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ON A FAMILY OF FIFTEENTH-ORDER DIFFERENCE EQUATIONS

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Abstract. Difference equations are mathematical tools that are useful in modeling diverse dynamic systems because they represent how a variable changes across discrete time increments. Applying symmetries to complicated difference equations can be a valuable tool for simplification. Transformations based on symmetry allow one to lower the order of difference equations, making them more comprehensible and solvable. The primary purpose of this project is to generalize and extend some results in [A. M. Ahmeda, S. Mohammadya, L. Aljoufia, Expressions and dynamical behavior of solutions of a class of rational difference equations of fifteenth-order, J. Math. Computer Sci. 25 (2022) 10–22] using symmetries.

1. Introduction

The study of difference equations has attracted the attention of many researchers. They are employed to model phenomena in which the variable is discrete. Sophus Lie (1842-1899) pioneered the notion of continuous symmetry in the nineteenth century. He invented and applied symmetry analysis to differential equations between 1872 and 1899. This ground-breaking theory paved way for algorithms to be used to solve differential equations in a systematic manner. Maeda demonstrated in 1987 that an enhanced version of Lie's approach could also be used to solve ordinary difference equations.

In this study, we embark on a quest to unveil the profound connection between difference equations and Lie symmetry analysis. We will explore how the application of symmetry principles can clarify the underlying structure of discrete dynamical systems and provide new tools for their analysis. Through a series of calculations and theoretical developments, we will demonstrate the power of symmetries in uncovering hidden patterns in the difference equation understudy, ultimately advancing our understanding of the behavior of the solution.

This work is inspired by the work of Ahmeda et. al. [1], where the authors studied the difference equations

$$x_{n+1} = \frac{x_{n-14}}{\pm 1 \pm x_n x_{n-2} x_{n-5} x_{n-8} x_{n-11} x_{n-14}}. \tag{1}$$

They obtained the expressions and dynamical behavior of solutions of (1) using mostly proof by induction. We use symmetry methods to solve for the generalized difference equation below

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$$x_{n+1}=\frac{x_{n-14}}{a_n+b_nx_nx_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}}, \tag{2}$$

where A_n and B_n are real numbers.

1.1. **Preliminaries on construction of symmetries for difference equations.** To begin, consider the fifteenth-order ordinary difference equation

$$x_{n+15}=\omega(n,x_n,x_{n+3},x_{n+6},x_{n+9},x_{n+12}),\quad \frac{\partial\omega}{\partial x_n}\neq 0, \tag{3}$$

where ω represents a smooth function and n an independent variable. The general solution of (3) depends on arbitrary variables and may be expressed as

$$x_n=f(n,c_1,\cdots,c_{15}). \tag{4}$$

Definition 1. *The forward shift operator is given by*

$$S:n\mapsto n+1,\qquad S^ix_n=x_{n+i}. \tag{5}$$

The fifteenth-order ordinary difference equation (3) admits a symmetry generator X given by

$$X=Q\frac{\partial}{\partial x_n}+S^3Q\frac{\partial}{\partial x_{n+3}}+S^6Q\frac{\partial}{\partial x_{n+6}}+S^9Q\frac{\partial}{\partial x_{n+9}}+S^{12}Q\frac{\partial}{\partial x_{n+12}} \tag{6}$$

that meets the symmetry requirement

$$\mathcal{S}^{(15)}Q-X\omega=0. \tag{7}$$

The function $Q=Q(n,x_n)$ is known as the characteristic of the group of transformations. For more details on this, please see [7].

2. Main Results

We will employ Lie point symmetry in this chapter to obtain generic solutions to the fifteen-order ordinary difference equation (2). Given the definitions and notation utilized in this work, we consider the equivalent fifteenth-order ordinary difference equation

$$x_{n+15}=\frac{x_n}{A_n+B_nx_nx_{n+3}x_{n+6}x_{n+9}x_{n+12}} \tag{8}$$

of (2). Applying the symmetry condition (7) to (8), we get

$$S^{15}Q-\left(Q\frac{\partial\omega}{\partial x_n}+S^3Q\frac{\partial w}{\partial x_{n+3}}+S^6Q\frac{\partial w}{\partial x_{n+6}}+S^9Q\frac{\partial w}{\partial x_{n+9}}+S^{12}Q\frac{\partial w}{\partial x_{n+12}}\right)=0 \tag{9}$$

where ω is the right hand side expression in (8) and noting that $\omega_{,y}$ denotes the partial derivative of ω to y. Applying the operator

$$L=\frac{\partial}{\partial x_n}+\frac{A_n}{B_nx_n^2x_{n+6}x_{n+9}x_{n+12}}\frac{\partial}{\partial x_{n+3}} \tag{10}$$

to (9), we obtain

$$\begin{aligned}
&\frac{A_nQ(n+12,x_{n+12})}{x_{n+12}(A_n+B_nx_nx_{n+3}x_{n+6}x_{n+9}x_{n+12})^2}+\frac{A_nQ(n+9,x_{n+9})}{x_{n+9}(A_n+B_nx_nx_{n+3}x_{n+6}x_{n+9}x_{n+12})^2}\\
&+\frac{A_nQ(n+6,x_{n+6})}{x_{n+6}(A_n+B_nx_nx_{n+3}x_{n+6}x_{n+9}x_{n+12})^2}+\frac{A_nQ'(n+3,x_{n+3})}{(A_n+B_nx_nx_{n+3}x_{n+6}x_{n+9}x_{n+12})^2}\\
&+\frac{A_nQ'(n,x_n)}{(A_n+B_nx_nx_{n+3}x_{n+6}x_{n+9}x_{n+12})^2}+\frac{2A_nQ(n,x_n)}{x_n(A_n+B_nx_nx_{n+3}x_{n+6}x_{n+9}x_{n+12})^2}=0 \qquad (11)
\end{aligned}$$

or simply

$$\begin{aligned}
&2x_{n+6}x_{n+9}x_{n+12}Q(n,x_n)-x_nx_{n+6}x_{n+9}x_{n+12}(Q'(n,x_n)-Q'(n+3,x_{n+3}))+\\
&x_n(x_{n+9}x_{n+12}Q(n+6,x_{n+6})+x_{n+6}x_{n+12}Q(n+9,x_{n+9})\\
&+x_{n+6}x_{n+9}Q(n+12,x_{n+12}))=0. \qquad (12)
\end{aligned}$$

To get around the difficulty of dealing with different arguments, we differentiate (12) twice in relation to x_n. Thus, we get the following:

$$x_nx_{n+6}x_{n+9}x_{n+12}Q'''(n,x_n)=0. \qquad (13)$$

The generic solution is

$$Q(n,x_n)=\gamma_nx_n^2+\alpha_nx_n+\beta_n \qquad (14)$$

where α_n, β_n and γ_n are some functions of n.
By substituting $Q(n,x_n)$ in the symmetry condition (7), we get that $\gamma_n=\beta_n=0$ and α_n must satisfy

$$\begin{cases}\alpha_{n+3}+\alpha_{n+6}+\alpha_{n+9}+\alpha_{n+12}+\alpha_{n+15}=0\\ \alpha_n-\alpha_{n+15}=0.\end{cases} \qquad (15)$$

Hence, the symmetry generator is of the form

$$X=Q(n,x_n)\frac{\partial}{\partial x_n}=\alpha_nx_n\frac{\partial}{\partial x_n}, \qquad (16)$$

where , thanks to (15),

$$\alpha_n+\alpha_{n+3}+\alpha_{n+6}+\alpha_{n+9}+\alpha_{n+12}=0. \qquad (17)$$

The solutions of (17) are

$$\alpha_n=e^{\frac{i(2kn\pi)}{15}}, \qquad (18)$$

$k=1,2,3,4,6,7,8,9,11,12,13,14$. It follows form (16) and (18) that

$$X_k=e^{i\frac{2kn\pi}{15}}x_n\frac{\partial}{\partial x_n}, \qquad (19)$$

$k=1,2,3,4,6,7,8,9,11,12,13,14$, are symmetries of (8). Introducing the canonical coordinate

$$S_n=\int\frac{1}{Q(n,x_n)}dx_n=\int\frac{1}{\alpha_nx_n}dx_n, \qquad (20)$$

we get

$$S_n\alpha_n=\ln|x_n|. \qquad (21)$$

Let $\tilde{F}_n = S_n\alpha_n + S_{n+3}\alpha_{n+3} + S_{n+6}\alpha_{n+6} + S_{n+9}\alpha_{n+9} + S_{n+12}\alpha_{n+12}$ and

$$F_n = e^{-\tilde{F}_n}. \tag{22}$$

As a result, we obtain $\tilde{F}_n = \ln(x_n x_{n+3} x_{n+6} x_{n+9} x_{n+12})$ and

$$F_n = \frac{1}{x_n x_{n+3} x_{n+6} x_{n+9} x_{n+12}}. \tag{23}$$

Now shifting equation (23) three times and substituting the expression of x_{n+15} given in (8), we get

$$F_{n+3} = A_n F_n + B_n. \tag{24}$$

By iterating (24), we get the following:

$$F_{3n+j} = F_j\left(\prod_{t=0}^{n-1} A_{3t+j}\right) + \sum_{i=0}^{n-1}\left(B_{3i+j}\prod_{k_2=i+1}^{n-1} A_{3k_2+j}\right), \quad j = 0, 1, 2. \tag{25}$$

Also,

$$x_{n+15} = \frac{F_n}{F_{n+3}} x_n \tag{26}$$

whose iteration yields

$$\begin{aligned} x_{15n+k} =& x_k\left(\prod_{s=0}^{n-1}\frac{F_{15s+k}}{F_{15s+k+3}}\right), \quad k = 0, 1, 2, \cdots, 14 \\ =& x_k\left(\prod_{s=0}^{n-1}\frac{F_{15s+3\lfloor\frac{k}{3}\rfloor+\tau(k)}}{F_{15s+3\lfloor\frac{k}{3}\rfloor+\tau(k)+3}}\right) \\ =& x_k\left(\prod_{s=0}^{n-1}\frac{F_{3(5s+\lfloor\frac{k}{3}\rfloor)+\tau(k)}}{F_{3(5s+1+\lfloor\frac{k}{3}\rfloor)+\tau(k)}}\right) \end{aligned} \tag{27}$$

since we can always write any integer in the form $a = 3\lfloor\frac{a}{3}\rfloor + \tau(i)$, where $\tau(i)$ is the remainder when a is divided by 3. Substituting (25) into equation (27), we obtain

$$x_{15n+k} = x_k\prod_{s=0}^{n-1}\frac{F_{\tau(k)}\left(\prod_{t=0}^{5s+\lfloor\frac{k}{3}\rfloor-1} A_{3t+\tau(k)}\right) + \sum_{i=0}^{5s+\lfloor\frac{k}{3}\rfloor-1}\left(B_{3i+\tau(k)}\prod_{k_2=i+1}^{5s+\lfloor\frac{k}{3}\rfloor-1} A_{3k_2+\tau(k)}\right)}{F_{\tau(k)}\left(\prod_{t=0}^{5s+\lfloor\frac{k}{3}\rfloor} A_{3t+\tau(k)}\right) + \sum_{i=0}^{5s+\lfloor\frac{k}{3}\rfloor}\left(B_{3i+\tau(k)}\prod_{k_2=i+1}^{5s+\lfloor\frac{k}{3}\rfloor} A_{3k_2+\tau(k)}\right)} \tag{28}$$

$$= x_k\prod_{s=0}^{n-1}\frac{\left(\prod_{t=0}^{5s+\lfloor\frac{k}{3}\rfloor-1} A_{3t+\tau(k)}\right) + \sum_{i=0}^{5s+\lfloor\frac{k}{3}\rfloor-1}\left(\frac{B_{3i+\tau(k)}}{F_{\tau(k)}}\prod_{k_2=i+1}^{5s+\lfloor\frac{k}{3}\rfloor-1} A_{3k_2+\tau(k)}\right)}{F_{\tau(k)}\left(\prod_{t=0}^{5s+\lfloor\frac{k}{3}\rfloor} A_{3t+\tau(k)}\right) + \sum_{i=0}^{5s+\lfloor\frac{k}{3}\rfloor}\left(\frac{B_{3i+\tau(k)}}{F_{\tau(k)}}\prod_{k_2=i+1}^{5s+\lfloor\frac{k}{3}\rfloor} A_{3k_2+\tau(k)}\right)}, \tag{29}$$

where $1/F_{\tau(k)} = x_{\tau(k)} x_{\tau(k)+3} x_{\tau(k)+6} x_{\tau(k)+9} x_{\tau(k)+12}$.

3. Case where $A_n = A$ and $B_n = B$

Substituting $A_n = A$ and $B_n = B$ into equation (28), we obtain

$$x_{15n+k} = x_k \left(\prod_{s=0}^{n-1} \frac{A^{5s+\lfloor \frac{k}{3} \rfloor} + \frac{B}{F_{\tau(k)}} \sum_{s=0}^{5s+\lfloor \frac{k}{3} \rfloor -1} A^s}{A^{5s+1+\lfloor \frac{k}{3} \rfloor} + \frac{B}{F_{\tau(k)}} \sum_{s=0}^{5s+\lfloor \frac{k}{3} \rfloor} A^s} \right) \tag{30}$$

$$= x_k \left(\prod_{s=0}^{n-1} \frac{A^{5s+\lfloor \frac{k}{3} \rfloor} + B x_{\tau(k)} x_{\tau(k)+3} x_{\tau(k)+6} x_{\tau(k)+9} x_{\tau(k)+12} \sum_{s=0}^{5s+\lfloor \frac{k}{3} \rfloor -1} A^s}{A^{5s+1+\lfloor \frac{k}{3} \rfloor} + B x_{\tau(k)} x_{\tau(k)+3} x_{\tau(k)+6} x_{\tau(k)+9} x_{\tau(k)+12} \sum_{s=0}^{5s+\lfloor \frac{k}{3} \rfloor} A^s} \right) \tag{31}$$

$$= \begin{cases} x_k \prod_{s=0}^{n-1} \frac{1+B(5s+\lfloor \frac{k}{3} \rfloor) x_{\tau(k)} x_{\tau(k)+3} x_{\tau(k)+6} x_{\tau(k)+9} x_{\tau(k)+12}}{1+B(5s+\lfloor \frac{k}{3} \rfloor +1) x_{\tau(k)} x_{\tau(k)+3} x_{\tau(k)+6} x_{\tau(k)+9} x_{\tau(k)+12}}, \text{ when } A = 1; \\ \\ x_k \prod_{s=0}^{n-1} \frac{A^{5s+\lfloor \frac{k}{3} \rfloor} + \frac{B(1-A^{5s+\lfloor \frac{k}{3} \rfloor})}{1-A} x_{\tau(k)} x_{\tau(k)+3} x_{\tau(k)+6} x_{\tau(k)+9} x_{\tau(k)+12}}{A^{5s+1+\lfloor \frac{k}{3} \rfloor} + \frac{B(1-A^{5s+1+\lfloor \frac{k}{3} \rfloor})}{1-A} x_{\tau(k)} x_{\tau(k)+3} x_{\tau(k)+6} x_{\tau(k)+9} x_{\tau(k)+12}}, \text{ when } A \neq 1 \end{cases} \tag{32}$$

for $k = 0, 1, 2 \cdots, 11$.

3.1. Case where $A = -1$.

For the special case $A = -1$, we have that

$$x_{15n+k} = x_k \left(\prod_{s=0}^{n-1} \frac{(-1)^s + \frac{B((-1)^{\lfloor \frac{k}{3} \rfloor} - (-1)^s)}{2} x_{\tau(k)} x_{\tau(k)+3} x_{\tau(k)+6} x_{\tau(k)+9} x_{\tau(k)+12}}{-(-1)^s + \frac{B((-1)^{\lfloor \frac{k}{3} \rfloor} + (-1)^s)}{2} x_{\tau(k)} x_{\tau(k)+3} x_{\tau(k)+6} x_{\tau(k)+9} x_{\tau(k)+12}} \right). \tag{33}$$

This means that

$$x_{15n+k} = \begin{cases} x_k & \text{if } n \text{ even} \\ \frac{x_k}{-1+Bx_k x_{k+3} x_{k+6} x_{k+9} x_{k+12}} & \text{if } n \text{ odd} \end{cases}, k = 0, 1, 2; \tag{34a}$$

$$x_{15n+k} = \begin{cases} x_k & \text{if } n \text{ even} \\ x_k(-1 + Bx_{k-3} x_k x_{k+3} x_{k+6} x_{k+9}) & \text{if } n \text{ odd} \end{cases}, k = 3, 4, 5; \tag{34b}$$

$$x_{15n+k} = \begin{cases} x_k & \text{if } n \text{ even} \\ \frac{x_k}{-1+Bx_{k-6} x_{k-3} x_k x_{k+3} x_{k+6}} & \text{if } n \text{ odd} \end{cases}, k = 6, 7, 8; \tag{34c}$$

$$x_{15n+k} = \begin{cases} x_k & \text{if } n \text{ even} \\ x_k(-1 + Bx_{k-9} x_{k-6} x_{k-3} x_k x_{k+3}) & \text{if } n \text{ odd} \end{cases}, k = 9, 10, 11; \tag{34d}$$

$$x_{15n+k} = \begin{cases} x_k & \text{if } n \text{ even} \\ x_k(-1 + Bx_{k-12} x_{k-9} x_{k-6} x_{k-3} x_k) & \text{if } n \text{ odd} \end{cases}, k = 12, 13, 14. \tag{34e}$$

3.2. **Special cases in literature.** Remember that we shifted equation (2) forward 14 times to get (8), the solution of which is provided by (32). We now reverse the equations in (32) 14 times to find the solution of the difference equation (2), which is given by

$$x_{15n+k-14} = x_{k-14}\prod_{s=0}^{n-1}\frac{1+B(5s+\lfloor\frac{k}{3}\rfloor)x_{\tau(k)-14}x_{\tau(k)-11}x_{\tau(k)-8}x_{\tau(k)-5}x_{\tau(k)-2}}{1+B(5s+\lfloor\frac{k}{3}\rfloor+1)x_{\tau(k)-14}x_{\tau(k)-11}x_{\tau(k)-8}x_{\tau(k)-5}x_{\tau(k)-2}}. \tag{35}$$

when $A=1$; and

$$x_{15n+k-14} = x_{k-14}\prod_{s=0}^{n-1}\frac{A^{5s+\lfloor\frac{k}{3}\rfloor}+\frac{B(1-A^{5s+\lfloor\frac{k}{3}\rfloor})}{1-A}x_{\tau(k)-14}x_{\tau(k)-11}x_{\tau(k)-8}x_{\tau(k)-5}x_{\tau(k)-2}}{A^{5s+1+\lfloor\frac{k}{3}\rfloor}+\frac{B(1-A^{5s+1+\lfloor\frac{k}{3}\rfloor})}{1-A}x_{\tau(k)-14}x_{\tau(k)-11}x_{\tau(k)-8}x_{\tau(k)-5}x_{\tau(k)-2}} \tag{36}$$

when $A \neq 1$.

If we let

$$j = 14-k, \tag{37a}$$

then

$$\lfloor\frac{j}{3}\rfloor = 4-\lfloor\frac{k}{3}\rfloor \tag{37b}$$

and

$$\tau(j) = 2-\tau(k) \tag{37c}$$

for $k=0,1,\ldots,14$. It follows from (35), (36) and (37) that

$$x_{15n-j} = x_{-j}\prod_{s=0}^{n-1}\frac{1+B(5s+4-\lfloor\frac{j}{3}\rfloor)x_{-\tau(j)-12}x_{-\tau(j)-9}x_{-\tau(j)-6}x_{-\tau(j)-3}x_{-\tau(j)}}{1+B(5s+5-\lfloor\frac{j}{3}\rfloor)x_{-\tau(j)-12}x_{-\tau(j)-9}x_{-\tau(j)-6}x_{-\tau(j)-3}x_{-\tau(j)}}, \tag{38}$$

when $A=1$; and

$$x_{15n-j} = x_{-j}\prod_{s=0}^{n-1}\frac{A^{5s+4-\lfloor\frac{j}{3}\rfloor}+\frac{B(1-A^{5s+4-\lfloor\frac{j}{3}\rfloor})}{1-A}x_{-\tau(j)-12}x_{-\tau(j)-9}x_{-\tau(j)-6}x_{-\tau(j)-3}x_{-\tau(j)}}{A^{5s+5-\lfloor\frac{j}{3}\rfloor}+\frac{B(1-A^{5s+5-\lfloor\frac{j}{3}\rfloor})}{1-A}x_{-\tau(j)-12}x_{-\tau(j)-9}x_{-\tau(j)-6}x_{-\tau(j)-3}x_{-\tau(j)}} \tag{39}$$

when $A \neq 1$.

3.2.1. *Case where $A=1$ and $B=1$.* Let $M_j = 5-\lfloor\frac{j}{3}\rfloor$, $P_j = \prod_{k=0}^{4} a_{mod(j,3)+3k}$ and $x_{-j} = a_j$. We have

$$P_j = \prod_{k=0}^{4} a_{mod(j,3)+3k} = x_{-\tau(j)-12}x_{-\tau(j)-9}x_{-\tau(j)-6}x_{-\tau(j)-3}x_{-\tau(j)}. \tag{40}$$

Then equation 38 becomes

$$x_{15n-j} = a_j\prod_{s=0}^{n-1}\left(\frac{1+(5s+M_j-1)P_j}{1+(5s+M_j)P_j}\right) \tag{41}$$

which is the same as Theorem 2.1 in Lama's paper given by the equation below:

$$x_{15n-k} = a_k\prod_{i=0}^{n-1}\left(\frac{1+(5i+M_k-1)P_k}{1+(5i+M_k)P_k}\right). \tag{42}$$

3.2.2. *Case where $A = 1$ and $B = -1$.* Similarly, when $A = 1$ and $B = -1$, equation (38) is given by

$$x_{15n-j} = a_j \prod_{s=0}^{n-1} \left(\frac{-1 + (5s + M_j - 1)P_j}{-1 + (5s + M_j)P_j} \right) \tag{43}$$

which is the same as Theorem 3.1 in Lama's paper as given below:

$$x_{15n-k} = a_k \prod_{i=0}^{n-1} \left(\frac{-1 + (5i + M_k - 1)P_k}{-1 + (5i + M_k)P_k} \right). \tag{44}$$

3.2.3. *Case where $A = -1$ and $B = 1$.* When we let $A = -1$ and $B = 1$, equation (39) becomes

$$x_{15n-j} = a_j \prod_{s=0}^{n-1} \frac{(-1)^{5s+M_j-1} + \frac{(1-(-1)^{5s+M_j-1})}{2} P_j}{(-1)^{5s+M_j} + \frac{(1-(-1)^{5s+M_j})}{2} P_j} \tag{45}$$

which is the same as Theorem 4.1 in Lama's paper given by

$$x_{15n-k} = a_k \prod_{i=0}^{n-1} \frac{(-1)^{5i+M_k-1} + \frac{(1-(-1)^{5i+M_k-1})}{2} P_k}{(-1)^{5i+M_k} + \frac{(1-(-1)^{5i+M_k})}{2} P_k}. \tag{46}$$

Note that $M_j = 5 - \lfloor \frac{j}{3} \rfloor$, $P_j = \prod_{k=0}^{4} a_{mod(j,3)+3k}$ and $x_{-j} = a_j$.

3.2.4. *Case where $A = -1$ and $B = -1$.* Similarly, when $A = 1$ and $B = -1$, equation (39) gives

$$x_{15n-j} = a_j \prod_{s=0}^{n-1} \frac{(-1)^{5s+M_j-1} + \frac{(-1-(-1)^{5s+M_j-2})}{2} P_j}{(-1)^{5s+M_j} + \frac{(-1-(-1)^{5s+M_j-1})}{2} P_j} \tag{47}$$

which is the same as Theorem 4.5 in Lama's paper given by

$$x_{15n-k} = a_k \prod_{i=0}^{n-1} \frac{(-1)^{5i+M_k-1} + \frac{(-1-(-1)^{5i+M_k-2})}{2} P_k}{(-1)^{5i+M_k} + \frac{(-1-(-1)^{5i+M_k-1})}{2} P_k}. \tag{48}$$

4. Numerical Examples

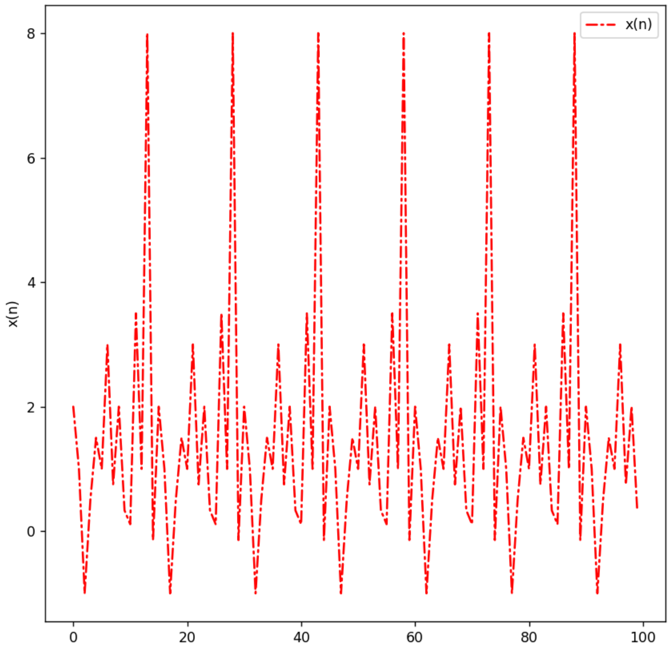

Figure 1. Graph of $x_{n+15} = \frac{x_n}{4-3x_nx_{n+3}x_{n+6}x_{n+9}x_{n+12}}$.

4.1. **Example 1.** Figure 1 depicts the graph of (8) with the initial conditions $x_0 = 2; x_1 = 1; x_2 = -1; x_3 = 1/2; x_4 = 3/2; x_5 = 1; x_6 = 3; x_7 = 3/4; x_8 = 2; x_9 = 1/3; x_{10} = 1/9; x_{11} = 7/2; x_{12} = 1; x_{13} = 8; x_{14} = -1/7$ satisfying

$$x_{\tau(j)}x_{\tau(j)+3}x_{\tau(j)+6}x_{\tau(j)+9}x_{\tau(j)+12} = \frac{1-A}{B} \tag{49}$$

and

$$x_i \neq x_{i+3}, x_i \neq x_{i+5}. \tag{50}$$

The answer is, as predicted, 15-periodic.

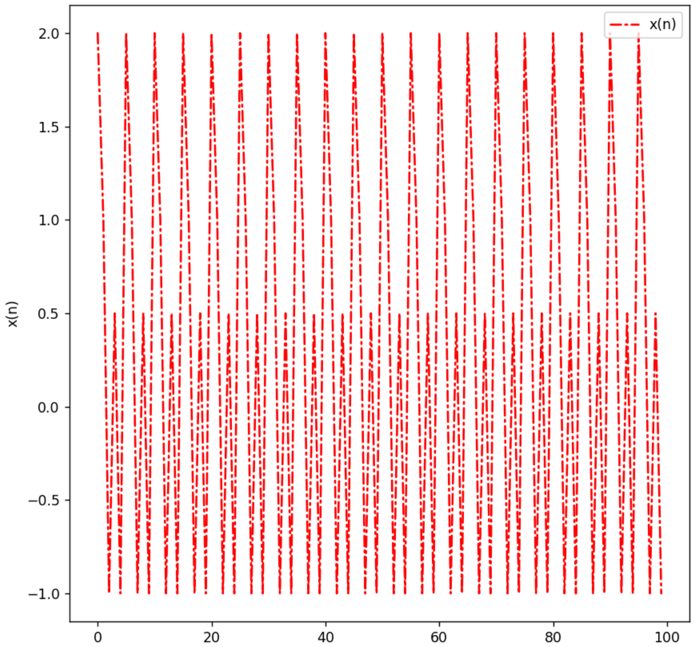

FIGURE 2. Graph of $x_{n+15} = \frac{x_n}{4-3x_n x_{n+3} x_{n+6} x_{n+9} x_{n+12}}$.

4.2. **Example 2.** Figure 2 depicts the graph of (8) with the initial conditions $x_0 = 2; x_1 = 1; x_2 = -1; x_3 = 1/2; x_4 = -1; x_5 = 2; x_6 = 1; x_7 = -1; x_8 = 1/2; x_9 = -1; x_{10} = 2; x_{11} = 1; x_{12} = -1; x_{13} = 1/2; x_{14} = -1$ satisfying

$$x_{\tau(j)} x_{\tau(j)+3} x_{\tau(j)+6} x_{\tau(j)+9} x_{\tau(j)+12} = \frac{1-A}{B} \tag{51}$$

and

$$x_i \neq x_{i+3}, x_i = x_{i+5}. \tag{52}$$

The answer is, as predicted, 5-periodic.

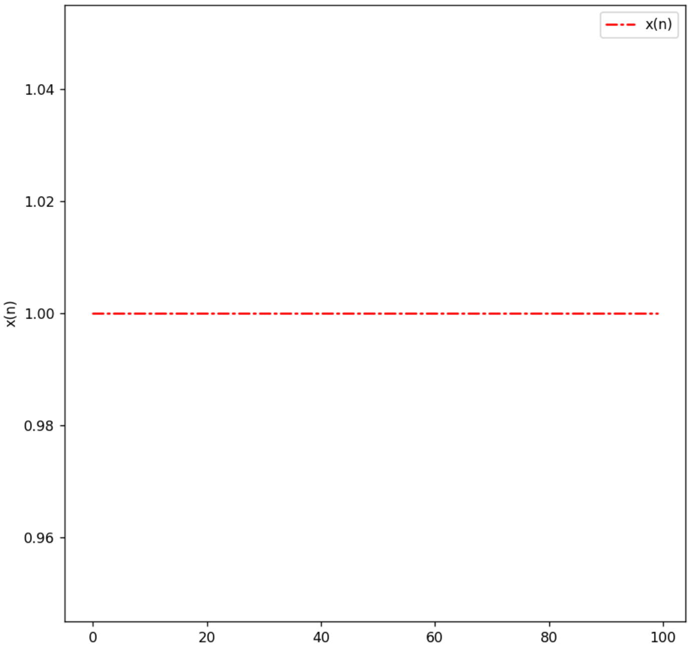

FIGURE 3. Graph of $x_{n+15} = \frac{x_n}{4-3x_n x_{n+3} x_{n+6} x_{n+9} x_{n+12}}$.

4.3. **Example 3.** Figure 3 depicts the graph of (8) with the initial conditions $x_0 = 1; x_1 = 1; x_2 = 1; x_3 = 1; x_4 = 1; x_5 = 1; x_6 = 1; x_7 = 1; x_8 = 1; x_9 = 1; x_{10} = 1; x_{11} = 1; x_{12} = 1; x_{13} = 1; x_{14} = 1$ satisfying

$$x^5_{\tau(j)} = \frac{1-A}{B} \tag{53}$$

and

$$x_i = x_{i+3}. \tag{54}$$

The answer is, as predicted, 1-periodic.

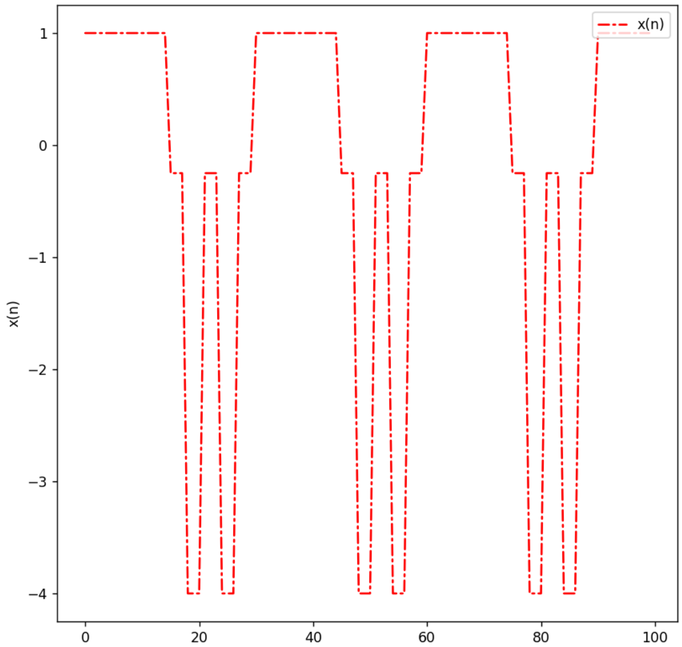

FIGURE 4. Graph of $x_{n+15} = \frac{x_n}{-1+Bx_nx_{n+3}x_{n+6}x_{n+9}x_{n+12}}$.

4.4. **Example 4.** Figure 4 depicts the graph of (8) with the initial conditions $x_0 = 1; x_1 = 1; x_2 = 1; x_3 = 1; x_4 = 1; x_5 = 1; x_6 = 1; x_7 = 1; x_8 = 1; x_9 = 1; x_{10} = 1; x_{11} = 1; x_{12} = 1; x_{13} = 1; x_{14} = 1$ satisfying

$$x^5_{\tau(j)} \neq \frac{2}{B} \tag{55}$$

and

$$x_i = x_j. \tag{56}$$

The answer is, as predicted, 30-periodic.

5. Conclusion

This investigation into the symmetry and exact Solutions of a fifteenth-Order difference equation has produced results. The major goal was to confirm and extend the findings of Lama et al. [1]. As a matter of fact, this goal has been achieved, it has been proved that the findings of this study are consistent with Lama's work through analysis and mathematical inquiry.

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