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MODIFIED MONOTONE HYBRID INERTIAL ALGORITHM FOR GENERALIZED NONEXPANSIVE MAPS, MAXIMAL MONOTONE OPERATORS AND GENERALIZED MIXED EQUILIBRIUM PROBLEMS

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Abstract. The aim of this article is to Propose a modified monotone hybrid inertial algorithm for approximating a common fixed point of a family of generalized nonexpansive maps, maximal monotone operators and a solution of a system of generalized mixed equilibrium problems in Banach spaces. Strong convergence of sequences generated by the propose algorithm has been establish in respect to this problems. Our theorems are significant improvement of some recent results.

1. Introduction

Let E be a real Banach space and C be a nonempty closed convex subset of a real Banach space E with $\| . \|$ and E^* as a norm and dual space of E respectively. A mapping $T : C \longrightarrow C$ is said to be nonexpansive if $\| Tx - Ty \| \leq \| x - y \|$ for all $x, y \in C$. $F(T) = \{x \in C : Tx = x\}$ denote the set of fixed point of T and $\mathbb{R}$ denote the set of real number. We present $GMEP$ [1] as the generalized mixed equilibrium problem: find an element $x \in C$ such that

$$B(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \forall y \in C,$$

where $B : C \times C \longrightarrow \mathbb{R}$ is a bifunction, $A : C \longrightarrow E^*$ is a nonlinear mapping and $\varphi : C \longrightarrow \mathbb{R}$ is a real valued function. The set of solutions of generalized mixed equilibrium problem is giving by

$$GMEP(B, A, \varphi) = \{x \in C : B(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \forall y \in C\}.$$

Notice that if $A \equiv 0$ and $\varphi \equiv 0$, then the generalized mixed equilibrium problem ($GMEP$) reduces to equilibrium problem, denoted by EP [3]: find an element $x \in C$ such that

$$B(x, y) \geq 0, \forall y \in C.$$

The solutions set of equilibrium problems is given by

$$EP(B) = \{x \in C : B(x, y) \geq 0, \forall y \in C\}.$$

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Many problems arising in physics, economics and optimization can be reduced to finding solutions of some equilibrium problems which is cornerstone in the field of search in science and engineering (for details see Blum and Oettli [3], Combettes and Hirstoage [7]).

Incorporating an inertial term in the iterative procedure accelerate the rate of convergence of the sequence generated by the algorithm, which was first suggested and studied by Polyak [22] for solving a smooth convex minimization problems (for details see [6, 8, 9]).

An operator $M \subset E \times E^*$ is called monotone if $\langle q-p, q^*-p^* \rangle \geq 0$, whenever $(q, q^*), (p, p^*) \in M$. A monotone M is called maximal if its graph $G(M) = \{(q, p) : p \in Mq\}$ is not properly contained in the graph of any other monotone operator. Recall that if M is a maximal monotone operator, $M^{-1}0 = \{q \in D(M) : 0 \in Mq\}$ is closed and convex, where $D(M) = \{q \in E : Mq \neq \emptyset\}$. For each $r > 0$, suppose that M is monotone, then we consider $J_r : R(I + rM) \longrightarrow D(M)$ defined by $J_r = (I + rM)^{-1}$ as nonexpansive mapping, where J_r is denoted as the resolvent of M (See for example, Browder [4] and Minty [13]). Furthermore, the monotone operators in Banach space can be formulated as follows: find $q \in E$ such that $0 \in Mq$. We denote $M^{-1}0$ as the set of all point $q \in E$ such that $0 \in Mq$.

In 2003 Nakajo and Takahashi [25], consider T as nonexpansive self mappings of a nonempty closed convex subset of C and established strong convergence theorem using modified Mann iterative procedure in Hilbert spaces. In 2006 Martinez-Yanes and Xu [20], presented modified Ishikawa iterative algorithm and proved strong convergence results by considering T as a nonexpansive self mapping and C as a nonempty, closed convex subset of a Hilbert space H. In 2008 Qin and Su [23], suggested a modified iterative procedure for approximating nonexpansive mapping in Hilbert space. A strong convergence results have been proved using monotone hybrid method.

In 2012 Klin-eam et al [16], consider the following monotone hybrid algorithm for solving family of generalized nonexpansive mapping in a Banach space:

$$\begin{cases} x_1 = x \in C,\ C_0 = Q_0 = C \ arbitrarily; \\ v_n = \gamma_n x_n + (1 - \gamma_n) T_n x_n; \\ C_n = \{v \in C_{n-1} \cap Q_{n-1} : \phi(v_n, v) \leq \phi(x_n, v)\}; \\ Q_n = \{v \in C_{n-1} \cap Q_{n-1} : \langle x - x_1, Jx_n - Jv \rangle \geq 0\}; \\ x_{n+1} = R_{C_n \cap Q_n} x, \forall n \in \mathbb{N}. \end{cases}$$

They established strong convergence of the sequence $\{x_n\}$ generated by the above algorithm.

In 2020 Chidume et al [6], approximated the set of zeros of maximal monotone operators and common fixed point of a countable family of relatively nonexpansive mappings in Banach spaces by considering hybrid inertial procedure as follows:

$$\begin{cases} x_0, x_1 \in K,\ K_0 = K; \\ \mu_n = x_n + \theta_n (x_n - x_{n-1}); \\ y_n = J^{-1}((1 - \sigma) J\mu_n + \sigma JSJ_{r_n} \mu_n); \\ z_n = J^{-1}((1 - \vartheta) J\mu_n + \vartheta JTy_n); \\ C_{n+1} = \{\mu \in C_n : \phi(\mu, z_n) \leq \phi(\mu, \mu_n); \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \forall n \in \mathbb{N} \cup \{0\}. \end{cases}$$

It has been proved that the sequence $\{x_n\}$ converges strongly to $\Pi_\Gamma x_0$.

Recently, Umar et al [19], introduced and studied the following inertial algorithm of generalized-f- projection technique for approximating a solution of a system of generalized mixed equilibrium problems, and maximal monotone operators in Banach space

$$\begin{cases} x_0 \in C_0 = E; \\ \vartheta_n = x_n + \alpha_n(x_n - x_{n-1}); \\ u_n = J^{-1}(\theta_n J\vartheta_n + (1-\theta_n)JJ_{r_n}\vartheta_n); \\ z_n \in C \text{ such that } f(z_n, y) + \langle \Psi z_n, y - z_n\rangle + \Theta(y) - \Theta(z_n) \\ +\frac{1}{r_n}\langle y - z_n, Jz_n - Ju_n\rangle \geq 0, \ \forall y \in C; \\ C_{n+1} = \{v \in C_n : G(v, Jz_n) \leq G(v, J\vartheta_n)\} : \\ x_{n+1} = \Pi^f_{C_{n+1}} x_0, \forall n \in \mathbb{N} \cup \{0\}. \end{cases}$$

The authors prove that $\{x_n\}$ converges strongly $\Pi^f_\Gamma x_0$

Very recently, Alizadeh and Morudlou [2], consider the following CQ algorithm for finding a family of non-self generalized nonexpansive mapping in Banach spaces, using monotone hybrid method

$$\begin{cases} x_1 = x_0 \in C, \ C_0 = Q_0 = C; \\ y_n = \alpha_n x_n + (1-\alpha_n)T_n x_n; \\ u_n = \theta_n y_n + (1-\theta_n)T_n x_n; \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(u_n, z) \leq \phi(x_n, z)\}; \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x - x_n, Jx_n - Jz\rangle\}; \\ x_{n+1} = R_{C_n \cap Q_n} x, \forall n \in \mathbb{N}. \end{cases}$$

It has been proved that the sequence $\{x_n\}$ converges strongly to $R_{F(\Gamma)}x$.

Motivated by the study of the class of a family of generalized nonexpansive mappings using monotone hybrid algorithm, it is our purpose to propose and study a modified monotone hybrid inertial algorithm for finding a common fixed point of a family of generalized nonexpansive mappings, maximal monotone operators and a solution of a system generalized mixed equilibrium problems in Banach spaces. Furthermore, a strong convergence theorem in respect to this problems has been establish. Also, our results improve and extend the result of Chidume et al [6], Umar et al [19], Alizadeh, Morudlou [2] and some recent results announced in the literature.

2. Preliminaries

Let E be a real Banach space with its dual space denoted by E^*, suppose that $\mathbb{R}$ and $\mathbb{N}$ denotes the set of real number and positive integer respectively. For any $\{x_n\} \subset E$ and a point $x \in E$, we consider $x_n \longrightarrow x$ and $x_n \rightharpoonup x$ as strong and weak convergences respectively. Furthermore, for any $\{x_n\} \subset E$, then E is said to satisfies Kadec - Klee property if $x_n \rightharpoonup x$ and $\| x_n \| \longrightarrow \| x \|$ which led to $x_n \longrightarrow x, \ \forall x \in E$. A mapping $J : E \longrightarrow 2^{E^*}$ is said to be normalized duality if:

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\|x^*\|, \|x^*\| = \|x\|\}, \forall x \in E.$$

Let $G := \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then E is said to be smooth if the $\lim_{t\to 0} \frac{\| x + ty \| - \| x \|}{t}$ exists for all $x, y \in G$, it is also said to be uniformly smooth if the limit exists uniformly in $x, y \in G$. A Banach space E said to be strictly convex if $\frac{\| x + y \|}{2} < 1$ for

all $x, y \in G$ with $\|x\| = \|y\| = 1$ and $x \neq y$ and E is said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\dfrac{\| x + y \|}{2} \leq 1 - \delta$ for all $x, y \in G$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$.
The function $\rho_E : [0, \infty) \longrightarrow [0, \infty)$ defined by

$$\rho_E(t) = sup\{\frac{\| x + y \| + \| x - y \|}{2} - 1; \|x\| = 1 : x \in G, \|y\| \leq t\}.$$

is called the modulus of smoothness of E. Also, the function $\delta : [0, 2] \longrightarrow [0, 1]$ defined by

$$\delta(\varepsilon) = inf\{1 - \|\frac{x + y}{2}\| : x, y \in G, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\}.$$

is called modulus of convexity of E. Now, consider E as a smooth Banach space, then a map $\phi : E \times E \longrightarrow \mathbb{R}$ is said to be Lyapunov functional if :

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2, \forall x, y \in E. \tag{2.1}$$

Recall that by the framework of Hilbert space H, (2.1) reduces to $\phi(x, y) = \|x - y\|^2, \forall x, y \in H$. Also for all $x, y, \in E$, it has been observe from (2.1) that the following properties hold:

$$(\|y\| - \|x\|)^2 \leq \phi(x, y) \leq (\|y\| + \|x\|)^2, \tag{2.2}$$

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle, \tag{2.3}$$

and

$$\phi(x, y) \leq \|x\|\|Jx - Jy\| + \|y\|\|x - y\|. \tag{2.4}$$

Let C be denote as the closed subset of a Banach space E, a mapping $T : C \longrightarrow E$ is said to be generalized nonexpansive if $F(T) \neq \emptyset$ and $\phi(Tx, v^*) \leq \phi(x, v^*)$ for all $x \in C,\ v^* \in F(T)$. If $x_n \longrightarrow x$ and $Tx_n \longrightarrow y$ whenever $Tx = y$, then a mapping T in E is said to be closed. A mapping $R : E \longrightarrow C$ defined by $R\big((Rx + t(x - Rx)\big) = R, \forall x \in E,\ t \geq 0$ is called Sunny, also is called retraction [2,12] if $Rx = x, \forall x \in C$. The retraction R which is sunny and nonexpansive is called sunny nonexpansive retraction from E onto C. Consider C as a nonempty closed subset of a smooth Banach space E, suppose that there exists a sunny generalized retraction R from E onto C, then C is called sunny generalized nonexpansive retract of E [10].

Lemma 2.1. *([14]) Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \| x_n - y_n \|= 0$.*

Remark 2.2. Suppose that $\{x_n\}$ and $\{y_n\}$ are bounded, then by considering (2.4) it is observe that the converse of Lemma 2.1 is also true.

Lemma 2.3. *([11]) Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C, let $z \in C$ and $x \in E$. Then the following hold*
(i) $z = Rx$ if and only if $\langle x - z, Jy - Jz\rangle \leq 0$ for all $y \in C$;
(ii) $\phi(x, Rx) + \phi(Rx, z) \leq \phi(x, z)$.

Lemma 2.4. *([17]) Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E.Then the following are equivalent:*
(i) C is a sunny generalized nonexpansive retract of E;
(ii) JC is closed and convex.

Lemma 2.5. *([5]) Let E be a uniformly convex Banach space. For arbitrary $r > 0$, let $B_r(0) := \{\|x \in E : \|x\| \leq r\}$. Then for any given sequence $\{x_n\}_{n=1}^{\infty} \subset B_r(0)$ and for any given sequence $\{\gamma_n\}_{n=1}^{\infty}$ of positive numbers such that $\sum_{n=1}^{\infty} \gamma_n = 1$, there exists a continuous strictly increasing convex function*

$$g : [0, 2r] \longrightarrow \mathbb{R},\ \ g(0) = 0$$

such that for any positive integers i, j with $i < j$, the following inequality holds:

$$\|\sum_{n=1}^{\infty} \gamma_n x_n\|^2 \leq \sum_{n=1}^{\infty} \gamma_n \|x_n\|^2 - \gamma_i \gamma_j g(\|x_i - x_j\|).$$

Lemma 2.6. *([14]) Let E be a u uniformly convex and smooth Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \longrightarrow [0, \infty)$ such that $g(0) = 0$ with*

$$g(\| x - y \|) \leq \phi(x, y),$$

for all $x, y \in B_r(0) = \{z \in E : \|z\| \leq 1\}$,

Lemma 2.7. *([12]) Let E be a smooth and strictly convex Banach space, let $z \in E$ and $\{t_i\}_{i=1}^{m} \subset (0, 1)$ with $\sum_{i=1}^{m} t_i = 1$. If $\{x_i\}_{i=1}^{m}$ is a finite sequence in E such that*

$$\phi\big(\sum_{i=1}^{m}(t_i x_i, z)\big) = \sum_{i=1}^{m} t_i \phi(x_i, z),$$

then $x_1 = x_2 = ... = x_m$.

Lemma 2.8. *([15]) Let E be a smooth, strictly convex and reflexive Banach space and let $M \subset E \times E^*$ be a monotone operator. Then M is maximal if and only if $R(J + rM) = E^*$ for all $r > 0$.*

Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E with $M \subset E \times E^*$ as a monotone operator satisfying

$$D(M) \subset C \subset J^{-1}\big(\bigcap_{r>0} R(J + rM)\big).$$

Then, the resolvent $J_r : C \longrightarrow D(M)$ of M can be define by

$$J_r x = \{z \in D(M) : Jx \in Jz + rMz\}, \forall x \in C.$$

Recall that $J_r x$ consists of one point. Now, for $r > 0$, the Yosida approximation $M_r : C \longrightarrow E^*$ is defined by

$$M_r x = (Jx - JJ_r x)/r,\ \forall x \in C.$$

Lemma 2.9. *([15, 17]) Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E and let $M \subset E \times E^*$ be a monotone operator satisfying*

$$D(M) \subset C \subset J^{-1}\big(\bigcap_{r>0} R(J + rM)\big).$$

Observe that for $r > 0$, let J_r and M_r be the resolvent and the Yosida approximation of M, respectively. Then, the following hold:
(i) $\phi(u, J_r x) + \phi(J_r x, x) \leq \phi(u, x)$, for all $x \in C,\ u \in M^{-1}0$;
(ii) $\phi(J_r x, M_r x) \in M$, for all $x \in C$; where $(x, x^*) \in M$ denotes the value of x^* at $x(x^* \in Mx)$.
(iii) $F(J_r) = M^{-1}0$.

Assumption B: The bifunction $B : C \times C \longrightarrow \mathbb{R}$ satisfies the following assumptions [3]:
(B_1) $B(x, x) = 0, \forall x \in C$;
(B_2) B is monotone, 1.e, $B(x, y) + B(y, x) \leq 0,\ \forall x, y \in C$;
(B_3) for each $x, y, z \in C, \limsup_{\lambda \to 0} B(\lambda z + (1-\lambda)x, y) \leq B(x, y)$;
(B_4) for each $x \in C, y \mapsto B(x, y)$ is convex and lower semicontinuous.

Lemma 2.10. *([3, 24]) Let E be a smooth, strictly convex and reflexive Banach space, and C be a nonempty closed convex subset of E. Let $B : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying the conditions $(B_1) - (B_4)$. For any given number $r > 0$ and any given point $x \in E$, then there exists $z \in C$ such that*

$$B(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in C.$$

Substituting x with $J^{-1}(Jx - rAx)$, then there exists $z \in C$ such that

$$B(z, y) + \langle y - z, Az\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0. \forall y \in C.$$

where A is a monotone mapping from C into E^*.

Lemma 2.11. *([24, 26]) Let C be a nonempty closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E. Let $A : C \longrightarrow E^*$ be a continuous and monotone mapping, $\varphi : C \longrightarrow \mathbb{R}$ be a proper convex and lower semi-continuous function and $B : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying the conditions $(B_1) - (B_4)$. For any given number $r > 0$ and any given point $x \in E$, a mapping $T_r : E \longrightarrow C$ is define by*

$$T_r(x) = \{z \in C : B(z, y) + \varphi(y) - \varphi(z) + \langle y - z, Az\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in C\}, \forall x \in E,$$

for all $x \in C$. The mapping T_r has the following properties:
(m_1) T_r is single-valued;
(m_2) T_r is a firmly nonexpansive - type mapping, for all $x \in E, y \in C$

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle$$

(m_3) $F(T_r) = GMEP(B, A, \varphi)$;
(m_4) $GMEP(B, A, \varphi)$ is a closed convex set of C.
(m_5) $\phi(v^*, T_r x) + \phi(T_r x, x) \leq \phi(v^*, x),\ \forall v^* \in F(T_r),\ x \in E$.

3. NST - Condition

Let C be a nonempty subset of a real Banach space E. Consider $\{T_n\}$ ans Υ as two families of the generalized nonexpansive mappings of C onto E such that $\cap_{n=1}^{\infty} F(T_n) = F(\Upsilon) \neq \emptyset$, where $F(T_n)$ is the set of all fixed points of T_n and the set of all common fixed point of Υ is denoted by $F(\Upsilon)$. Then, the sequence $\{T_n\}$ satisfies $NST-$ condition [25] with Υ if for each bounded sequence $\{x_n\}$ in C such that

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0 \Longrightarrow \lim_{n\to\infty} \|x_n - T x_n\| = 0, \forall T \in \Upsilon.$$

Notrice that, if $\Upsilon = \{T_n\}$, that is if Υ possess one mapping T, then $\{T_n\}$ satisfies the $NST-$ condition with $\{T\}$. For all $n \in \mathbb{N}$, putting $T_n = T$ satisfies $NST-$ condition with $\Upsilon = \{T\}$.

Lemma 3.1. *Let C be a subset of a uniformly smooth and uniformly convex Banach space E and S_1, S_2, ..., S_N are generalized nonexpansive mappings of C into E such that $\cap_{j=1}^{N} F(S_j) \neq \emptyset$. Notice that the sequence $\{\gamma_{j,n}\} \subset [0,1]$ satisfying:*
(a) $\sum_{j=0}^{N} \gamma_{j,n} = 1$
(b) $\liminf_{n\to\infty} \gamma_{0,n}\gamma_{j,n} > 0$, *for all* $j \in \{1, 2, \ , \dots \ , N\}$.
Consider $T_n : C \longrightarrow E$ as a mapping, for each $n \in \mathbb{N}$ defined by

$$T_n x = \gamma_{0,n} x + \sum_{j=1}^{N} \gamma_{j,n} S_j x, \forall x \in C.$$

Then, $\{T_n\}$ is a countable family of generalized nonexpansive mappings satisfies $NST-$ condition with $\Upsilon = \{S_1, S_2 \dots , S_N\}$.

Proof. First,we show that $\cap_{n=1}^{\infty} F(T_n) = F(\Upsilon)$, T_n are generalized nonexpansive mappings for all $n \in \mathbb{N}$. Observe that

$$F(\Upsilon) = \cap_{j=1}^{N} F(S_j) \subset \cap_{n=1}^{\infty} F(T_n).$$

Suppose that $v^* \in \cap_{j=1}^{N} F(S_j)$, then by Lemma 2.5 we obtain the following estimates:

$$\begin{aligned}
\phi(T_n x, v^*) &= \phi(\gamma_{0,n} x + \sum_{j=1}^{N} \gamma_{j,n} S_j x, v^*) \\
&= \| \gamma_{0,n} x + \sum_{j=1}^{N} \gamma_{j,n} S_j x \|^2 - 2\langle \gamma_{0,n} x + \sum_{j=1}^{N} \gamma_{j,n} S_j x, J v^* \rangle + \|v^*\|^2 \\
&\leq \gamma_{0,n} \| x \|^2 + \sum_{j=1}^{N} \gamma_{j,n} \| S_j x \|^2 - 2\gamma_{0,n} \langle x, J v^* \rangle - 2 \sum_{j=1}^{N} \gamma_{j,n} \langle S_j x, J v^* \rangle + \|v^*\|^2 \\
&= \gamma_{0,n} \phi(x, v^*) + \sum_{j=1}^{N} \gamma_{j,n} \phi(S_j x, v^*) \\
&\leq \gamma_{0,n} \phi(x, v^*) + \sum_{j=1}^{N} \gamma_{j,n} \phi(x, v^*) \\
&= (\gamma_{0,n} + \sum_{j=1}^{N} \gamma_{j,n}) \phi(x, v^*)
\end{aligned}$$

$$= \phi(x, v^*)$$

Also, for all $v \in F(T_n)$, we get

$$\begin{aligned}
\phi(v, v^*) &= \phi(T_n v, v^*) \\
&= \phi(\gamma_{0,n} v + \sum_{j=1}^{N} \gamma_{j,n} S_j v, v^*) \\
&= \| \gamma_{0,n} v + \sum_{j=1}^{N} \gamma_{j,n} S_j v \|^2 - 2\langle \gamma_{0,n} v + \sum_{j=1}^{N} \gamma_{j,n} S_j v, Jv^* \rangle + \|v^*\|^2 \\
&\leq \gamma_{0,n} \| v \|^2 + \sum_{j=1}^{N} \gamma_{j,n} \| S_j v \|^2 - 2\gamma_{0,n} \langle v, Jv^* \rangle - 2 \sum_{j=1}^{N} \gamma_{j,n} \langle S_j v, Jv^* \rangle + \|v^*\|^2 \\
&= \gamma_{0,n} \phi(v, v^*) + \sum_{j=1}^{N} \gamma_{j,n} \phi(S_j v, v^*) \\
&\leq \gamma_{0,n} \phi(v, v^*) + \sum_{j=1}^{N} \gamma_{j,n} \phi(v, v^*) \\
&= \big(\gamma_{0,n} + \sum_{j=1}^{N} \gamma_{j,n}\big) \phi(v, v^*) \\
&= \phi(v, v^*)
\end{aligned}$$

Thus, by Lemma 2.7, we obtain $v = T_n v = S_1 v = S_2 v = \ldots, S_N v$. Therefore $F(T_n) \subset \cap_{j=1}^{N} F(S_j), \forall n \in \mathbb{N}$. Hence $\cap_{n=1}^{\infty} F(T_n) = F(\Upsilon)$.

Next, we show that $\{T_n\}$ satisfies $NST-$ condition with $\{S_1, S_2, \ldots, S_N\}$. To show this, we presume that the sequence $\{x_n\}$ is bounded in C such that $\lim_{n\to\infty} \| x_n - T_n x_n \| = 0$. Now based on lemma 2.5 and for all $v^* \in \cap_{n=1}^{\infty} F(T_n)$, we get the following estimate:

$$\begin{aligned}
\phi(T_n x_n, v^*) &= \phi(\gamma_{0,n} x_n + \sum_{j=1}^{N} \gamma_{j,n} S_j x_n, v^*) \\
&= \| \gamma_{0,n} x_n + \sum_{j=1}^{N} \gamma_{j,n} S_j x_n \|^2 - 2\langle \gamma_{0,n} x_n + \sum_{j=1}^{N} \gamma_{j,n} S_j x_n, Jv^* \rangle + \|v^*\|^2 \\
&\leq \gamma_{0,n} \| x_n \|^2 + \sum_{j=1}^{N} \gamma_{j,n} \| S_j x_n \|^2 - 2\gamma_{0,n} \langle x_n, Jv^* \rangle \\
&- 2 \sum_{j=1}^{N} \gamma_{j,n} \langle S_j x_n, Jv^* \rangle + \|v^*\|^2 - \gamma_{0,n}\gamma_{j,n} g(\|x_n - S_j x_n\|) \\
&= \gamma_{0,n} \phi(x_n, v^*) + \sum_{j=1}^{N} \gamma_{j,n} \phi(S_j x_n, v^*) - \gamma_{0,n}\gamma_{j,n} g(\|x_n - S_j x_n\|) \\
&\leq \gamma_{0,n} \phi(x_n, v^*) + \sum_{j=1}^{N} \gamma_{j,n} \phi(x_n, v^*) - \gamma_{0,n}\gamma_{j,n} g(\|x_n - S_j x_n\|)
\end{aligned}$$

$$
\begin{aligned}
&= \Big(\gamma_{0,n} + \sum_{j=1}^{N} \gamma_{j,n}\Big)\phi(x_n, v^*) - \gamma_{0,n}\gamma_{j,n} g(\|x_n - S_j x_n\|) \\
&= \phi(x_n, v^*) - \gamma_{0,n}\gamma_{j,n} g(\|x_n - S_j x_n\|).
\end{aligned}
$$

This implies that

$$
\gamma_{0,n}\gamma_{j,n} g(\| x_n - S_j x_n \|) \leq \phi(x_n, v^*) - \phi(T_n x_n, v^*) \tag{3.1}
$$

Let $\{\|x_{n_k} - S_j x_{n_k}\|\}$ be an arbitrary subsequence of $\{\|x_n - S_j x_n\|\}$. From the boundedness of $\{x_{n_k}\}$, there exists a subsequence $\{x_{n_i'}\}$ of $\{x_{n_k}\}$ such that

$$
\lim_{i\to\infty} \phi(x_{n_i'}, v^*) = \limsup_{k\to\infty} \phi(x_{n_k}, v^*) = a.
$$

If follows from the properties (2.3) and (2.4) of ϕ that

$$
\begin{aligned}
\phi(x_{n_i'}, v^*) &= \phi(x_{n_i'}, T_{n_i'} x_{n_i'}) + \phi(T_{n_i'} x_{n_i'}, v^*) + 2\langle x_{n_i'} - T_{n_i'} x_{n_i'}, JT_{n_i'} x_{n_i'} - Jv^*\rangle \\
&\leq \phi(T_{n_i'} x_{n_i'}, v^*) + \| x_{n_i'} \| \| Jx_{n_i'} - JT_{n_i'} x_{n_i'} \| + \| T_{n_i'} x_{n_i'} - x_{n_i'} \| \| T_{n_i'} x_{n_i'} \| \\
&+ 2 \| x_{n_i'} - T_{n_i'} x_{n_i'} \| \| JT_{n_i'} x_{n_i'} - Jv^* \| .
\end{aligned} \tag{3.2}
$$

Following the fact that $\lim\limits_{n\to\infty} \| x_n - T_n x_n \| = 0$ and J is uniformly norm-to-norm continuous on bounded subsets of E, get

$$
\lim_{n\to\infty} \| Jx_n - JT_n x_n \| = 0
$$

Taking the advantage of limit inferior on (3.2), we have

$$
a = \liminf_{i\to\infty} \phi(x_{n_i'}, v^*) \leq \liminf_{i\to\infty} \phi(T_{n_i'} x_{n_i'}, v^*).
$$

Furthermore, since T_n is generalized nonexpansive mapping and $\phi(T_n x_n, v^*) \leq \phi(x_n, v^*), \forall n \in \mathbb{N}$. Hence, we obtain

$$
\limsup_{i\to\infty} \phi(T_{n_i'} x_{n_i'}, v^*) \leq \limsup_{i\to\infty} \phi(x_{n_i'}, v^*) = a
$$

Therefore

$$
\lim_{i\to\infty} \phi(x_{n_i'}, v^*) = \lim_{i\to\infty} \phi(T_{n_i'} x_{n_i'}, v^*) = a.
$$

Taking the advantage of $\lim\limits_{n\to\infty} \gamma_{0,n}\gamma_{j,n} > 0$ and (4.1), we conclude that

$$
\lim_{n\to\infty} g(\| x_{n_i'} - S_j x_{n_i'} \|) = 0.
$$

It follows from the properties of the function g that

$$
\lim_{i\to\infty} \| x_{n_i'} - S_j x_{n_i'} \| = 0.
$$

This lead to

$$
\lim_{n\to\infty} \| x_n - S_j x_n \| = 0, \forall j \in \{1, 2, ..., N\}.
$$

□

4. Main Results

Theorem 4.1. *Let C be a nonempty closed and convex subset of a uniformly smooth and uniformly convex Banach space E such that JC is closed convex. Let $\{B_j\}_{j=1}^{\infty} : C \times C \longrightarrow \mathbb{R}$ be a sequence of bifunctions satisfying assumptions $(B_1) - (B_4)$, let a nonlinear mapping $\{A_j\}_{j=1}^{\infty} : C \longrightarrow E^*$ be a sequence of continuous and monotone, and $\{\varphi_j\}_{j=1}^{\infty} : C \longrightarrow \mathbb{R}$ be a sequence of convex and lower semi-continuous function. Let $\{M_j\}_{j=1}^{\infty} \subset E \times E^*$ be a sequence of maximal monotone operators satisfying $D(M_j) \subset C$ and $J_r = (J + rM_j)^{-1}J$, for all $r > 0$. Let $\{T_n\} : C \longrightarrow E$ be a countable family of generalized nonexpansive mappings and Υ be a family of closed generalized nonexpansive mappings from C into E such that $\Gamma := \big(\cap_{n=1}^{\infty} F(T_n)\big) \cap \big(\cap_{j=1}^{\infty} M_j^{-1}0\big) \cap \big(\cap_{j=1}^{\infty} GMEP(B_j, A_j, \varphi_j)\big) \neq \emptyset$, where $\cap_{n=1}^{\infty} F(T_n) = F(\Upsilon) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$
(4.1)\qquad \begin{cases}
x_1 = x \in E, \ C_0 = Q_0 = E;\\
w_n = x_n + \theta_n(x_n - x_{n-1});\\
y_n = \rho_n w_n + (1-\rho_n) T_n J_{r_n} w_n;\\
z_n = \sigma_n y_n + (1-\sigma_n) T_n J_{r_n} w_n;\\
u_n \in C \ \textit{such that} \ B_j(u_n, y) + \langle A_j u_n, y - u_n\rangle + \varphi_j(y) - \varphi_j(u_n)\\
+\dfrac{1}{r_{j,n}}\langle y - u_n, Ju_n - Jz_n\rangle \geq 0, \forall y \in C;\\
C_n = \{u \in C_{n-1} \cap Q_{n-1} : \phi(u_n, u) \leq \phi(w_n, u)\};\\
Q_n = \{u \in C_{n-1} \cap Q_{n-1} : \langle x - x_n, Jx_n - Ju\rangle \geq 0\};\\
x_{n+1} = R_{C_n \cap Q_n} x, \ \forall n \in \mathbb{N},
\end{cases}
$$

where $\theta_n \subset (0,1)$ and $\{\rho_n\}$, $\{\sigma_n\}$ are sequence in $[0,1]$ such that $\liminf_{n\to\infty}(1-\rho_n) > 0$, $\liminf_{n\to\infty}(1-\sigma_n) > 0$ and $\{r_{j,n}\} \subset [a, \infty)$ for some $a > 0$, $\forall j \geq 1$ Then, the sequence $\{x_n\}$ converges strongly to $R_\Gamma x$, where R_Γ is the sunny generalized nonexpansive retraction of E onto Γ.

Proof. Let $\Phi_j : C \times C \longrightarrow \mathbb{R}$ and $T_{j,r} : E \longrightarrow C$ be two functions defined by

$$\Phi_j(z, y) = B_j(z, y) + \langle A_j z, y - z\rangle + \varphi_j(y) - \varphi_j(z), \ \forall z, y \in C$$

and

$$T_{j,r}(x) = \{z \in C : \Phi_j(z, y) + \frac{1}{r_{j,n}}\langle y - z, Jz - Jx\rangle \geq 0, \ \forall y \in C\},$$
$$\forall j \geq 1, \ x \in E,$$

respectively. Notice that the two functions Φ_j and $T_{j,r}$ satisfies Assumptions $(B1) - (B4)$ and properties $(m_1) - (m_5)$ (see [24, 26]) respectively.

The proof is consider in several steps as follows:

Step 1 : we show that for each $n \in \mathbb{N}$, JC_n and JQ_n are closed and convex. Taking the advantage of the definition of C_n and Q_n that JC_n is closed and JQ_n is closed and convex for all $n \in \mathbb{N}$. Now from the definition of C_n, we have $\phi(u_n, u) \leq \phi(w_n, u)$ which is equivalent to

$$\| w_n \|^2 - \| u_n \|^2 - 2\langle w_n - u_n, Ju\rangle \geq 0, \ \forall u \in JC_n,$$

implies that JC_n is convex. Since J is one-to-one and for each $n \in \mathbb{N}$, it follows that $J(C_n \cap Q_n) = JC_n \cap JQ_n$is closed and convex. Using Lemma 2.4, it can be observe that $C_n \cap Q_n$ is a sunny generalized nonexpansive retract of E. It is obvious that $\Gamma \subset C = C_0 \cap Q_0$.

Step 2 : we show that $\Gamma \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. Suppose that $\Gamma \subset C_{n-1} \cap Q_{n-1}$ for some $n \in \mathbb{N}$. Let $v^* \in \Gamma$, $u_n = T_{r_n} z_n$ and $\mu_n = J_{r_n} w_n$. Since $\{T_n\}$ are generalized nonexpansive mappings for all $n \in \mathbb{N}$, then we obtain the following estimate:

$$\begin{aligned}
\phi(y_n, v^*) &= \phi(\rho_n w_n + (1-\rho_n) T_n \mu_n, v^*) \\
&= \| \rho_n w_n + (1-\rho_n) T_n \mu_n \|^2 - 2\langle \rho_n w_n + (1-\rho_n) T_n \mu_n, Jv^* \rangle + \|v^*\|^2 \\
&\leq \rho_n \| w_n \|^2 + (1-\rho_n) \| T_n \mu_n \|^2 - 2\rho_n \langle w_n, Jv^* \rangle - 2(1-\rho_n) \langle T_n \mu_n, Jv^* \rangle \\
&+ \|v^*\|^2 \\
&= \rho_n \phi(w_n, v^*) + (1-\rho_n) \phi(T_n \mu_n, v^*) \\
&\leq \rho_n \phi(w_n, v^*) + (1-\rho_n) \phi(\mu_n, v^*) \qquad (4.2) \\
&= \rho_n \phi(w_n, v^*) + (1-\rho_n) \phi(J_{r_n} w_n, v^*) \\
&\leq \rho_n \phi(w_n, v^*) + (1-\rho_n) \phi(w_n, v^*),
\end{aligned}$$

which implies that

$$\phi(y_n, v^*) \leq \phi(w_n, v^*) \qquad (4.3)$$

Observe that

$$\begin{aligned}
\phi(u_n, v^*) &= \phi(T_{r_n} z_n, v^*) \\
&\leq \phi(z_n, v^*) \\
&= \phi(\sigma_n y_n + (1-\sigma_n) T_n \mu_n, v^*) \\
&= \| \sigma_n y_n + (1-\sigma_n) T_n \mu_n \|^2 - 2\langle \sigma_n y_n + (1-\sigma_n) T_n \mu_n, Jv^* \rangle + \|v^*\|^2 \\
&\leq \sigma_n \| y_n \|^2 + (1-\sigma_n) \| T_n \mu_n \|^2 - 2\sigma_n \langle y_n, Jv^* \rangle - 2(1-\sigma_n) \langle T_n \mu_n, Jv^* \rangle \\
&+ \|v^*\|^2 \\
&= \sigma_n \phi(y_n, v^*) + (1-\sigma_n) \phi(T_n \mu_n, v^*) \\
&\leq \sigma_n \phi(y_n, v^*) + (1-\sigma_n) \phi(\mu_n, v^*) \\
&= \sigma_n \phi(y_n, v^*) + (1-\sigma_n) \phi(J_{r_n} w_n, v^*) \\
&\leq \sigma_n \phi(y_n, v^*) + (1-\sigma_n) \phi(w_n, v^*) \\
&= \phi(w_n, v^*), \qquad (4.4)
\end{aligned}$$

implies that $v^* \in C_n, \ \forall n \in \mathbb{N}$. Therefore $\Gamma \subset C_n$. Now since $x_n = R_{C_{n-1} \cap Q_{n-1}}$, the it follows from Lemma 2.3(i) that

$$\langle x - x_n, Jx_n - Jv^* \rangle \geq 0, \ \forall v^* \in C_{n-1} \cap Q_{n-1} \qquad (4.5)$$

Also, since $\Gamma \in C_{n-1} \cap Q_{n-1}$, we obtain

$$\langle x - x_n, Jx_n - Jv^* \rangle \geq 0, \ \forall v^* \in \Gamma$$

Moreover, by definition of Q_n, we conclude that $\Gamma \subset Q_n$. Hence $\Gamma \subset C_n \cap Q_n, \ \forall n \in \mathbb{N}$. Implies that $\{x_n\}$ is well defined.

Step 3 : we show that $\{x_n\}$ is Cauchy and $x_n \longrightarrow \vartheta$ (as $n \to \infty$). Now, taking the advantage of the definition of Q_n, we observe that $x_n = R_{Q_n}x$. Also, based on Lemma 2.3, we get

$$\phi(x, x_n) = \phi(x, R_{Q_n}x) \leq \phi(x, v^*) - \phi(R_{Q_n}x, v^*) \leq \phi(x, v^*), \quad \forall v^* \in \Gamma \subset Q_n. \tag{4.6}$$

Implies that $\{\phi(x, x_n)\}$ is bounded and so $\{x_n\}$, $\{w_n\}$, $\{y_n\}$, $\{z_n\}$, $\{u_n\}$ and $\{T_n x_n\}$ It follows from the definition of R_{Q_n} that

$$\phi(x, x_n) \leq (x, x_{n+1}), \ \forall n \in \mathbb{N}. \tag{4.7}$$

Since $x_{n+1} = R_{C_n \cap Q_n}x \in C_n \cap Q_n \subset Q_n$ and $x_n = R_{Q_n}x$. Therefore $\lim_{n\to\infty}\{\phi(x, x_n)\}$ exists. We notice that for each $n \in \mathbb{N}$ and for every positive integer k, using Lemma 2.3 and $x_n = R_{Q_n}x$, we obtain

$$\begin{aligned} \phi(x_n, x_{n+k}) &= \phi(R_{Q_n}x, x_{n+k}) \\ &\leq \phi(x, x_{n+k}) - \phi(x, R_{Q_n}x) \\ &= \phi(x, x_{n+k}) - \phi(x, x_n) \end{aligned} \tag{4.8}$$

By taking the limit in (4.8) implies that

$$\lim_{n\to\infty} \phi(x_n, x_{n+k}) = 0. \tag{4.9}$$

This lead to

$$\lim_{n\to\infty} \phi(x_n, x_{n+1}) = 0. \tag{4.10}$$

Since E is uniformly convex and smooth, using Lemma 2.1 and (4.10), we obtain

$$\lim_{n\to\infty} \| x_n - x_{n+1} \| = 0. \tag{4.11}$$

It follows from Lemma 2.6 that for $m, n \in \mathbb{N}$ with $m > n$, there exists a strictly increasing, convex and continuous function $g : [0, \infty) \longrightarrow [0, \infty)$ with $g(0) = 0$ such that

$$g(\| x_n - x_m \|) \leq \phi(x_n, x_m) \leq \phi(x, x_m) - \phi(x, x_n)$$

It can be observe that from the properties of g we conclude that the sequence $\{x_n\}$ is Cauchy. Therefore there exists a point $\vartheta \in C$ such that

$$\lim_{n\to\infty} x_n = \vartheta. \tag{4.12}$$

Step 4 : we show that $\vartheta \in \cap_{n=1}^{\infty} F(T_n)$. It can be observe from the definition of w_n that

$$\| w_n - x_n \| = \| \theta_n(x_n - x_{n-1}) \| \leq \| x_n - x_{n-1} \|$$

By (4.11), we get

$$\lim_{n\to\infty} \| w_n - x_n \| = 0. \tag{4.13}$$

Using (4.12) and (4.13), we obtain

$$\lim_{n\to\infty} w_n = \vartheta. \tag{4.14}$$

We notice from (4.11) and (4.13) that

$$\lim_{n\to\infty} \| x_{n+1} - w_n \| = 0. \tag{4.15}$$

Since $\{w_n\}$ is bounded and by (4.15) and Remark 2.2, we conclude that

$$\lim_{n\to\infty} \phi(w_n, x_{n+1}) = 0. \tag{4.16}$$

By considering the definition of C_n and $x_{n+1} = R_{C_n \cap Q_n} x \in C_n$, we have

$$\phi(u_n, x_{n+1}) \leq \phi(w_n, x_{n+1}), \ \forall n \in \mathbb{N}. \tag{4.17}$$

Using (4.16) in (4.17), we obtain

$$\lim_{n\to\infty} \phi(u_n, x_{n+1}) = 0. \tag{4.18}$$

Since E is uniformly convex and smooth, using (4.18) and Lemma 2.1, we get

$$\lim_{n\to\infty} \| u_n - x_{n+1} \| = 0. \tag{4.19}$$

Taking into account that

$$\| x_n - u_n \| \leq \| x_n - x_{n+1} \| + \| x_{n+1} - u_n \| \tag{4.20}$$

Putting (4.11) and (4.19) in (4.20), we obtain

$$\lim_{n\to\infty} \| x_n - u_n \| = 0. \tag{4.21}$$

Using (4.12) and (4.21), we get

$$\lim_{n\to\infty} u_n = \vartheta. \tag{4.22}$$

It follows from the definition of C_n and $x_{n+1} = R_{C_n \cap Q_n} x \in C_n$ that

$$\lim_{n\to\infty} \phi(z_n, x_{n+1}) \leq \phi(w_n, x_{n+1}), \ \forall n \in \mathbb{N}. \tag{4.23}$$

By putting (4.16) in (4.23), we observe that

$$\lim_{n\to\infty} \phi(z_n, x_{n+1}) = 0. \tag{4.24}$$

Taking the advantage of uniformly convexity and smoothness of E, (4.24) and Lemma 2.1, we conclude that

$$\lim_{n\to\infty} \| z_n - x_{n+1} \| = 0. \tag{4.25}$$

From the triangular inequality, we have

$$\| z_n - x_n \| \leq \| z_n - x_{n+1} \| + \| x_{n+1} - x_n \| . \tag{4.26}$$

Using (4.11) and (4.25) in (4.26), we get

$$\lim_{n\to\infty} \| z_n - x_n \| = 0. \tag{4.27}$$

By (4.12) and (4.27), implies that

$$\lim_{n\to\infty} z_n = \vartheta. \tag{4.28}$$

From (4.21) and (4.27), we obtain

$$\lim_{n\to\infty} \| u_n - z_n \| = 0. \tag{4.29}$$

Since J is uniformly norm- to- norm continuous on bounded sets, we conclude that

$$\lim_{n\to\infty} \| Ju_n - Jz_n \| = 0. \tag{4.30}$$

Also, since $x_{n+1} = R_{C_n \cap Q_n} x \in C_n$, it follows that

$$\phi(y_n, x_{n+1}) \leq \phi(w_n, x_{n+1}), \forall n \in \mathbb{N}. \tag{4.31}$$

By using (4.16) and (4.31), we have

$$\lim_{n\to\infty} \phi(y_n, x_{n+1}) = 0. \tag{4.32}$$

Also, by Lemma 2.1, (4.32), and uniformly convexity and smoothness of E, we obtain

$$\lim_{n\to\infty} \| y_n - x_{n+1} \| = 0. \tag{4.33}$$

Notice that

$$\| y_n - x_n \| \leq \| y_n - x_{n+1} \| + \| x_{n+1} - x_n \| . \tag{4.34}$$

By using (4.11) and (4.33) in (4.34), we get

$$\lim_{n\to\infty} \| y_n - x_n \| = 0. \tag{4.35}$$

From (4.12) and (4.35), we conclude that

$$\lim_{n\to\infty} y_n = \vartheta. \tag{4.36}$$

It follows from (4.13) and (4.35) that

$$\lim_{n\to\infty} \| w_n - y_n \| = 0. \tag{4.37}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \| Jw_n - Jy_n \| = 0. \tag{4.38}$$

It follows from the inequality (4.2) that

$$\phi(y_n, v^*) \leq \rho_n \phi(w_n, v^*) + (1 - \rho_n)\phi(\mu_n, v^*).$$

Implies that

$$\phi(y_n, v^*) - \rho_n \phi(w_n, v^*) \leq (1 - \rho_n)\phi(\mu_n, v^*).$$

So

$$\frac{1}{(1-\rho_n)}(\phi(y_n, v^*) - \rho_n \phi(w_n, v^*)) \leq \phi(\mu_n, v^*). \tag{4.39}$$

Now, since $\mu_n = J_{r_n} w_n$ and by Lemma 2.9, we have

$$\phi(w_n, \mu_n) = \phi(w_n, J_{r_n} w_n) \leq \phi(w_n, v^*) - \phi(J_{r_n} w_n, v^*) = \phi(w_n, v^*) - \phi(\mu_n, v^*)$$

Therefore by (4.39), we obtain

$$\begin{aligned}
\phi(w_n,\mu_n) &\leq \phi(w_n,v^*)-\phi(\mu_n,v^*)\\
&\leq \phi(w_n,v^*)-\frac{1}{(1-\rho_n)}\big(\phi(y_n,v^*)-\rho_n\phi(w_n,v^*)\big)\\
&= \frac{1}{(1-\rho_n)}\big(\phi(w_n,v^*)-\phi(y_n,v^*)\big)\\
&= \frac{1}{(1-\rho_n)}\big(\parallel w_n\parallel^2-\parallel y_n\parallel^2-2\langle v^*,Jw_n-Jy_n\rangle\big)\\
&\leq \frac{1}{(1-\rho_n)}\big[\,|\parallel w_n\parallel-\parallel y_n\parallel|\,(\parallel w_n\parallel+\parallel y_n\parallel)+2\parallel v^*\parallel\parallel Jw_n-Jy_n\parallel\big]\\
&\leq \frac{1}{(1-\rho_n)}\big[\parallel w_n-y_n\parallel(\parallel w_n\parallel+\parallel y_n\parallel)+2\parallel v^*\parallel\parallel Jw_n-Jy_n\parallel\big].
\end{aligned}$$

It follows from (4.37) and (4.38) that

$$\lim_{n\to\infty}\phi(w_n,\mu_n)=0. \tag{4.40}$$

From Lemma 2.1, we obtain

$$\lim_{n\to\infty}\parallel w_n-\mu_n\parallel=0. \tag{4.41}$$

By the uniformly convexity of J on bounded sets and (4.41), we get

$$\lim_{n\to\infty}\parallel Jw_n-J\mu_n\parallel=0. \tag{4.42}$$

$$\parallel \mu_n-x_{n+1}\parallel\leq\parallel \mu_n-w_n\parallel+\parallel w_n-x_{n+1}\parallel$$

By (4.15) and (4.41) , we arrive at

$$\lim_{n\to\infty}\parallel \mu_n-x_{n+1}\parallel=0. \tag{4.43}$$

Taking the advantage (4.14) and (4.41), implies that

$$\lim_{n\to\infty}\mu_n=\vartheta. \tag{4.44}$$

From (4.1), we observe that

$$\begin{aligned}
\parallel x_{n+1}-z_n\parallel &= \parallel x_{n+1}-(\sigma_n y_n+(1-\sigma_n)T_n\mu_n\parallel\\
&= \parallel x_{n+1}-\sigma_n y_n-(1-\sigma_n)T_n\mu_n\parallel\\
&= \parallel (1-\sigma_n)(x_{n+1}-T_n\mu_n)-\sigma_n(y_n-x_{n+1})\parallel\\
&\geq (1-\sigma_n)\parallel x_{n+1}-T_n\mu_n\parallel-\sigma_n\parallel y_n-x_{n+1}\parallel.
\end{aligned}$$

This implies that

$$\parallel x_{n+1}-T_n\mu_n\parallel\leq\frac{1}{(1-\sigma_n)}(\parallel x_{n+1}-z_n\parallel+\sigma_n\parallel y_n-x_{n+1}\parallel). \tag{4.45}$$

Since $\liminf\limits_{n\to\infty}(1-\sigma_n)>0$, Using (4.25) and (4.33) in (4.45), we obtain

$$\lim_{n\to\infty}\parallel x_{n+1}-T_n\mu_n\parallel=0. \tag{4.46}$$

Taking into account that

$$\parallel \mu_n-T_n\mu_n\parallel\leq\parallel \mu_n-x_{n+1}\parallel+\parallel x_{n+1}-T_n\mu_n\parallel \tag{4.47}$$

Putting (4.43) and (4.46) in (4.47), we get

$$\lim_{n\to\infty} \parallel \mu_n - T_n\mu_n \parallel = 0.$$

Therefore, since $\{T_n\}$ satisfies $NST-$ condition, we conclude that

$$\lim_{n\to\infty} \parallel \mu_n - T\mu_n \parallel = 0, \forall T \in \Upsilon.$$

Also, since T is closed and $\mu_n \longrightarrow \vartheta$, we obtain that ϑ is a fixed point of T, implies that $\vartheta \in F(T_n), \forall n \geq 1$. Therefore

$$\vartheta \in \cap_{n=1}^{\infty} F(T_n) = F(\Upsilon).$$

Step 5 : we show that $\vartheta \in \cap_{j=1}^{\infty} M_j^{-1}0$. Now, since $\{w_n\}$ is bounded, there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \longrightarrow \vartheta$. Also since $\lim_{n\to\infty} \parallel w_n - \mu_n \parallel = 0$ and $\lim_{n\to\infty} \parallel w_n - y_n \parallel = 0$, then we conclude that $\mu_{n_k} \longrightarrow \vartheta$. and $y_{n_k} \longrightarrow \vartheta$. respectively. Taking the advantage of $r_{j,n} \geq 0$, $\mu_n = J_{r_n} w_n$, (4.42) and Lemma 2.8, we have

$$\lim_{n\to\infty} \frac{1}{r_{j,n}} \parallel Jw_n - J\mu_n \parallel = 0, \ \forall j \geq 1.$$

Therefore

$$\begin{aligned}
\lim_{n\to\infty} \parallel M_{jr_{j,n}} w_n \parallel &= \lim_{n\to\infty} \frac{1}{r_{j,n}} \parallel Jw_n - JJ_{r_n} w_n \parallel, \ \forall j \geq 1 \\
&= \lim_{n\to\infty} \frac{1}{r_{j,n}} \parallel Jw_n - J\mu_n \parallel \\
&= 0.
\end{aligned}$$

From the fact that M_j, $\forall j \geq 1$ are monotone and by Lemma 2.9(ii), we have

$$\langle \varpi - \mu_n, \varpi^* - M_{jr_{j,n}} w_n \rangle \geq 0, \ \forall n \in \mathbb{N}.$$

Implies that

$$\langle \varpi - \mu_{n_k}, \varpi^* - M_{jr_{j,n_k}} w_{n_k} \rangle = \langle \varpi - \vartheta, \varpi^* \rangle \geq 0.$$

Therefore, it follows from the maximality of M_j that $\vartheta \in M_j^{-1}0$, $\forall j \geq 1$. Hence

$$\vartheta \in \cap_{j=1}^{\infty} M_j^{-1}0.$$

Step 6 : we show that $\vartheta \in \cap_{j=1}^{\infty} GMEP(B_j, A_j, \varphi_j)$. Now for each $j \geq 1$ and $n \geq 1$, from equation $u_n = T_{r_{j,n}} z_n$, using (4.30) with $\{r_{j,n}\} \subset [a, \infty)$ for some $a > 0$, we obtain

$$\lim_{n\to\infty} \frac{\parallel Ju_n - Jz_n \parallel}{r_{j,n}} = 0, \ \forall j \geq 1.$$

By using $u_n = T_{r_{j,n}} z_n$, we get

$$\Phi_j(u_n, y) + \frac{1}{r_{j,n}} \langle y - u_n, Ju_n - Jz_n \rangle \geq 0, \ \forall y \in C,$$

where

$$\Phi_j(u_n, y) = B_j(u_n, y) + \langle A_j u_n, y - u_n \rangle + \varphi_j(y) - \varphi_j(u_n).$$

By applying the assumption B_2, we have for each $n \geq 1$ and $j \geq 1$ that

$$\frac{1}{r_{j,n}} \langle y - u_n, Ju_n - Jz_n \rangle \geq -\Phi_j(u_n, y) \geq \Phi_j(y, u_n), \ \forall y \in C.$$

Therefore, we obtain

$$\langle y-u_n, Ju_n-Jz_n\rangle \geq r_{j,n}\Phi_j(y,u_n),\ \forall y\in C.$$

This implies that

$$\| y-u_n \| \| Ju_n-Jz_n \| \geq r_{j,n}\Phi_j(y,u_n),\ \forall y\in C.$$

From (B_4), we conclude that

$$\Phi_j(y,\vartheta)\leq 0,\ \forall y\in C,\ j\geq 1.$$

Consider $y_\lambda=\lambda y+(1-\lambda)\vartheta,\ \forall\lambda\in(0,1]$. Since $y\in C$ and $\vartheta\in C$, then we have $y_\lambda\in C$ and $\Phi_j(y_\lambda,\vartheta)\leq 0$. Now $\forall y_\lambda\in C,\ j\geq 1$ and by the assumptions $(B_1)-(B_4)$, we obtain

$$\begin{aligned} 0 &= \Phi_j(y_\lambda,y_\lambda)\\ &\leq \lambda\Phi_j(y_\lambda,y)+(1-\lambda)\Phi_j(y_\lambda,\vartheta)\\ &\leq \lambda\Phi_j(y_\lambda,y). \end{aligned}$$

Dividing by λ, implies that

$$\Phi_j(y_\lambda,y)\geq 0,\ \forall y\in C.$$

Letting $\lambda\longrightarrow 0$ and by (B_3), we conclude that

$$\Phi_j(\vartheta,y)\geq 0,\ \forall y\in C.$$

This shows that $\vartheta\in GMEP(B_j,A_j,\varphi_j),\ \forall j\geq 1$. Therefore $\vartheta\in\cap_{j=1}^{\infty}GMEP(B_j,A_j,\varphi_j),\ \forall j\geq 1$. Hence

$$\vartheta\in\big(\cap_{n=1}^{\infty}F(T_n)\big)\cap\big(\cap_{j=1}^{\infty}M_j^{-1}0\big)\cap\big(\cap_{j=1}^{\infty}GMEP(B_j,A_j,\varphi_j)\big)$$

Step 7 : we show that $\vartheta=R_\Gamma x$. from Lemma 2.3(ii), we obtain

$$\phi(x,R_\Gamma x)\leq\phi(x,R_\Gamma x)+\phi(R_\Gamma x,\vartheta)\leq\phi(x,\vartheta).$$

Also, by Lemma 2.3(ii), $x_{n+1}=R_{C_n\cap Q_n}x$ and $\vartheta\in\Gamma\subset C_n\cap Q_n$, we obtain the following estimation:

$$\phi(x,x_{n+1})\leq\phi(x,x_{n+1})+\phi(x_{n+1},R_\Gamma x)\leq\phi(x,R_\Gamma x).$$

Therefore $\phi(x,\vartheta)\leq\phi(x,R_\Gamma x)$, since $x_n\longrightarrow\vartheta$. Hence $\phi(x,\vartheta)=\phi(x,R_\Gamma x)$. Now, by the uniqueness of $R_\Gamma x$, we conclude that $\vartheta=R_\Gamma x$. This completes the proof. □

Corollary 4.2. *Let C be a nonempty closed and convex subset of a uniformly smooth and uniformly convex Banach space E such that JC is closed convex. Let $\{B_j\}_{j=1}^{\infty}:C\times C\longrightarrow\mathbb{R}$ be a sequence of bifunctions satisfying assumptions $(B_1)-(B_4)$, let a nonlinear mapping $\{A_j\}_{j=1}^{\infty}:C\longrightarrow E^*$ be a sequence of continuous and monotone, and $\{\varphi_j\}_{j=1}^{\infty}:C\longrightarrow\mathbb{R}$ be a sequence of convex and lower semi-continuous function. Let $\{M_j\}_{j=1}^{\infty}\subset E\times E^*$ be a sequence of maximal monotone operators satisfying $D(M_j)\subset C$ and $J_r=(J+rM_j)^{-1}J$, for all $r>0$. Let $T:C\longrightarrow E$ be a generalized nonexpansive mappings such that $\Gamma:=$*

$F(T) \cap \big(\cap_{j=1}^{\infty} M_j^{-1}0\big) \cap \big(\cap_{j=1}^{\infty} GMEP(B_j, A_j, \varphi_j)\big) \neq \emptyset$, Let $\{x_n\}$ be a sequence generated by the following algorithm:

$$\begin{cases} x_1 = x \in E,\ C_0 = Q_0 = E; \\ w_n = x_n + \theta_n(x_n - x_{n-1}); \\ y_n = \rho_n w_n + (1-\rho_n)TJ_{r_n}w_n; \\ z_n = \sigma_n y_n + (1-\sigma_n)TJ_{r_n}w_n; \\ u_n \in C \text{ such that } B_j(u_n, y) + \langle A_j u_n, y - u_n\rangle + \varphi_j(y) - \varphi_j(u_n) \\ +\dfrac{1}{r_{j,n}}\langle y - u_n, Ju_n - Jz_n\rangle \geq 0, \forall y \in C; \\ C_n = \{u \in C_{n-1} \cap Q_{n-1} : \phi(u_n, u) \leq \phi(w_n, u)\}; \\ Q_n = \{u \in C_{n-1} \cap Q_{n-1} : \langle x - x_n, Jx_n - Ju\rangle \geq 0\}; \\ x_{n+1} = R_{C_n \cap Q_n}x,\ \forall n \in \mathbb{N}, \end{cases}$$

where $\theta_n \subset (0,1)$ and $\{\rho_n\},\ \{\sigma_n\}$ are sequence in $[0,1]$ such that $\liminf_{n\to\infty}(1-\rho_n) > 0,\ \liminf_{n\to\infty}(1-\sigma_n) > 0$ and $\{r_{j,n}\} \subset [a,\infty)$ for some $a > 0,\ \forall j \geq 1$ Then, the sequence $\{x_n\}$ converges strongly to $R_\Gamma x$.

Proof. For all $n \in \mathbb{N}$, setting $T_n = T$ in Theorem 4.1, we obtain the desired result. □

Corollary 4.3. *Suppose that $S_1,\ S_2,\ ...,\ S_N : C \longrightarrow E$ are generalized nonexpansive mappings. Let $\{B_j\}_{j=1}^{\infty} : C \times C \longrightarrow \mathbb{R}$ be a sequence of bifunctions satisfying assumptions $(B_1) - (B_4)$, let a nonlinear mapping $\{A_j\}_{j=1}^{\infty} : C \longrightarrow E^*$ be a sequence of continuous and monotone, and $\{\varphi_j\}_{j=1}^{\infty} : C \longrightarrow \mathbb{R}$ be a sequence of convex and lower semi-continuous function. Let $\{M_j\}_{j=1}^{\infty} \subset E \times E^*$ be a sequence of maximal monotone operators satisfying $D(M_j) \subset C$ and $J_r = (J + rM_j)^{-1}J$, for all $r > 0$ such that $\Gamma := \big(\cap_{j=1}^{N} F(S_j)\big) \cap \big(\cap_{j=1}^{\infty} M_j^{-1}0\big) \cap \big(\cap_{j=1}^{\infty} GMEP(B_j, A_j, \varphi_j)\big) \neq \emptyset$, Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases} x_1 = x \in E,\ C_0 = Q_0 = E; \\ w_n = x_n + \theta_n(x_n - x_{n-1}); \\ y_n = \rho_n w_n + (1-\rho_n)\big(\gamma_{0,n}w_n + \sum_{j=1}^{N}\gamma_{j,n}S_jJ_{r_n}w_n\big); \\ z_n = \sigma_n y_n + (1-\sigma_n)\big(\gamma_{0,n}w_n + \sum_{j=1}^{N}\gamma_{j,n}S_jJ_{r_n}w_n\big); \\ u_n \in C \text{ such that } B_j(u_n, y) + \langle A_j u_n, y - u_n\rangle + \varphi_j(y) - \varphi_j(u_n) \\ +\dfrac{1}{r_{j,n}}\langle y - u_n, Ju_n - Jz_n\rangle \geq 0, \forall y \in C; \\ C_n = \{u \in C_{n-1} \cap Q_{n-1} : \phi(u_n, u) \leq \phi(w_n, u)\}; \\ Q_n = \{u \in C_{n-1} \cap Q_{n-1} : \langle x - x_n, Jx_n - Ju\rangle \geq 0\}; \\ x_{n+1} = R_{C_n \cap Q_n}x,\ \forall n \in \mathbb{N}, \end{cases}$$

where $\theta_n \subset (0,1)$, $\{\rho_n\},\ \{\sigma_n\}$ and $\{\gamma_{j,n}\}$ are sequence in $[0,1]$, $\{r_{j,n}\} \subset [a,\infty)$ for some $a > 0,\ \forall j \geq 1$ and the following condition holds:

(a) $\liminf_{n\to\infty}(1-\rho_n) > 0$

(b) $\liminf_{n\to\infty}(1-\sigma_n) > 0$

(c) $\sum_{j=0}^{N}\gamma_{j,n} = 1$

(d) $\liminf_{n\to\infty}\gamma_{0,n}\gamma_{j,n} > 0$, *for all* $j \in \{1, 2, \ , ... \ , N\}$.
Then, the sequence $\{x_n\}$ *converges strongly to* $R_\Gamma x$.

5. Application

What we present in this section are some of the application of theorem 4.1 :

5.1. **Countable family of generalized nonexpansive mappings, maximal monotone operator and system of equilibrium problems.** Consider the sequence $\{x_n\}$ defined in theorem 4.1 converges strongly to $R_\Gamma x$ by setting $A \equiv 0$, $\varphi \equiv 0$ in theorem 4.1, where $\Gamma := \big(\cap_{n=1}^{\infty} F(T_n)\big) \cap \big(\cap_{j=1}^{\infty} M_j^{-1}0\big) \cap \big(\cap_{j=1}^{\infty} GMEP(B_j)\big) \neq \emptyset$ and $EP(B_j)$ is the set of solutions of the equilibrium problem for B.

5.2. **Countable family of generalized nonexpansive mappings, maximal monotone operator and system of convex optimization problems.** Consider the sequence $\{x_n\}$ defined in theorem 4.1 converges strongly to $R_\Gamma x$ by setting $B \equiv 0$, $A \equiv 0$ in theorem 4.1, where $\Gamma := \big(\cap_{n=1}^{\infty} F(T_n)\big) \cap \big(\cap_{j=1}^{\infty} M_j^{-1}0\big) \cap \big(\cap_{j=1}^{\infty} GMEP(\varphi_j)\big) \neq \emptyset$ and $CMP(\varphi_j)$ is the set of solutions of the convex optimization problem for φ.

5.3. **Countable family of generalized nonexpansive mappings, maximal monotone operator and system of variational inequalities problems.** Consider the sequence $\{x_n\}$ defined in theorem 4.1, converges strongly to $R_\Gamma x$ by setting $B \equiv 0$, $\varphi \equiv 0$ in theorem 4.1, where $\Gamma := \big(\cap_{n=1}^{\infty} F(T_n)\big) \cap \big(\cap_{j=1}^{\infty} M_j^{-1}0\big) \cap \big(\cap_{j=1}^{\infty} VIP(C, A_j)\big) \neq \emptyset$ and $VIP(C, A_j)$ is the set of solutions of the variational inequality problem for A over C.

Application in Hilbert space

We also present the application of theorem 4.1 in Hilbert space as follows:

Theorem 5.1. *Let C be a nonempty closed and convex subset of a Hilbert space H such that JC is closed convex. Let $\{B_j\}_{j=1}^{\infty} : C \times C \longrightarrow \mathbb{R}$ be a sequence of bifunctions satisfying assumptions $(B_1) - (B_4)$, let a nonlinear mapping $\{A_j\}_{j=1}^{\infty} : C \longrightarrow E^*$ be a sequence of continuous and monotone, and $\{\varphi_j\}_{j=1}^{\infty} : C \longrightarrow \mathbb{R}$ be a sequence of convex and lower semi-continuous function. Let $\{M_j\}_{j=1}^{\infty} \subset E \times E^*$ be a sequence of maximal monotone operators satisfying $D(M_j) \subset C$ and $J_r = (J + rM_j)^{-1}J$, for all $r > 0$. Let $\{T_n\}$, $\Upsilon : C \longrightarrow H$ be two family of generalized nonexpansive mappings such that $\Gamma := \big(\cap_{n=1}^{\infty} F(T_n)\big) \cap \big(\cap_{j=1}^{\infty} M_j^{-1}0\big) \cap \big(\cap_{j=1}^{\infty} GMEP(B_j, A_j, \varphi_j)\big) \neq \emptyset$, where $\cap_{n=1}^{\infty} F(T_n) = F(\Upsilon) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases} x_1 = x \in E, \ C_0 = Q_0 = C; \\ w_n = x_n + \theta_n(x_n - x_{n-1}); \\ y_n = \rho_n w_n + (1 - \rho_n) T_n J_{r_n} w_n; \\ z_n = \sigma_n y_n + (1 - \sigma_n) T_n J_{r_n} w_n; \\ u_n \in C \text{ such that } B_j(u_n, y) + \langle A_j u_n, y - u_n \rangle + \varphi_j(y) - \varphi_j(u_n) \\ + \dfrac{1}{r_{j,n}} \langle y - u_n, Ju_n - Jz_n \rangle \geq 0, \forall y \in C; \\ C_n = \{u \in C_{n-1} \cap Q_{n-1} : \| u_n - u \| \leq \| w_n - u \|; \\ Q_n = \{u \in C_{n-1} \cap Q_{n-1} : \langle x - x_n, x_n - u \rangle \geq 0\}; \\ x_{n+1} = P_{C_n \cap Q_n} x, \ \forall n \in \mathbb{N}, \end{cases}$$

where $\theta_n \subset (0,1)$ *and* $\{\rho_n\}$, $\{\sigma_n\}$ *are sequence in* $[0,1]$ *such that* $\liminf_{n\to\infty}(1-\rho_n) > 0$, $\liminf_{n\to\infty}(1-\sigma_n) > 0$ *and* $\{r_{j,n}\} \subset [a,\infty)$ *for some* $a > 0$, $\forall j \geq 1$ *Then, the sequence* $\{x_n\}$ *converges strongly to* $P_\Gamma x$, *where* P_Γ *is the metric projection from* C *onto* Γ.

Proof. From Theorem 4.1, we obtain the desired result by considering a nonexpansive mapping $T : C \longrightarrow H$ with a fixed point as generalized nonexpansive mapping. where J is the identity mapping and $\phi(x,y) = \| x - y \|^2$ for all $x, y \in H$. □

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