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OSTROWSKI'S TYPE INEQUALITIES BY USING THE MODIFIED 2-STEP LINEAR KERNEL

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Abstract. In this scholarly paper, we introduce an extension Ostrowski's Inequalities through the modified derived mathematical identity. Our work focuses on the new results for $\acute{y}' \in L_1, \acute{y}' \in L_2$ and $\acute{y}'' \in L_2$. To achive this, we employ our methodology such as Grüss inequality, Diaz-Metcalf inequality and Cauchy inequality. To achive the core of our finding, we use a modified 2-step linear kernel and also discussed some modified results. Lastly, we demonstrate the practical applications of our findings in the context of Cumulative Distributive Functions.

1. Introduction

The research on Ostrowski Type Inequalities has seen notable allowance from numerous researchers over the years. The year 1970, marked a pivotal moment when Metrinovic $[7]-[9]$ underscored the significance of inequalities and varified Ostrowski Type Inequalities with the domain of twice differentiable mappings. Subsequently, Barnet et al. [4] delved into thr realm of Ostrowski Type Inequalities for $L_p(c,d)$ and $L_1(c,d)$. Husain et al. [6] introduced comprehensive generalization of Ostrowski Type Inequalities and presented advanced estimates. Qayyum et al. $[10]-[15]$ extended these inequalities, providing a generalized form of Ostrowski Type Inequalities for twice derivable mappings, while Dragomir and Wang [5] introduced a classical method for validating Ostrowski Type Inequalities and reveal its applications for the first time. Additionally, Barnet et al. [4] introduced a novel concept by proving Ostrowski Type Inequalities using the β-function for 1^{st} and 2^{nd} differential mappings, applying their findings to numerical quadrature rules. Notably, the evolution of Ostrowski Type Inequalities initiated with 2-Step kernels and later expanded to 3-Step kernels e.g., $[1]-[3]$ and a few researchers e.g., $[13]-[14]$ focused on 5-Step kernels.

2. Main Findings

Lemma 1. *Let $\hat{H}:\left[\ddot{a},\grave{I}\right]\to\mathbb{R}$ be such that $\hat{H}'$ is absolutely continuous on $\left[\ddot{a},\grave{I}\right]$. Define the kernel $P(\bar{c},\bar{e})$ as:*

$$P(\bar{c},\bar{e})=\begin{cases}(\bar{e}-\ddot{a})+\left\{\bar{e}-\left(\ddot{a}+h\frac{\grave{I}-\ddot{a}}{2}\right)\right\}; & \bar{e}\in(\ddot{a},\bar{c}]\\ \left(\bar{e}-\grave{I}\right)+\left\{\bar{e}-\left(\grave{I}-h\frac{\grave{I}-\ddot{a}}{2}\right)\right\}; & \bar{e}\in\left(\bar{c},\grave{I}\right]\end{cases}\tag{1}$$

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$\forall \bar{c}\epsilon \left[\ddot{a}, \frac{\ddot{a}+\grave{I}}{2}\right]$ *and* $h\epsilon\,[0,1]$.

Proof. Then the following identity holds:

$$\int_{\ddot{a}}^{\grave{I}} P(\bar{c},\bar{e})\hat{H}'(\bar{e})\,d\bar{e} \tag{2}$$

$$= (2-h)\left(\grave{I}-\ddot{a}\right)\hat{H}(\bar{c}) + h\frac{\grave{I}-\ddot{a}}{2}\left(\hat{H}(\ddot{a})+\hat{H}\left(\grave{I}\right)\right) - 2\int_{\ddot{a}}^{\grave{I}}\hat{H}(\bar{e})\,d\bar{e}.$$

If we put $h=0$ in 2 we have the identity 3

$$\frac{1}{2\left(\grave{I}-\ddot{a}\right)}\int_{\ddot{a}}^{\grave{I}} P(\bar{c},\bar{e})\hat{H}'(\bar{e})\,d\bar{e} = \hat{H}(\bar{c}) - \frac{1}{\grave{I}-\ddot{a}}\int_{\ddot{a}}^{\grave{I}}\hat{H}(\bar{e})\,d\bar{e}. \tag{3}$$

By applying integration by parts on 1, we get the identity 2. □

Now we're going to use three different condition.

2.1. Case For L_∞ Norm.

Theorem 1. *Let* $\hat{H}:\left[\ddot{a},\grave{I}\right]\to\mathbb{R}$ *be differentiable mapping on* $\left(\ddot{a},\grave{I}\right)$ *and* $\hat{H}':\left(\ddot{a},\grave{I}\right)\to\mathbb{R}$ *is bounded i.e.,* $\left\|\hat{H}'\right\|_\infty = \sup_{\bar{e}\in\left(\ddot{a},\grave{I}\right)}\left|\hat{H}'(\bar{e})\right| < \infty$. *Then:*

$$\left|\hat{H}(\bar{c}) - \frac{1}{\grave{I}-\ddot{a}}\int_{\ddot{a}}^{\grave{I}}\hat{H}(\bar{e})\,d\bar{e}\right| \tag{4}$$

$$\leq \left(\grave{I}-\ddot{a}\right)\left[\frac{1}{8}\left[1+(1-h)^2\right] + \frac{\left(\bar{c}-\frac{\ddot{a}+\grave{I}}{2}\right)^2}{\left(\grave{I}-\ddot{a}\right)^2}\right]\left\|\hat{H}'\right\|_\infty$$

If we put $h=0$ *in* 4 *we have* 5

$$\left|\hat{H}(\bar{c}) - \frac{1}{\grave{I}-\ddot{a}}\int_{\ddot{a}}^{\grave{I}}\hat{H}(\bar{e})\,d\bar{e}\right| \tag{5}$$

$$\leq \left(\grave{I}-\ddot{a}\right)\left[\frac{1}{4} + \frac{\left(\bar{c}-\frac{\ddot{a}+\grave{I}}{2}\right)^2}{\left(\grave{I}-\ddot{a}\right)^2}\right]\left\|\hat{H}'\right\|_\infty.$$

$\forall\ \bar{e}\in\left[\ddot{a},\grave{I}\right]$, $\bar{c}\epsilon\left[\ddot{a},\frac{\ddot{a}+\grave{I}}{2}\right]$ *and* $h\epsilon\,[0,1]$.

Proof. As we have

$$\left|\hat{H}(\bar{c}) - \frac{1}{\grave{I}-\ddot{a}}\int_{\ddot{a}}^{\grave{I}}\hat{H}(\bar{e})\,d\bar{e}\right| = \frac{1}{2\left(\grave{I}-\ddot{a}\right)}\left|\int_{\ddot{a}}^{\grave{I}} P(\bar{c},\bar{e})\hat{H}'(\bar{e})\,d\bar{e}\right|.$$

$$\leq \frac{1}{2\left(\grave{I}-\ddot{a}\right)}\left\| \hat{H}'\right\|_{\infty} \int\limits_{\ddot{a}}^{\grave{I}} P(\bar{c},\bar{e})d\bar{e}.$$

$$= \frac{1}{2\left(\grave{I}-\ddot{a}\right)}\left\| \hat{H}'\right\|_{\infty}\left[\int\limits_{\ddot{a}}^{\bar{c}}\left[(\bar{e}-\ddot{a})+\left\{\bar{e}-\left(\ddot{a}+h\frac{\grave{I}-\ddot{a}}{2}\right)\right\}\right]d\bar{e}\right.$$
$$\left.+\int\limits_{\bar{c}}^{\grave{I}}\left[\left(\bar{e}-\grave{I}\right)+\left\{\bar{e}-\left(\grave{I}-h\frac{\grave{I}-\ddot{a}}{2}\right)\right\}\right]d\bar{e}\right].$$

$$= \frac{1}{2\left(\grave{I}-\ddot{a}\right)}\left\| \hat{H}'\right\|_{\infty}\left[\frac{(\bar{c}-\ddot{a})^2}{2}+\frac{\left(\bar{c}-\grave{I}\right)^2}{2}\right.$$
$$\left.+\frac{\left\{\bar{c}-\left(\ddot{a}+h\frac{\grave{I}-\ddot{a}}{2}\right)\right\}^2}{2}+\frac{\left\{\bar{c}-\left(\grave{I}-h\frac{\grave{I}-\ddot{a}}{2}\right)\right\}^2}{2}\right].$$

Now, observe that

$$\frac{(\bar{c}-\ddot{a})^2}{2}+\frac{\left(\bar{c}-\grave{I}\right)^2}{2}=\left(\bar{c}-\frac{\ddot{a}+\grave{I}}{2}\right)^2+\frac{1}{4}\left(\grave{I}-\ddot{a}\right)^2$$

and

$$\frac{\left\{\bar{c}-\left(\ddot{a}+h\frac{\grave{I}-\ddot{a}}{2}\right)\right\}^2}{2}+\frac{\left\{\bar{c}-\left(\grave{I}-h\frac{\grave{I}-\ddot{a}}{2}\right)\right\}^2}{2}$$
$$=\left(\bar{c}-\frac{\ddot{a}+\grave{I}}{2}\right)^2+\frac{1}{4}\left(\grave{I}-\ddot{a}\right)^2(1-h)^2.$$

By using above equatons we get the inequility 4. □

Corollary 1. *If we put $\bar{c}=\ddot{a}$ in 4 we have 6*

(6)
$$\left|\hat{H}\left(\ddot{a}\right)-\frac{1}{\grave{I}-\ddot{a}}\int\limits_{\ddot{a}}^{\grave{I}}\hat{H}\left(\bar{e}\right)d\bar{e}\right|$$
$$\leq\left(\grave{I}-\ddot{a}\right)\left[\frac{1}{8}\left[1+(1-h)^2\right]+\frac{1}{4}\right]\left\| \hat{H}'\right\|_{\infty}$$

If we put $h=0$ in 6 we have 7

(7)
$$\left|\hat{H}\left(\ddot{a}\right)-\frac{1}{\grave{I}-\ddot{a}}\int\limits_{\ddot{a}}^{\grave{I}}\hat{H}\left(\bar{e}\right)d\bar{e}\right|$$
$$\leq\frac{1}{2}\left(\grave{I}-\ddot{a}\right)\left\| \hat{H}'\right\|_{\infty}.$$

Corollary 2. *If we put* $\bar{c} = \grave{I}$ *in* 4 *we have* 8

$$\left| \hat{H}\left(\grave{I}\right) - \frac{1}{\grave{I} - \ddot{a}} \int\limits_{\ddot{a}}^{\grave{I}} \hat{H}\left(\bar{e}\right) d\bar{e} \right| \leq \left(\grave{I} - \ddot{a}\right) \left[\frac{1}{8} \left[1 + (1-h)^2\right] + \frac{1}{4} \right] \left\| \hat{H}' \right\|_{\infty} \tag{8}$$

If we put $h = 0$ *in* 8 *we have* 9

$$\left| \hat{H}\left(\grave{I}\right) - \frac{1}{\grave{I} - \ddot{a}} \int\limits_{\ddot{a}}^{\grave{I}} \hat{H}\left(\bar{e}\right) d\bar{e} \right| \leq \frac{1}{2} \left(\grave{I} - \ddot{a}\right) \left\| \hat{H}' \right\|_{\infty}. \tag{9}$$

2.2. Case For L_1 Norm.

Theorem 2. *Let* $\hat{H} : \left[\ddot{a}, \grave{I}\right] \to \mathbb{R}$ *be continuous on* $\left[\ddot{a}, \grave{I}\right]$, *single differentiable on* $\left(\ddot{a}, \grave{I}\right)$ *and* $\hat{H}' \in L_1\left(\ddot{a}, \grave{I}\right)$. *Then:*

$$\left| \hat{H}\left(\bar{c}\right) - \frac{1}{\grave{I} - \ddot{a}} \int\limits_{\ddot{a}}^{\grave{I}} \hat{H}\left(\bar{e}\right) d\bar{e} \right| \leq \left[\frac{1}{4}(2-h) + \frac{\bar{c} - \frac{\ddot{a} + \grave{I}}{2}}{\grave{I} - \ddot{a}} \right] \left\| \hat{H}' \right\|_{1}. \tag{10}$$

If we put $h = 0$ *in* 10 *we have* 11

$$\left| \hat{H}\left(\bar{c}\right) - \frac{1}{\grave{I} - \ddot{a}} \int\limits_{\ddot{a}}^{\grave{I}} \hat{H}\left(\bar{e}\right) d\bar{e} \right| \leq \left[\frac{1}{2} + \frac{\bar{c} - \frac{\ddot{a} + \grave{I}}{2}}{\grave{I} - \ddot{a}} \right] \left\| \hat{H}' \right\|_{1}. \tag{11}$$

$\forall\ \bar{e} \in \left[\ddot{a}, \grave{I}\right]$, $\bar{c} \epsilon \left[\ddot{a}, \frac{\ddot{a} + \grave{I}}{2}\right]$ *and* $h \epsilon [0, 1]$.

Proof. As we have

$$\left| \hat{H}\left(\bar{c}\right) - \frac{1}{\grave{I} - \ddot{a}} \int\limits_{\ddot{a}}^{\grave{I}} \hat{H}\left(\bar{e}\right) d\bar{e} \right| = \frac{1}{2\left(\grave{I} - \ddot{a}\right)} \left| \int\limits_{\ddot{a}}^{\grave{I}} P(\bar{c}, \bar{e}) \hat{H}'\left(\bar{e}\right) d\bar{e} \right|.$$

$$= \frac{1}{2\left(\grave{I} - \ddot{a}\right)} \left| \int\limits_{\ddot{a}}^{\bar{c}} \left[\left(\bar{e} - \ddot{a}\right) + \left\{ \bar{e} - \left(\ddot{a} + h\frac{\grave{I} - \ddot{a}}{2} \right) \right\} \right] \hat{H}'\left(\bar{e}\right) d\bar{e} \right.$$
$$\left. + \int\limits_{\bar{c}}^{\grave{I}} \left[\left(\bar{e} - \grave{I}\right) + \left\{ \bar{e} - \left(\grave{I} - h\frac{\grave{I} - \ddot{a}}{2} \right) \right\} \right] \hat{H}'\left(\bar{e}\right) d\bar{e} \right|.$$

$$\leq \frac{1}{2\left(\grave{I}-\ddot{a}\right)}\left[\int_{\ddot{a}}^{\bar{c}}(\bar{e}-\ddot{a})\left|\hat{H}'(\bar{e})\right|d\bar{e}+\int_{\ddot{a}}^{\bar{c}}\left\{\bar{e}-\left(\ddot{a}+h\frac{\grave{I}-\ddot{a}}{2}\right)\right\}\left|\hat{H}'(\bar{e})\right|d\bar{e}\right.$$

$$\left.+\int_{\bar{c}}^{\grave{I}}\left(\bar{e}-\grave{I}\right)\left|\hat{H}'(\bar{e})\right|d\bar{e}+\int_{\bar{c}}^{\grave{I}}\left\{\bar{e}-\left(\grave{I}-h\frac{\grave{I}-\ddot{a}}{2}\right)\right\}\left|\hat{H}'(\bar{e})\right|d\bar{e}\right].$$

$$=\frac{1}{2\left(\grave{I}-\ddot{a}\right)}\left[(\bar{c}-\ddot{a})\int_{\ddot{a}}^{\bar{c}}\left|\hat{H}'(\bar{e})\right|d\bar{e}+\left(\grave{I}-\bar{c}\right)\int_{\bar{c}}^{\grave{I}}\left|\hat{H}'(\bar{e})\right|d\bar{e}\right.$$

$$+\left\{\bar{c}-\left(\ddot{a}+h\frac{\grave{I}-\ddot{a}}{2}\right)\right\}\int_{\ddot{a}}^{\bar{c}}\left|\hat{H}'(\bar{e})\right|d\bar{e}$$

$$\left.+\left\{\left(\grave{I}-h\frac{\grave{I}-\ddot{a}}{2}\right)-\bar{c}\right\}\int_{\bar{c}}^{\grave{I}}\left|\hat{H}'(\bar{e})\right|d\bar{e}\right].$$

$$\leq \frac{1}{2\left(\grave{I}-\ddot{a}\right)}\left[\max\left\{(\bar{c}-\ddot{a}),\left(\grave{I}-\bar{c}\right)\right\}\right.$$

$$\left.+\max\left\{\left\{\bar{c}-\left(\ddot{a}+h\frac{\grave{I}-\ddot{a}}{2}\right)\right\},\left\{\left(\grave{I}-h\frac{\grave{I}-\ddot{a}}{2}\right)-\bar{c}\right\}\right\}\right]$$

$$\times\left[\int_{\ddot{a}}^{\bar{c}}\left|\hat{H}'(\bar{e})\right|d\bar{e}+\int_{\bar{c}}^{\grave{I}}\left|\hat{H}'(\bar{e})\right|d\bar{e}\right].$$

Now, observe that

$$\max\left\{(\bar{c}-\ddot{a}),\left(\grave{I}-\bar{c}\right)\right\}=\frac{\grave{I}-\ddot{a}}{2}+\left(\bar{c}-\frac{\ddot{a}+\grave{I}}{2}\right)$$

and

$$\max\left\{\left\{\bar{c}-\left(\ddot{a}+h\frac{\grave{I}-\ddot{a}}{2}\right)\right\},\left\{\left(\grave{I}-h\frac{\grave{I}-\ddot{a}}{2}\right)-\bar{c}\right\}\right\}$$

$$=\frac{\left(\grave{I}-\ddot{a}\right)(1-h)}{2}+\left(\bar{c}-\frac{\ddot{a}+\grave{I}}{2}\right).$$

By using above equatons we get the inequility 10. □

Corollary 3. *If we put $\bar{c}=\ddot{a}$ in 10 we have 12*

$$\left|\hat{H}(\ddot{a})-\frac{1}{\grave{I}-\ddot{a}}\int_{\ddot{a}}^{\grave{I}}\hat{H}(\bar{e})\,d\bar{e}\right| \tag{12}$$

$$\leq\left[\frac{1}{4}(2-h)-\frac{1}{2}\right]\left\|\hat{H}'\right\|_{1}.$$

If we put $h=0$ *in* 12 *we have* 13

$$\left|\hat{H}(\ddot{a})-\frac{1}{\grave{I}-\ddot{a}}\int\limits_{\ddot{a}}^{\grave{I}}\hat{H}(\bar{e})\,d\bar{e}\right| \leq 0. \tag{13}$$

Corollary 4. *If we put* $\bar{c}=\grave{I}$ *in* 10 *we have* 14

$$\left|\hat{H}\left(\grave{I}\right)-\frac{1}{\grave{I}-\ddot{a}}\int\limits_{\ddot{a}}^{\grave{I}}\hat{H}(\bar{e})\,d\bar{e}\right| \leq \left[\frac{1}{4}(2-h)+\frac{1}{2}\right]\left\|\hat{H}'\right\|_1. \tag{14}$$

If we put $h=0$ *in* 14 *we have* 15

$$\left|\hat{H}\left(\grave{I}\right)-\frac{1}{\grave{I}-\ddot{a}}\int\limits_{\ddot{a}}^{\grave{I}}\hat{H}(\bar{e})\,d\bar{e}\right| \leq \left\|\hat{H}'\right\|_1. \tag{15}$$

2.3. Case For L_p Norm.

Theorem 3. *Let* $\hat{H}:\left[\ddot{a},\grave{I}\right]\to\mathbb{R}$ *be continuous on* $\left[\ddot{a},\grave{I}\right]$, *single differentiable on* $\left(\ddot{a},\grave{I}\right)$ *and* $\hat{H}'\in L_p\left(\ddot{a},\grave{I}\right)$. *Then:*

$$\left|\hat{H}(\bar{c})-\frac{1}{\grave{I}-\ddot{a}}\int\limits_{\ddot{a}}^{\grave{I}}\hat{H}(\bar{e})\,d\bar{e}\right| \leq \frac{1}{2\left(\grave{I}-\ddot{a}\right)}\left[\frac{(\bar{c}-\ddot{a})^{q+1}}{q+1}\left\{1+\left(1-\frac{h\frac{\grave{I}-\ddot{a}}{2}}{\bar{c}-\ddot{a}}\right)\right\}^{q+1} + \frac{(\grave{I}-\bar{c})^{q+1}}{q+1}\left\{1+\left(1-\frac{h\frac{\grave{I}-\ddot{a}}{2}}{\grave{I}-\bar{c}}\right)\right\}^{q+1}\right]^{\frac{1}{q}}\left\|\hat{H}'\right\|_p. \tag{16}$$

If we put $h=0$ *in* 16 *we have* 17

$$\left|\hat{H}(\bar{c})-\frac{1}{\grave{I}-\ddot{a}}\int\limits_{\ddot{a}}^{\grave{I}}\hat{H}(\bar{e})\,d\bar{e}\right| \leq \frac{1}{\grave{I}-\ddot{a}}\left[\frac{(\bar{c}-\ddot{a})^{q+1}+\left(\grave{I}-\bar{c}\right)^{q+1}}{q+1}\right]^{\frac{1}{q}}\left\|\hat{H}'\right\|_p. \tag{17}$$

$\forall\ \bar{e}\in\left[\ddot{a},\grave{I}\right]$, $\bar{c}\epsilon\left[\ddot{a},\frac{\ddot{a}+\grave{I}}{2}\right]$ *and* $h\epsilon\,[0,1]$.

Proof. As we have

$$\left| \hat{H}(\bar{c}) - \frac{1}{\grave{I} - \ddot{a}} \int_{\ddot{a}}^{\grave{I}} \hat{H}(\bar{e})\, d\bar{e} \right| = \frac{1}{2\left(\grave{I} - \ddot{a}\right)} \left| \int_{\ddot{a}}^{\grave{I}} P(\bar{c}, \bar{e}) \hat{H}'(\bar{e})\, d\bar{e} \right|.$$

$$\leq \frac{1}{2\left(\grave{I} - \ddot{a}\right)} \left[\int_{\ddot{a}}^{\grave{I}} P^q(\bar{c}, \bar{e}) d\bar{e} \right]^{\frac{1}{q}} \left\| \hat{H}' \right\|_p.$$

$$= \frac{1}{2\left(\grave{I} - \ddot{a}\right)} \left[\int_{\ddot{a}}^{\bar{c}} (\bar{e} - \ddot{a})^q d\bar{e} + \int_{\ddot{a}}^{\bar{c}} \left\{ \bar{e} - \left(\ddot{a} + h\frac{\grave{I} - \ddot{a}}{2} \right) \right\}^q d\bar{e} \right.$$

$$\left. + \int_{\bar{c}}^{\grave{I}} (\bar{e} - \grave{I})^q d\bar{e} + \int_{\bar{c}}^{\grave{I}} (\left\{ \bar{e} - \left(\grave{I} - h\frac{\grave{I} - \ddot{a}}{2} \right) \right\}^q d\bar{e} \right]^{\frac{1}{q}} \left\| \hat{H}' \right\|_p.$$

Now, observe that

$$\int_{\ddot{a}}^{\bar{c}} (\bar{e} - \ddot{a})^q d\bar{e} + \int_{\bar{c}}^{\grave{I}} (\bar{e} - \grave{I})^q d\bar{e} = \frac{(\bar{c} - \ddot{a})^{q+1}}{q+1} + \frac{(\grave{I} - \bar{c})^{q+1}}{q+1}$$

and

$$\int_{\ddot{a}}^{\bar{c}} \left\{ \bar{e} - \left(\ddot{a} + h\frac{\grave{I} - \ddot{a}}{2} \right) \right\}^q d\bar{e} + \int_{\bar{c}}^{\grave{I}} (\left\{ \bar{e} - \left(\grave{I} - h\frac{\grave{I} - \ddot{a}}{2} \right) \right\}^q d\bar{e}$$

$$= \frac{\left\{ \bar{c} - \left(\ddot{a} + h\frac{\grave{I} - \ddot{a}}{2} \right) \right\}^{q+1}}{q+1} + \frac{\left\{ \left(\grave{I} - h\frac{\grave{I} - \ddot{a}}{2} \right) - \bar{c} \right\}^{q+1}}{q+1}.$$

By using above equatons we get the inequility 16. □

Corollary 5. *If we put* $\bar{c} = \ddot{a}$ *in* 16 *we have* 18

$$\left| \hat{H}(\ddot{a}) - \frac{1}{\grave{I} - \ddot{a}} \int_{\ddot{a}}^{\grave{I}} \hat{H}(\bar{e})\, d\bar{e} \right| \tag{18}$$

$$\leq \frac{1}{2\left(\grave{I} - \ddot{a}\right)} \left[\frac{(\grave{I} - \ddot{a})^{q+1}}{q+1} \left\{ 1 + \left(1 - \frac{h\frac{\grave{I} - \ddot{a}}{2}}{\grave{I} - \ddot{a}} \right) \right\}^{q+1} \right]^{\frac{1}{q}} \left\| \hat{H}' \right\|_p.$$

If we put $h = 0$ *in* 18 *we have* 19

$$\left| \hat{H}(\ddot{a}) - \frac{1}{\grave{I} - \ddot{a}} \int_{\ddot{a}}^{\grave{I}} \hat{H}(\bar{e})\, d\bar{e} \right| \tag{19}$$

$$\leq \frac{1}{\grave{I} - \ddot{a}} \left[\frac{\left(\grave{I} - \ddot{a} \right)^{q+1}}{q+1} \right]^{\frac{1}{q}} \left\| \hat{H}' \right\|_p.$$

Corollary 6. *If we put* $\bar{c} = \grave{I}$ *in* 10 *we have* 20

$$\left| \hat{H}\left(\grave{I}\right) - \frac{1}{\grave{I} - \ddot{a}} \int\limits_{\ddot{a}}^{\grave{I}} \hat{H}\left(\bar{e}\right) d\bar{e} \right| \tag{20}$$

$$\leq \frac{1}{2\left(\grave{I} - \ddot{a}\right)} \left[\frac{(\grave{I} - \ddot{a})^{q+1}}{q+1} \left\{ 1 + \left(1 - \frac{h\frac{\grave{I} - \ddot{a}}{2}}{\grave{I} - \ddot{a}} \right) \right\}^{q+1} \right]^{\frac{1}{q}} \left\| \hat{H}' \right\|_p .$$

If we put $h = 0$ *in* 20 *we have* 21

$$\left| \hat{H}\left(\grave{I}\right) - \frac{1}{\grave{I} - \ddot{a}} \int\limits_{\ddot{a}}^{\grave{I}} \hat{H}\left(\bar{e}\right) d\bar{e} \right| \tag{21}$$

$$\leq \frac{1}{\grave{I} - \ddot{a}} \left[\frac{\left(\grave{I} - \ddot{a}\right)^{q+1}}{q+1} \right]^{\frac{1}{q}} \left\| \hat{H}' \right\|_p .$$

3. An Application to Comulative Distribution Function

Let $\bar{c}$ be a random variable taking values in the finite interval $\left[\ddot{a}, \grave{I}\right]$ with the probability density function $F : \left[\ddot{a}, \grave{I}\right] \to [0, 1]$ and cumulative distributive function

$$F\left(\bar{c}\right) = \Pr\left(\bar{c} \leq \bar{c}\right) = \int_{\ddot{a}}^{\bar{c}} F\left(\bar{e}\right) d\bar{e}, \tag{22}$$

$$F\left(\grave{I}\right) = \Pr\left(\bar{c} \leq \grave{I}\right) = \int_{\ddot{a}}^{\grave{I}} F(u) du = 1. \tag{23}$$

Theorem 4. *With the assumption of Theorem* 1*, we have the following inequality wich holds*

$$\left| \frac{\grave{I} - E\left(\bar{c}\right)}{\grave{I} - \ddot{a}} - F\left(\bar{c}\right) \right| \tag{24}$$

$$\leq \left[\frac{1}{4}(2 - h) + \frac{\bar{c} - \frac{\ddot{a} + \grave{I}}{2}}{\grave{I} - \ddot{a}} \right] \left\| F' \right\|_1 .$$

If we put $h = 0$ *in* 22 *we have* 25

$$\left| \frac{\grave{I} - E\left(\bar{c}\right)}{\grave{I} - \ddot{a}} - F\left(\bar{c}\right) \right| \tag{25}$$

$$\leq \left[\frac{1}{2} + \frac{\bar{c} - \frac{\ddot{a} + \grave{I}}{2}}{\grave{I} - \ddot{a}} \right] \left\| F' \right\|_1 .$$

$\forall\ \bar{e} \in \left[\ddot{a}, \grave{I}\right]$, $\bar{c} \epsilon \left[\ddot{a}, \frac{\ddot{a} + \grave{I}}{2}\right]$ *and* $h \epsilon\, [0, 1]$. *Where* $E\left(\bar{c}\right)$ *is the expectation of* $\bar{c}$.

Proof. By 4 and 5, on choosing $\hat{H} = F$ and using the fact

$$E(\bar{c}) = \int_{\hat{h}}^{\breve{a}} \dot{g} dF(\dot{g}) = \grave{I} - \int_{\hat{h}}^{\breve{a}} F\left(\dot{g}\right) d\dot{g}.$$

We obtain 24 and 25. □

Corollary 7. *If we put* $\bar{c} = \ddot{a}$ *in* 24 *we have* 26

$$\left|\frac{\grave{I} - E(\bar{c})}{\grave{I} - \ddot{a}} - F(\ddot{a})\right| \leq \left(\grave{I} - \ddot{a}\right)\left[\frac{1}{8}\left[1 + (1-h)^2\right] + \frac{1}{4}\right]\left\| F'\right\|_\infty \tag{26}$$

If we put $h = 0$ *in* 26 *we have* 27

$$\left|\frac{\grave{I} - E(\bar{c})}{\grave{I} - \ddot{a}} - F(\ddot{a})\right| \leq \frac{1}{2}\left(\grave{I} - \ddot{a}\right)\left\| F'\right\|_\infty . \tag{27}$$

Corollary 8. *If we put* $\bar{c} = \grave{I}$ *in* 24 *we have* 28

$$\left|\frac{\grave{I} - E(\bar{c})}{\grave{I} - \ddot{a}} - F\left(\grave{I}\right)\right| \leq \left(\grave{I} - \ddot{a}\right)\left[\frac{1}{8}\left[1 + (1-h)^2\right] + \frac{1}{4}\right]\left\| F'\right\|_\infty \tag{28}$$

If we put $h = 0$ *in* 28 *we have* 29

$$\left|\frac{\grave{I} - E(\bar{c})}{\grave{I} - \ddot{a}} - F\left(\grave{I}\right)\right| \leq \frac{1}{2}\left(\grave{I} - \ddot{a}\right)\left\| F'\right\|_\infty . \tag{29}$$

Theorem 5. *With the assumption of Theorem* 2*, we have the following inequality wich holds*

$$\left|\frac{\grave{I} - E(\bar{c})}{\grave{I} - \ddot{a}} - F(\bar{c})\right| \leq \left[\frac{1}{4}(2-h) + \frac{\bar{c} - \frac{\ddot{a}+\grave{I}}{2}}{\grave{I} - \ddot{a}}\right]\left\| F'\right\|_1 . \tag{30}$$

If we put $h = 0$ *in* 30 *we have* 31

$$\left|\frac{\grave{I} - E(\bar{c})}{\grave{I} - \ddot{a}} - F(\bar{c})\right| \leq \left[\frac{1}{2} + \frac{\bar{c} - \frac{\ddot{a}+\grave{I}}{2}}{\grave{I} - \ddot{a}}\right]\left\| F'\right\|_1 . \tag{31}$$

$\forall\ \bar{e} \in \left[\ddot{a}, \grave{I}\right]$, $\bar{c}\epsilon\left[\ddot{a}, \frac{\ddot{a}+\grave{I}}{2}\right]$ *and* $h\epsilon\,[0,1]$. *Where* $E(\bar{c})$ *is the expectation of* $\bar{c}$.

Proof. By using 10 − 11 and the same condition that we use in above theorem, we get the required inequility 30and 31. □

Corollary 9. *If we put* $\bar{c} = \ddot{a}$ *in* 30 *we have* 32

$$\left|\frac{\grave{I} - E(\bar{c})}{\grave{I} - \ddot{a}} - F(\ddot{a})\right| \leq \left[\frac{1}{4}(2-h) - \frac{1}{2}\right]\left\| F'\right\|_1 . \tag{32}$$

If we put $h = 0$ *in* 32 *we have* 33

$$\left|\frac{\grave{I} - E(\bar{c})}{\grave{I} - \ddot{a}} - F(\ddot{a})\right| \leq 0. \tag{33}$$

Corollary 10. *If we put* $\bar{c} = \grave{I}$ *in* 30 *we have* 34

$$\left|\frac{\grave{I} - E(\bar{c})}{\grave{I} - \ddot{a}} - F\left(\grave{I}\right)\right| \leq \left[\frac{1}{4}(2-h) + \frac{1}{2}\right] \left\| F' \right\|_1 . \tag{34}$$

If we put $h = 0$ *in* 34 *we have* 35

$$\left|\frac{\grave{I} - E(\bar{c})}{\grave{I} - \ddot{a}} - F\left(\grave{I}\right)\right| \leq \left\| F' \right\|_1 . \tag{35}$$

Theorem 6. *With the assumption of Theorem* 3, *we have the following inequality wich holds*

$$\left|\frac{\grave{I} - E(\bar{c})}{\grave{I} - \ddot{a}} - F(\bar{c})\right| \leq \left[\frac{1}{4}(2-h) + \frac{\bar{c} - \frac{\ddot{a}+\grave{I}}{2}}{\grave{I} - \ddot{a}}\right] \left\| F' \right\|_1 . \tag{36}$$

If we put $h = 0$ *in* 36 *we have* 37

$$\left|\frac{\grave{I} - E(\bar{c})}{\grave{I} - \ddot{a}} - F(\bar{c})\right| \leq \left[\frac{1}{2} + \frac{\bar{c} - \frac{\ddot{a}+\grave{I}}{2}}{\grave{I} - \ddot{a}}\right] \left\| F' \right\|_1 . \tag{37}$$

$\forall\ \bar{e} \in \left[\ddot{a}, \grave{I}\right]$, $\bar{c}\epsilon \left[\ddot{a}, \frac{\ddot{a}+\grave{I}}{2}\right]$ *and* $h\epsilon\,[0,1]$. *Where* $E(\bar{c})$ *is the expectation of* $\bar{c}$.

Proof. By using $16 - 17$ and the same condition that we use in theorem 4, we get the required inequility 36and 37. □

Corollary 11. *If we put* $\bar{c} = \ddot{a}$ *in* 36 *we have* 38

$$\left|\frac{\grave{I} - E(\bar{c})}{\grave{I} - \ddot{a}} - F(\ddot{a})\right| \leq \frac{1}{2\left(\grave{I} - \ddot{a}\right)} \left[\frac{(\grave{I} - \ddot{a})^{q+1}}{q+1} \left\{1 + \left(1 - \frac{h\frac{\grave{I}-\ddot{a}}{2}}{\grave{I} - \ddot{a}}\right)\right\}^{q+1}\right]^{\frac{1}{q}} \left\| F' \right\|_p . \tag{38}$$

If we put $h = 0$ *in* 38 *we have* 39

$$\left|\frac{\grave{I} - E(\bar{c})}{\grave{I} - \ddot{a}} - F(\ddot{a})\right| \leq \frac{1}{\grave{I} - \ddot{a}}\left[\frac{\left(\grave{I} - \ddot{a}\right)^{q+1}}{q+1}\right]^{\frac{1}{q}} \left\| F' \right\|_p . \tag{39}$$

Corollary 12. *If we put* $\bar{c} = \grave{I}$ *in* 36 *we have* 40

$$\left|\frac{\grave{I} - E(\bar{c})}{\grave{I} - \ddot{a}} - F\left(\grave{I}\right)\right| \leq \frac{1}{2\left(\grave{I} - \ddot{a}\right)}\left[\frac{(\grave{I} - \ddot{a})^{q+1}}{q+1}\left\{1 + \left(1 - \frac{h\frac{\grave{I}-\ddot{a}}{2}}{\grave{I} - \ddot{a}}\right)\right\}^{q+1}\right]^{\frac{1}{q}} \left\| F' \right\|_p . \tag{40}$$

If we put $h = 0$ *in* 40 *we have* 41

$$\left|\frac{\grave{I} - E(\bar{c})}{\grave{I} - \ddot{a}} - F\left(\grave{I}\right)\right| \leq \frac{1}{\grave{I} - \ddot{a}}\left[\frac{\left(\grave{I} - \ddot{a}\right)^{q+1}}{q+1}\right]^{\frac{1}{q}} \left\| F' \right\|_p . \tag{41}$$

4. Conclusion

In this paper, we constructed the modified Ostrowski's type inequalities for various norms by using well known inequalities. Furthermore, we also discussed some perturbed results. Notably, we developed a new peano kernel i.e., modified 2-step linear kernel. Finally, we applied the outcomes of our result to the domain of numerical integration and also apply an application to Comulative Distribution Function.

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