k} f_{n-k} = \sum_{k=-\infty}^{0} u_k^- \sum_{l=1-k}^{\infty} \alpha_l f_l \\
&= \sum_{k=-\infty}^{0} \frac{1}{|\mu_-|} \sum_{l=1-k}^{\infty} \alpha_l f_l + \sum_{k=-\infty}^{0} r_k^- \sum_{l=1-k}^{\infty} \alpha_l f_l =: I_1 + I_2.
\end{aligned}$$

The following estimates hold true:

$$I_1 = \sum_{l=1}^{\infty} \alpha_l f_l \sum_{1-l}^{0} \frac{1}{|\mu_-|} = \frac{1}{|\mu_-|} \sum_{l=1}^{\infty} l\alpha_l f_l < \infty,$$

$$I_2 \leq \|R_-\| \sum_{l=1}^{\infty} \alpha_l f_l < \infty,$$

that is, $F_+ \in W^{<\alpha>}$. The proof of Lemma 2 is complete. □

Proof of Lemma 3. In our case, $\mathsf{R} = \{w \in \mathbb{C} : 1 \le |w| \le \rho_2\}$ By Lemma 2, $F_+ \in W^{<\alpha>}$. Let $\rho_2 > 1$. It follows from the assumption $\widehat{F}(\rho_2) < 1$ and (8) that $\widehat{F}_+(\rho_2) < 1$. Therefore, $\widehat{F}_+(w) < 1$ for all $w \in \mathsf{R}$, whence $\widehat{\delta}_0(w) - \widehat{F}_+(w) = 1 - \widehat{F}_+(w) \ne 0$ for all $w \in \mathsf{R}$. This means that the element $\delta_0 - F_+ \in W^{<\alpha>}$ is invertible: $U_+ = (\delta_0 - F_+)^{-1} \in W^{<\alpha>}$. If $\rho_2 = 1$, then $|\widehat{F}_+(w)| \le F_+(\mathbb{Z}) < 1$ for all $w \in \mathsf{R}$. Therefore, the element $\delta_0 - F_+ \in W^{<\alpha>}$ is also invertible in $W^{<\alpha>}$. Relation $U_+ \in W^{<\alpha>}$ is proved. □

6. Homogeneous equation. Solutions with arbitrary limits

The solution z of equation (3) obtained by successive approximations in Theorem 1 is not unique. Consider the homogeneous equation

$$Z_j = \sum_{k=0}^{\infty} f_{j-k}\, Z_k, \qquad j \in \mathbb{Z}_+, \tag{12}$$

where $Z = \{Z_j\}_{j=0}^{\infty}$ is an unknown sequence and $F = \{f_j\}_{j=-\infty}^{\infty}$ is an arithmetic probability distribution with span 1 which generates a random walk $\{S_n\}$ with drift to $-\infty$. The random variable $\overline{S} := \sup_{n\ge 0} S_n$ is finite with probability one. Denote by $M_j = \mathsf{P}(\overline{S} \le j)$, $j \in \mathbb{Z}_+$, the distribution function of the discrete random variable $\overline{S}$. The sequence $M = \{M_j\}_{j=0}^{\infty}$ satisfies equation (12) (see [1, Chapter XII, § 3, Example c]); moreover, $M_j \to 1$ as $j \to \infty$. The sequence $Z_c = \{Z_c(j)\}_{j=0}^{\infty} := z + cM$, where $c \in \mathbb{C}$, is also a solution to equation (3). It is clear that if the conditions of Theorem 2 are satisfied, then $Z_c(j) \to c$ as $j \to \infty$. Let us give a rate of convergence.

Theorem 5. *Let the hyptheses of Theorem 4 be fulfilled. Then*

$$Z_c(j) - c = o\left(\frac{1}{\alpha_j}\right) \qquad \text{as } j \to \infty.$$

Proof. Let us show that the distribution of the supremum $\overline{S}$ coincides with the renewal measure U_+, up to a constant factor. Lemma 1 in [1, Chapter XVIII, § 3], applied to the symmetrical random walk $\{-S_n\}$, asserts that for $|\xi| < 1$

$$\log \frac{1}{1 - \mathsf{E}\big(\xi^{\overline{\mathscr{T}}_-} e^{s\overline{\mathscr{H}}_-}\big)} = \sum_{n=1}^{\infty} \frac{\xi^n}{n} \int_{-\infty}^{0-} e^{sx}\, F^{n*}(dx),$$

since in our case the point $(\overline{\mathscr{T}}_-, -\overline{\mathscr{H}}_-)$ is the point of the first hit by the random walk $\{-S_n\}$ of the open interval $(0,\infty)$. Similarly, according to Lemma 2 in [1, Chapter XVIII, § 3]

$$\log \frac{1}{1 - \mathsf{E}\big(\xi^{\overline{\mathscr{T}}_+} e^{s\overline{\mathscr{H}}_+}\big)} = \sum_{n=1}^{\infty} \frac{\xi^n}{n} \int_{0-}^{\infty} e^{sx}\, F^{n*}(dx), \tag{13}$$

By Theorem 2 in [1, Chapter XVIII, § 5],

$$\mathsf{E}e^{s\overline{S}} = \exp\left\{\sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} (e^{sx} - 1)\, F^{n*}(dx)\right\}.$$

Put $\xi = 1$ in (13). We have

$$\int_{0-}^{\infty} e^{sx}\, U_+(dx) = \left[1 - \int_{0-}^{\infty} e^{sx}\, F_+(dx)\right]^{-1} = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} e^{sx}\, F^{n*}(dx)\right].$$

Take $a = \exp\{\sum_{n=1}^{\infty} \mathsf{P}(S_n \geq 0)/n\}$. Then

$$a\mathsf{E}e^{s\overline{S}} = \int_{0-}^{\infty} e^{sx}\, U_+(dx), \qquad \Re s = 0,$$

that is, the distribution of the random variable $\overline{S}$ coincides with the measure U_+/a. The assertion of the theorem now follows from the available estimates:

$$\begin{aligned} |Z_c(j) - c| &= |z_j + cM_j - c| \leq |z_j| + |c||M_j - c| \\ &= |z_j| + |c|\mathsf{P}(\overline{S} > j) = |z_j| + \frac{|c|}{a} \sum_{k=j+1}^{\infty} u_k^+ \\ &\leq |z_j| + \frac{|c|}{a} \cdot \frac{1}{\alpha_j} \sum_{k=j+1}^{\infty} \alpha_k u_k^+ = o\left(\frac{1}{\alpha_j}\right) \qquad \text{as } j \to \infty, \end{aligned}$$

since $z_j = o(1/\alpha_j)$ as $j \to \infty$ by Theorem 4 and $U_+ = \{u_n^+\}_{n=-\infty}^{\infty} \in W^{<\alpha>}$ according to Lemma 3; here $u_n^+ \equiv 0$ for $n < 0$. $\square$

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