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EXISTENCE AND STABILITY OF POSITIVE WEAK SOLUTIONS FOR A CLASS OF CHEMICALLY REACTING SYSTEMS

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ABSTRACT. In this article, we study the existence and nonexistence results of positive weak solutions for semilinear elliptic system of the form:

$$\begin{cases} -\Delta u = \lambda a(x)[f(u,v) - \frac{1}{u^{\alpha}}], & x \in \Omega, \\ -\Delta v = \lambda b(x)[g(u,v) - \frac{1}{v^{\beta}}], & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases}$$

where λ is a positive parameter, $\alpha,\beta \in (0,1)$ and $\Omega \subset \mathbf{R}^n (n > 1)$ is a bounded domain with smooth boundary $\partial\Omega$. Here f, g are C^1 non-decreasing functions such that f, g: $[0,\infty) \times [0,\infty) \to [0,\infty)$; $f(u,v) > 0$, $g(u,v) > 0$ for $u, v > 0$ and $a(x)$, $b(x)$ are C^1 sign-changing functions that are probably negative near the boundary. In particular, on $f(0,0)$ or $g(0,0)$ there is no any sign conditions. Our approach is based on the sub-super solutions method. Also, under some certain conditions, we study the stability and instability properties of the positive weak solution for the system under consideration.

1. Introduction

In the present article, we discuss the existence results and stability of positive weak solutions for the following semilinear elliptic system:

$$\begin{cases} -\Delta u = \lambda a(x)[f(u,v) - \frac{1}{u^{\alpha}}], & x \in \Omega, \\ -\Delta v = \lambda b(x)[g(u,v) - \frac{1}{v^{\beta}}], & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Δu is the Laplacian operator, λ is a positive parameter, $a(x)$, $b(x)$ are C^1 sign-changing functions that are probably negative near the boundary, $\alpha,\beta \in (0,1)$ and $\Omega \subset \mathbf{R}^n (n > 1)$ is a bounded domain with smooth boundary $\partial\Omega$. Here f, g are C^1 non-decreasing functions such that f, g: $[0,\infty) \times [0,\infty) \to [0,\infty)$; $f(u,v) > 0$, $g(u,v) > 0$ for $u, v > 0$. In particular, on $f(0,0)$ or $g(0,0)$ there is no any sign conditions.

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Systems of singular equations such as (1) represent the stationary counterpart of general evolutionary problems of the form:

$$(2)\qquad \begin{cases} u_t = \eta\Delta u + \lambda a(x)[f(u,v) - \frac{1}{u^\alpha}], & x \in \Omega, \\ v_t = \delta\Delta v + \lambda b(x)[g(u,v) - \frac{1}{v^\beta}], & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases}$$

where $\eta, \delta > 0$ are positive parameters. System (2) is inspired by some significant applications in chemically reacting systems, where u denotes the density of an activator chemical substance while v denotes an inhibitor. Diffusion rates of u and v are respectively slow and rapid, which are converted to a small η and large δ (see [3]).

Lately, similar problems have been discussed in [6, 7, 13–15, 18, 23, 25, 27, 31]. The authors studied in [31] the model problem:

$$(3)\qquad \begin{cases} -\Delta u + \frac{1}{u^\alpha} = \lambda u^p, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $\alpha > 0$, $\lambda > 0$, $p > 0$ and $\Omega \subset \mathbf{R}^n (n \geq 1)$ is a bounded domain with $C^{2+\gamma}$ boundary for some $\gamma \in (0,1)$. This problem appeared in the context of the chemical heterogeneous catalysts as well as in non-Newtonian fluids. Their results are the following theorems:

Theorem A. If $\alpha, p \in (0,1)$, then there exists $0 < \bar{\lambda} < \infty$ that is if $\lambda > \bar{\lambda}$ then (3) has at least one solution $u_\lambda \in H_0^1(\Omega) \cap C(\bar{\Omega}) \cap C^{2+\gamma}(\Omega)$ satisfying $u_\lambda^{-\alpha} \in L^1(\Omega)$ and if $\lambda < \bar{\lambda}$ then (3) has no solution in $C(\bar{\Omega}) \cap C^2(\Omega)$.

Theorem B. If $\alpha \geq 1$, then (3) has no solution in $C(\bar{\Omega}) \cap C^2(\Omega)$ when p and λ are positive.

Diaz, Morel and Oswald established an essential and adequate existence condition for the solutions of the system:

$$(4)\qquad \begin{cases} -\Delta u + \frac{1}{u^\alpha} = f & x \in \Omega, \\ u^{-\alpha} \in L^1(\Omega), u > 0 & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases}$$

where $f \geq 0, f \in L^1(\Omega)$ and $0 < \alpha < 1$. They have shown that system (4) has a solution $u \in H_0^1(\Omega)$ if $\int_\Omega f\phi_1 dx$ is large enough and (4) has no solution if $\int_\Omega f\phi_1 dx$ is small enough (see [7]). In [25], the authors analyzed the positive solutions for the semilinear elliptic system:

$$(5)\qquad \begin{cases} -\Delta u = \lambda[f(u) - \frac{1}{u^\alpha}], & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $f \in C^2(0,\infty)$, $f(0) \geq 0$, $f' > 0$, $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$, $\alpha \in (0,1)$, $\lambda > 0$ and $\Omega \subset \mathbf{R}^n (n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$. When $n = 1$, they discussed the multiplicity and uniqueness results by using the quadrature method, while for $n > 1$ they used the sub-super solutions method to establish their existence results.

Finally, in [27], the authors studied the existence of a positive weak solution for the following semilinear elliptic system:

$$(6)\qquad \begin{cases} -\Delta u = \lambda a(x)[f(v) - \frac{1}{u^{\alpha}}], & x \in \Omega, \\ -\Delta v = \lambda b(x)[g(u) - \frac{1}{v^{\beta}}], & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases}$$

where $f, g \in C^1[0,\infty)$ are non-decreasing functions such that $f(u), g(u) > 0$ for $u > 0$, $\lim_{u\to\infty} \frac{f(Mg(u))}{u} = 0$ for every $M > 0$ and $a(x), b(x)$ are C^1 sign-changing functions satisfy certain additional conditions.

The first goal of our article is to extend the study of system (6) to system (1) with C^1 sign-changing weight functions $a(x)$, $b(x)$ and non-decreasing functions f, g satisfying

$$\lim_{x\to\infty} \frac{f(x, Mg(x,x))}{x} = 0 \quad \text{for every } M > 0, \quad \lim_{x\to\infty} \frac{g(x,x)}{x} = 0.$$

On the other hand, several authors are keen on studying the stability and instability of positive solutions of linear [1], semilinear [11,22,24,30], semiposiotne [2,5,29] and fractional [16] systems, as a result of many applications in Newtonian fluids, in Fluid mechanics, in reaction-diffusion problems, in population dynamics, glaciology, etc.; see [4,16] and their references.

Brown and Shivaji [5] studied the stability and instability of positive solutions to the system:

$$(7)\qquad \begin{cases} -\Delta u = \lambda f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

such that every non-negative solution of (7) is unstable if f is a smooth function such that $f(0) < 0$ (semipositone), $f' > 0$ and $f'' \geq 0$ for $u > 0$. In [29], Tertikas proved the statement in the non-monotone case. Shivaji and Maya [24] reduced the problem to the monotone case via decomposition of f to a monotone and linear function involving $f(0)$ and $f'(0)$. A direct proof of the result given by Karatson and Simon [11]. This could be summarized as every positive solution of (7) is unstable if $f'' > 0$ and $f(0) \leq 0$ while every positive solution of (7) is stable if $f'' < 0$ and $f(0) \geq 0$.

The second goal of our article is to extend these results to system (1) under specific conditions. We refer to [2, 10, 12, 17, 19, 20, 30] for additional results of stability and instability on elliptic systems.

We consider the following eigenvalue problem to accurately state our existence results

$$(8)\qquad \begin{cases} -\Delta \phi = \lambda \phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Assume $\lambda_1 > 0$ be the first eigenvalue of (8), ϕ_1 be the corresponding eigenfunction such that $\phi_1(x) > 0$ in Ω and $\|\phi_1\|_\infty = 1$. We consider δ, μ, $m > 0$ be such that

$$(9)\qquad \mu \leq \phi_1 \leq 1, \quad x \in \Omega - \bar{\Omega}_\delta,$$

$$(10)\qquad \frac{2}{1+s}(1 - \frac{2s}{1+s})|\nabla\phi_1|^2 \geq m, \quad x \in \bar{\Omega}_\delta,$$

for $s=\alpha,\beta$, where $\bar{\Omega}_\delta := \{x\in\Omega \,|\, d(x,\partial\Omega)\le\delta\}$. This possible since $|\nabla\phi_1|\neq 0$ on $\partial\Omega$ while $\phi_1=0$ on $\partial\Omega$ by Hopf's lemma. Furthermore, to discuss our existence results, let $e\in W_0^{1,2}(\Omega)$ be the weak solution of

$$(11)\qquad \begin{cases} -\Delta e = 1, & x\in\Omega,\\ e=0, & x\in\partial\Omega. \end{cases}$$

It is common that $e>0$ in Ω, $\frac{\partial e}{\partial n}<0$ on $\partial\Omega$ such that n is the outward unit normal vector to $\partial\Omega$ (See [9, 25]). In $\bar{\Omega}_\delta$, we suppose that $a(x),b(x)<0$, but in $\Omega-\bar{\Omega}_\delta$, $a(x),b(x)>0$. To be more specific, let $\underline{a}_0,\underline{b}_0,\underline{a}_1,\underline{b}_1,\bar{a}_0,\bar{b}_0,\bar{a}_1,\bar{b}_1>0$ be such that $-\underline{a}_0\le a(x)\le-\bar{a}_0$, $-\underline{b}_0\le b(x)\le-\bar{b}_0$ in $\bar{\Omega}_\delta$, and $\underline{a}_1\le a(x)\le\bar{a}_1$, $\underline{b}_1\le b(x)\le\bar{b}_1$ in $\Omega-\bar{\Omega}_\delta$.

2. Existence and nonexistence results

In this section, to establish our existence results we use the sub-super solutions method. Also, by the help of Young inequality we have the boundedness of the parameter λ where system (1) has no positive weak solution.

Definition 2.1.(*Positive weak solution*):
A pair of positive functions (u,v) is called a positive weak solution of (1) if $u,v\in W_0^{1,2}(\Omega)$ and

$$\begin{cases} -\Delta u = \lambda a(x)[f(u,v)-\frac{1}{u^\alpha}], & x\in\Omega,\\ -\Delta v = \lambda b(x)[g(u,v)-\frac{1}{v^\beta}], & x\in\Omega,\\ u=0=v, & x\in\partial\Omega. \end{cases}$$

Definition 2.2.(*Positive weak subsolution*):
A pair of positive functions (ψ_1,ψ_2) is called a positive weak subsolution of (1) if $\psi_1,\psi_2\in W_0^{1,2}(\Omega)$ and

$$\begin{cases} -\Delta \psi_1 \le \lambda a(x)[f(\psi_1,\psi_2)-\frac{1}{\psi_1^\alpha}], & x\in\Omega,\\ -\Delta \psi_2 \le \lambda b(x)[g(\psi_1,\psi_2)-\frac{1}{\psi_2^\beta}], & x\in\Omega,\\ \psi_1=0=\psi_2, & x\in\partial\Omega. \end{cases}$$

Definition 2.3.(*Positive weak supersolution*):
A pair of positive functions (z_1,z_2) is called a positive weak supersolution of (1) if $z_1,z_2\in W_0^{1,2}(\Omega)$ and

$$\begin{cases} -\Delta z_1 \ge \lambda a(x)[f(z_1,z_2)-\frac{1}{z_1^\alpha}], & x\in\Omega,\\ -\Delta z_2 \ge \lambda b(x)[g(z_1,z_2)-\frac{1}{z_2^\beta}], & x\in\Omega,\\ z_1=0=z_2, & x\in\partial\Omega. \end{cases}$$

Hence the following results hold.

Lemma 2.4.(See [6]): Assume there exist a subsolution (ψ_1,ψ_2) and a supersolution (z_1,z_2) of (1) such that $\psi_1\le z_1$ and $\psi_2\le z_2$. Then (1) has solution (u,v) such that $\psi_1\le u\le z_1$ and $\psi_2\le v\le z_2$.

To establish our results we assume the following:

(H1) $f,\ g:\ [0,\infty)\times[0,\infty)\to[0,\infty)$ are C^1 non-decreasing functions where $f(u,v)>0$, $g(u,v)>0$ for $u,v>0$ and $\lim_{u,v\to\infty} f(u,v)=\lim_{u,v\to\infty} g(u,v)=\infty$,

(H2) $\lim_{x\to\infty}\frac{f(x,Mg(x,x))}{x}=0$ for every $M>0$ and $\lim_{x\to\infty}\frac{g(x,x)}{x}=0$,
(H3) Let us assume that we have $\epsilon>0$ such that :
(i) $N=f(\frac{\mu\epsilon}{2},\frac{\mu\epsilon}{2})-(\frac{2}{\mu\epsilon})^{\alpha}>0$ and $M=g(\frac{\mu\epsilon}{2},\frac{\mu\epsilon}{2})-(\frac{2}{\mu\epsilon})^{\beta}>0$,
(ii) $\frac{\lambda_1 f(\epsilon,\epsilon)}{m}\le min\{\frac{2^{\alpha-1}(\alpha+1)}{\epsilon^{\alpha}},\frac{N\underline{a}_1(\alpha+1)}{2\underline{a}_0},\frac{2^{\beta-1}\underline{b}_0(\beta+1)}{\underline{a}_0\epsilon^{\beta}},\frac{M\underline{b}_1(\beta+1)}{2\underline{a}_0}\}$,
(iii) $\frac{\lambda_1 g(\epsilon,\epsilon)}{m}\le min\{\frac{2^{\beta-1}(\beta+1)}{\epsilon^{\beta}},\frac{N\underline{a}_1(\alpha+1)}{2\underline{b}_0},\frac{2^{\alpha-1}\underline{a}_0(\alpha+1)}{\underline{b}_0\epsilon^{\alpha}},\frac{M\underline{b}_1(\beta+1)}{2\underline{b}_0}\}$.
(H4) There exists $f_0,g_0>0$ such that $f(u,v)\le f_0u^{k_1}v^{l_1}$ and $g(u,v)\le g_0u^{l_2}v^{k_2}$ where k_1,k_2,l_1,l_2 are positive parameters such that k_1,$k_2\in(0,1)$ and $l_2+k_2<max\{1,\frac{1}{l_1}\}$.

To be more specific we define $\lambda_*(\epsilon)$ and $\lambda^*(\epsilon)$ by

$\lambda^*=min\left\{\frac{m\epsilon}{2\underline{a}_0f(\epsilon,\epsilon)},\frac{m\epsilon}{2\underline{b}_0g(\epsilon,\epsilon)}\right\}$ and $\lambda_*=max\left\{\frac{\lambda_1\epsilon^{\alpha+1}}{2^{\alpha}\underline{a}_0(\alpha+1)},\frac{\lambda_1\epsilon^{\beta+1}}{2^{\beta}\underline{b}_0(\beta+1)},\frac{\lambda_1\epsilon}{N\underline{a}_1(\alpha+1)},\frac{\lambda_1\epsilon}{M\underline{b}_1(\beta+1)}\right\}$.

Example 2.5. Let $f(u,v)=[v^k+(uv)^l-1]$ and $g(u,v)=[u^{\omega}+(uv)^{\frac{\tau}{2}}-1]$ where k,l,ω,τ are positive parameters. So, it is clear that the hypotheses of (H1)-(H3) satisfied by f,g if $\max\{\omega,\tau\}k<1$, $\max\{\omega,\tau\}<1$ and $(\max\{\omega,\tau\}+1)l<1$.

Remark 2.6. By (H3) we conclude that $\lambda_*<\lambda^*$.

Now we can state our existence results.

Theorem 2.7. There exists a positive weak solution of (1) for every $\lambda_*(\epsilon)\le\lambda\le\lambda^*(\epsilon)$ if the assumptions (H1)-(H3) are satisfied.

proof. We shall verify that $(\psi_1,\psi_2)=\left(\frac{\epsilon}{2}\phi_1^{\frac{2}{1+\alpha}},\frac{\epsilon}{2}\phi_1^{\frac{2}{1+\beta}}\right)$ is a positive weak subsolution of (1). A calculations shows that $\nabla\psi_1=\left(\frac{\epsilon}{1+\alpha}\right)\phi_1^{\frac{1-\alpha}{1+\alpha}}\nabla\phi_1$, and hence

$$\begin{aligned}-\Delta\psi_1&=-\nabla(\nabla\psi_1)=\nabla\left((\frac{\epsilon}{1+\alpha})\phi_1^{\frac{1-\alpha}{1+\alpha}}\nabla\phi_1\right)\\&=\frac{-\epsilon}{1+\alpha}\left\{(\frac{1-\alpha}{1+\alpha})\phi_1^{\frac{-2\alpha}{1+\alpha}}|\nabla\phi_1|^2+\phi_1^{\frac{1-\alpha}{1+\alpha}}\Delta\phi_1\right\}\\&=\frac{\epsilon}{1+\alpha}\left\{\phi_1^{\frac{1-\alpha}{1+\alpha}}(-\Delta\phi_1)-(\frac{1-\alpha}{1+\alpha})\phi_1^{\frac{-2\alpha}{1+\alpha}}|\nabla\phi_1|^2\right\}\\&=\frac{\epsilon}{1+\alpha}\left\{\lambda_1\phi_1^{\frac{2}{1+\alpha}}-(\frac{1-\alpha}{1+\alpha})\phi_1^{\frac{-2\alpha}{1+\alpha}}|\nabla\phi_1|^2\right\}.\end{aligned}\tag{12}$$

Similarly,

$$-\Delta\psi_2=\frac{\epsilon}{1+\beta}\left\{\lambda_1\phi_1^{\frac{2}{1+\beta}}-(\frac{1-\beta}{1+\beta})\phi_1^{\frac{-2\beta}{1+\beta}}|\nabla\phi_1|^2\right\}.$$

Firstly, we study the case when $x\in\bar{\Omega}_{\delta}$. For $s=\alpha$ in (10) one can get

$$\frac{-\epsilon}{1+\alpha}\phi_1^{\frac{-2\alpha}{1+\alpha}}(\frac{1-\alpha}{1+\alpha})|\nabla\phi_1|^2\le\frac{-m\epsilon}{2},$$

and since $\lambda\le\lambda^*$, we have

$$\frac{-m\epsilon}{2}\le-\lambda\underline{a}_0f(\epsilon,\epsilon)\le-\lambda\underline{a}_0f(\frac{\epsilon}{2}\phi_1^{\frac{2}{1+\alpha}},\frac{\epsilon}{2}\phi_1^{\frac{2}{1+\beta}}),$$

and so

$$\frac{-\epsilon}{1+\alpha}\phi_1^{\frac{-2\alpha}{1+\alpha}}(\frac{1-\alpha}{1+\alpha})|\nabla\phi_1|^2\le-\lambda\underline{a}_0f(\frac{\epsilon}{2}\phi_1^{\frac{2}{1+\alpha}},\frac{\epsilon}{2}\phi_1^{\frac{2}{1+\beta}}).\tag{13}$$

Also, since $\lambda\ge\lambda_*$, we have

$$\frac{\epsilon}{1+\alpha}\lambda_1\phi_1^{\frac{2}{1+\alpha}}\le\frac{\lambda_1\epsilon}{1+\alpha}\le\frac{\lambda\underline{a}_0}{(\frac{\epsilon}{2})^{\alpha}}\le\frac{\lambda\underline{a}_0}{(\frac{\epsilon}{2}\phi_1^{\frac{2}{1+\alpha}})^{\alpha}}\tag{14}$$

Combining (12-14) we see that

$$\begin{aligned}
-\Delta\psi_1 &\leq \frac{\lambda\underline{a}_0}{(\frac{\epsilon}{2}\phi_1^{\frac{2}{1+\alpha}})^\alpha} - \lambda\underline{a}_0 f(\frac{\epsilon}{2}\phi_1^{\frac{2}{1+\alpha}}, \frac{\epsilon}{2}\phi_1^{\frac{2}{1+\beta}}) \\
&= -\lambda\underline{a}_0\left[f(\frac{\epsilon}{2}\phi_1^{\frac{2}{1+\alpha}}, \frac{\epsilon}{2}\phi_1^{\frac{2}{1+\beta}}) - \frac{1}{(\frac{\epsilon}{2}\phi_1^{\frac{2}{1+\alpha}})^\alpha}\right] \\
&\leq \lambda a(x)[f(\psi_1,\psi_2) - \frac{1}{\psi_1{}^\alpha}].
\end{aligned}$$

Furthermore, on $\Omega - \bar{\Omega}_\delta$ we have $\mu \leq \phi_1 \leq 1$, $\underline{a}_1 \leq a(x)$ and $\underline{b}_1 \leq b(x)$. Since $\lambda_* \leq \lambda$, then

$$\frac{\lambda_1\epsilon}{N\underline{a}_1(\alpha+1)} \leq \lambda.$$

Now, we have

$$\begin{aligned}
-\Delta\psi_1 &= \frac{\epsilon}{1+\alpha}\left\{\lambda_1\phi_1^{\frac{2}{1+\alpha}} - (\frac{1-\alpha}{1+\alpha})\phi_1^{\frac{-2\alpha}{1+\alpha}}|\nabla\phi_1|^2\right\} \\
&\leq \frac{\lambda_1\epsilon\,\phi_1^{\frac{2}{1+\alpha}}}{1+\alpha} \leq \lambda\underline{a}_1 N \\
&= \lambda\underline{a}_1\left[f(\frac{\mu\epsilon}{2}, \frac{\mu\epsilon}{2}) - (\frac{2}{\mu\epsilon})^\alpha\right] \\
&\leq \lambda\underline{a}_1\left[f(\frac{\epsilon}{2}\phi_1^{\frac{2}{1+\alpha}}, \frac{\epsilon}{2}\phi_1^{\frac{2}{1+\beta}}) - \frac{1}{(\frac{\epsilon}{2}\phi_1^{\frac{2}{1+\alpha}})^\alpha}\right] \\
&\leq \lambda a(x)\left[f(\psi_1,\psi_2) - \frac{1}{\psi_1^\alpha}\right].
\end{aligned}$$

Similarly, we can get

$$-\Delta\psi_2 \leq \lambda b(x)[g(\psi_1,\psi_2) - \frac{1}{\psi_2^\beta}],$$

i.e., (ψ_1,ψ_2) is a positive weak subsolution of (1).

Next, we show that there exists a large enough c thus,

$$(z_1,z_2) = \left(ce(x), \lambda\mu_b\left[g(c\mu_e, c\mu_e)\right]e(x)\right)$$

is a positive weak supersolution of (1), where $\mu_e = \|e(x)\|_\infty$, $\mu_a = \|a(x)\|_\infty$ and $\mu_b = \|b(x)\|_\infty$. Now by (H2) we are able to take c large enough so that

$$c \geq \lambda\mu_a f\left(c\mu_e, \lambda\mu_b[g(c\mu_e, c\mu_e)]\mu_e\right). \tag{15}$$

Then, using (11) and (15) we have

$$\begin{aligned}
-\Delta z_1 = -\Delta(ce(x)) = c &\geq \lambda\mu_a f\left(c\mu_e, \lambda\mu_b[g(c\mu_e, c\mu_e)]\mu_e\right) \\
&\geq \lambda a(x) f\left(ce(x), \lambda\mu_b[g(c\mu_e, c\mu_e)]e(x)\right) \\
&= \lambda a(x) f(z_1,z_2) \\
&\geq \lambda a(x)[f(z_1,z_2) - \frac{1}{z_1^\alpha}].
\end{aligned}$$

Also, by (H2) we can take $c \geq \lambda\mu_b g(c\mu_e, c\mu_e)$. Then we have

$$\begin{aligned}
-\Delta z_2 &= -\Delta\big(\lambda\mu_b\, g(c\mu_e, c\mu_e)e(x)\big)\\
&= \lambda\mu_b\, g(c\mu_e, c\mu_e)\\
&\geq \lambda b(x)\, g(ce(x), ce(x))\\
&\geq \lambda b(x)\, g\big(ce(x), \lambda\mu_b\, g(c\mu_e, c\mu_e)e(x)\big)\\
&= \lambda b(x) g(z_1, z_2)\\
&\geq \lambda b(x)\big[g(z_1, z_2) - \frac{1}{z_2^{\beta}}\big].
\end{aligned}$$

i.e., (z_1, z_2) is a positive weak supersolution of (1) with $\psi_i \leq z_i$ for large c and $i = 1, 2$. Therefore, there exists a positive weak solution (u, v) of (1) thus $\psi_1 \leq u \leq z_1$ and $\psi_2 \leq v \leq z_2$. Hence, the proof is completed.

Theorem 2.8. Let (H4) holds with $l_i + k_i = 1$, $i = 1, 2$, then system (1) has no positive weak solution for every $\lambda \in (\frac{-\lambda_1}{2t}, \frac{\lambda_1}{2s})$, where $s = max\{f_0\overline{a}_1, g_0\overline{b}_1\}$ and $t = min\{f_0\overline{a}_0, g_0\overline{b}_0\}$.

Proof. Assume that system (1) has a positive weak solution (u, v). We will eventually arrive at a contradiction in order to prove Theorem 2.8. If the first equation of (1) is multiplied by u, then by Young inequality we have

$$\int_{\Omega} |\nabla u|^2 dx \leq \lambda \int_{\Omega} f_0 a(x)(\frac{u^2}{\mu_1} + \frac{v^2}{\mu_2})dx, \tag{16}$$

with $\mu_1 = \frac{2}{1+k_1} > 1$ and $\mu_2 = \frac{2}{1-k_1} > 1$. Similarly, we have

$$\int_{\Omega} |\nabla v|^2 dx \leq \lambda \int_{\Omega} g_0 b(x)(\frac{u^2}{\theta_1} + \frac{v^2}{\theta_2})dx, \tag{17}$$

with $\theta_1 = \frac{2}{1-k_2} > 1$ and $\theta_2 = \frac{2}{1+k_2} > 1$. Note that

$$\lambda_1 \int_{\Omega} u^2 dx \leq \int_{\Omega} |\nabla u|^2 dx, \quad \lambda_1 \int_{\Omega} v^2 dx \leq \int_{\Omega} |\nabla v|^2 dx. \tag{18}$$

Combining (16)-(18), we obtain

$$\lambda_1 \int_{\Omega} u^2 dx + \lambda_1 \int_{\Omega} v^2 dx \leq \lambda\Big[\int_{\Omega} \Big(\frac{f_0 a(x)}{\mu_1} + \frac{g_0 b(x)}{\theta_1}\Big)u^2 dx + \int_{\Omega} \Big(\frac{f_0 a(x)}{\mu_2} + \frac{g_0 b(x)}{\theta_2}\Big)v^2 dx\Big]. \tag{19}$$

Now, on $\Omega - \bar{\Omega}_\delta$ we have $a(x) \leq \overline{a}_1, b(x) \leq \overline{b}_1$, then (19) becomes

$$(\lambda_1 - 2\lambda s)\int_{\Omega} u^2 dx + (\lambda_1 - 2\lambda s)\int_{\Omega} v^2 dx \leq 0, \tag{20}$$

where $s = max\{f_0\overline{a}_1, g_0\overline{b}_1\}$, which is a contradiction if $\lambda < \frac{\lambda_1}{2s}$.
Similarly, on $\bar{\Omega}_\delta$ we have $a(x) \leq -\overline{a}_0, b(x) \leq -\overline{b}_0$, and then (19) becomes

$$(\lambda_1 + 2\lambda t)\int_{\Omega} u^2 dx + (\lambda_1 + 2\lambda t)\int_{\Omega} v^2 dx \leq 0, \tag{21}$$

where $t = min\{f_0\overline{a}_0, g_0\overline{b}_0\}$, which is a contradiction if $\lambda > -\frac{\lambda_1}{2t}$. Hence, the proof is completed.

3. Stability and instability results

This section deals with the stability and instability of the positive weak solution (u,v) of system (1) under specific conditions.

The linearized equation about (u,v), where (u,v) any positive weak solution of (1) is

$$\begin{cases} -\Delta\phi - \lambda a(x)[f_u(u,v) + \frac{\alpha}{u^{1+\alpha}}]\phi - \lambda a(x) f_v(u,v)\psi = \mu\phi, & x \in \Omega, \\ -\Delta\psi - \lambda b(x) g_u(u,v)\phi - \lambda b(x)[g_v(u,v) + \frac{\beta}{v^{1+\beta}}]\psi = \mu\psi, & x \in \Omega, \\ \phi = 0 = \psi, & x \in \partial\Omega, \end{cases} \tag{22}$$

where $f_u(u,v)$ represents the partial derivative of $f(u,v)$ with respect to u (see [8]). Let (ϕ_1, ψ_1) be the corresponding eigenfunction to the principal eigenvalue μ_1. We take ϕ_1, ψ_1 such that $\phi_1, \psi_1 > 0$ in Ω (see [21, 28]).

Definition 4.1. A solution (u,v) of (1) is a stable solution if all eigenvalues of (22) are strictly positive, which can be inferred if the principal eigenvalue $\mu_1 > 0$. In contrast, (u,v) is unstable.

To establish our results we assume the following:

$$f_v(u,v),\ g_u(u,v) > 0, \quad \text{for } u, v > 0, \tag{23}$$

$$u \mapsto \frac{f(u,v) - u^{-\alpha}}{u} \quad \text{is strictly non-decreasing at } u \quad \forall v > 0, \tag{24}$$

$$v \mapsto \frac{g(u,v) - v^{-\beta}}{v} \quad \text{is strictly non-decreasing at } v \quad \forall u > 0, \tag{25}$$

Theorem 4.2. Assume (23)-(25) hold, then the positive weak solution (u_0, v_0) of system (1) is unstable in $\Omega - \bar{\Omega}_\delta$ and stable in $\bar{\Omega}_\delta$.

Proof. Let (u_0, v_0) be any positive weak solution of (1). We multiply the first and second equation of (1) by ϕ_1, ψ_1, respectively and integrate over Ω yields

$$-\int_\Omega \phi_1(x)\Delta u_0 dx - \lambda \int_\Omega \phi_1(x) a(x)[f(u_0, v_0) - \frac{1}{{u_0}^\alpha}]dx = 0, \tag{26}$$

and

$$-\int_\Omega \psi_1(x)\Delta v_0 dx - \lambda \int_\Omega \psi_1(x) b(x)[g(u_0, v_0) - \frac{1}{{v_0}^\beta}]dx = 0. \tag{27}$$

On the other side, we multiply the first and second equation of (22) by $-u_0$, $-v_0$, respectively and integrate over Ω yields

$$\begin{aligned} \int_\Omega u_0 \Delta\phi_1 dx + \lambda \int_\Omega \phi_1(x) a(x)\big[f_u(u_0, v_0) + \frac{\alpha}{{u_0}^{1+\alpha}}\big] u_0 dx + \lambda \int_\Omega \psi_1(x) a(x) f_v(u_0, v_0) u_0 dx \\ = -\mu_1 \int_\Omega u_0 \phi_1(x) dx, \end{aligned} \tag{28}$$

and

$$\begin{aligned} \int_\Omega v_0 \Delta\psi_1 dx + \lambda \int_\Omega \psi_1(x) b(x)\big[g_v(u_0, v_0) + \frac{\beta}{{v_0}^{1+\beta}}\big] v_0 dx + \lambda \int_\Omega \phi_1(x) b(x) g_u(u_0, v_0) v_0 dx \\ = -\mu_1 \int_\Omega v_0 \psi_1(x) dx. \end{aligned} \tag{29}$$

Combining (26) to (29) we get

$$
\begin{aligned}
&\int_\Omega [u_0\Delta\phi_1 - \phi_1(x)\Delta u_0]dx + \int_\Omega [v_0\Delta\psi_1 - \psi_1(x)\Delta v_0]dx \\
&+\lambda\int_\Omega \phi_1(x)a(x)\left[u_0\left(f_u(u_0,v_0) + \frac{\alpha}{{u_0}^{1+\alpha}}\right) - \left(f(u_0,v_0) - \frac{1}{{u_0}^{\alpha}}\right)\right]dx \\
&+\lambda\int_\Omega \psi_1(x)b(x)\left[v_0\left(g_v(u_0,v_0) + \frac{\beta}{{v_0}^{1+\beta}}\right) - \left(g(u_0,v_0) - \frac{1}{{v_0}^{\beta}}\right)\right]dx \\
&+\lambda\int_\Omega a(x)\psi_1(x)f_v(u_0,v_0)u_0dx + \lambda\int_\Omega b(x)\phi_1(x)g_u(u_0,v_0)v_0dx \\
&= -\mu_1\int_\Omega [u_0\phi_1(x) + v_0\psi_1(x)]dx.
\end{aligned} \tag{30}
$$

But by the Green's first identity

$$\int_\Omega u_0\Delta\phi_1 dx = -\int_\Omega \nabla u_0\cdot\nabla\phi_1 dx, \tag{31}$$

$$\int_\Omega \phi_1(x)\Delta u_0 dx = -\int_\Omega \nabla u_0\cdot\nabla\phi_1 dx, \tag{32}$$

and

$$\int_\Omega v_0\Delta\psi_1 dx = -\int_\Omega \nabla v_0\cdot\nabla\psi_1 dx, \tag{33}$$

$$\int_\Omega \psi_1(x)\Delta v_0 dx = -\int_\Omega \nabla v_0\cdot\nabla\psi_1 dx. \tag{34}$$

By using (31)-(34) in (30) we get

$$
\begin{aligned}
&\lambda\int_\Omega \phi_1(x)a(x)\left[u_0f_u(u_0,v_0) - f(u_0,v_0) + \frac{1+\alpha}{{u_0}^{\alpha}}\right]dx + \lambda\int_\Omega \psi_1(x)a(x)f_v(u_0,v_0)u_0dx \\
&+\lambda\int_\Omega \psi_1(x)b(x)\left[v_0g_v(u_0,v_0) - g(u_0,v_0) + \frac{1+\beta}{v_0^{\beta}}\right]dx + \lambda\int_\Omega \phi_1(x)b(x)g_u(u_0,v_0)v_0dx \\
&= -\,\mu_1\int_\Omega [u_0\phi_1(x) + v_0\psi_1(x)]dx.
\end{aligned} \tag{35}
$$

Also, since $\frac{f(u_0,v_0)-{u_0}^{-\alpha}}{u_0}$ is strictly non-decreasing at u_0 $\forall\, v_0 > 0$, then we get

$$\frac{u_0f_u(u_0,v_0) - f(u_0,v_0) + (1+\alpha){u_0}^{-\alpha}}{u_0^2} > 0 \quad \text{for} \quad u_0, v_0 > 0, \tag{36}$$

and since $\frac{g(u_0,v_0)-{v_0}^{-\beta}}{v_0}$ is strictly non-decreasing at v_0 $\forall\, u_0 > 0$, then we get

$$\frac{v_0g_v(u_0,v_0) - g(u_0,v_0) + (1+\beta){v_0}^{-\beta}}{v_0^2} > 0 \quad \text{for} \quad u_0, v_0 > 0. \tag{37}$$

On $\Omega - \bar{\Omega}_\delta$ we have $a(x), b(x) > 0$, by using (36) and (37) in (35) we get

$$-\mu_1\int_\Omega [u_0\phi_1(x) + v_0\psi_1(x)]dx > 0. \tag{38}$$

Then, $\mu_1 < 0$ and the solution is unstable.
Similarly, on $\bar{\Omega}_\delta$ we have $a(x), b(x) < 0$, by using (36) and (37) in (35) we get

$$-\mu_1 \int_\Omega [u_0\phi_1(x) + v_0\psi_1(x)]dx < 0, \tag{39}$$

and hence $\mu_1 > 0$ and the solution is stable. Hence, the proof is completed.

Remark 4.3. According to the previous theorem, the stability of the positive weak solution depends on the domain besides the assumptions given by (23)-(25).

Remark 4.4. If the conditions (23)-(25) replaced by the following conditions

$$f_v(u,v),\ g_u(u,v) < 0, \quad \text{for } u,v > 0, \tag{40}$$

$$u \mapsto f(u,v)u^{-1} - u^{-\alpha-1} \quad \text{is strictly non-increasing at } u \quad \forall v > 0, \tag{41}$$

$$v \mapsto g(u,v)v^{-1} - v^{-\beta-1} \quad \text{is strictly non-increasing at } v \quad \forall u > 0, \tag{42}$$

we conclude:

Corollary 4.5. If (40)-(42) hold, then every positive weak solution (u,v) of system (1) is stable in $\Omega - \bar{\Omega}_\delta$ and unstable in $\bar{\Omega}_\delta$.

Proof. The proof procedure is analogous to the proof of Theorem 4.2.

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