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AN INVESTIGATION OF THE SOLUTIONS FOR RATIONAL SYSTEMS OF DIFFERENCE EQUATIONS OF ORDER TWO

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Abstract. This paper is concerned to investigate the form of the solutions of the following systems of difference equations of second order.

$$\mu_{n+1} = \frac{\mu_n \omega_{n-1}}{\pm\omega_{n-1} \pm \mu_{n-1}}, \quad \omega_{n+1} = \frac{\omega_n \mu_{n-1}}{\pm\mu_{n-1} \pm \omega_{n-1}}, \quad n = 0, 1, ...,$$

where the initial conditions are arbitrary nonzero real numbers. Moreover, we verify our theoretical outcomes at the end with some numerical applications and draw them using some mathematical programs to illustrate the results.

1. Introduction

This paper is devoted to investigate the behavior of the solution of the following system of difference equation

$$\mu_{n+1} = \frac{\mu_n \omega_{n-1}}{\pm\omega_{n-1} \pm \mu_{n-1}}, \quad \omega_{n+1} = \frac{\omega_n \mu_{n-1}}{\pm\mu_{n-1} \pm \omega_{n-1}}, \quad n = 0, 1, ..., \tag{1}$$

where the initial conditions are arbitrary nonzero real numbers. For the last decade, there have been a lot of research activities related to the qualitative analysis of rational difference equations and systems of order greater than one. One of the reasons for this is that it is widely used to investigate equations arising in mathematical models describing real-life phenomena. For example, biology, economics, probability theory, etc.

There are many papers related to difference equation systems of order greater than one; for example, The form of the solutions to the rational difference equations system

$$\mu_{n+1} = \frac{A\mu_n + \omega_n}{\mu_{n-p}}, \qquad \omega_{n+1} = \frac{A + \mu_n}{\omega_{n-q}},$$

have obtained by Papaschinopoulos and Schinas in [1].

Grove et al. [2] studied existence and behavior of solutions of the rational system

$$\mu_{n+1} = \frac{a}{\mu_n} + \frac{b}{\omega_n}, \qquad \omega_{n+1} = \frac{c}{\mu_n} + \frac{d}{\omega_n}$$

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Yang et al. [3] investigated the behavior of the solutions of the system

$$\mu_n = \frac{a}{\omega_{n-p}}, \quad \omega_n = \frac{b\omega_{n-p}}{\mu_{n-q}\omega_{n-q}}.$$

Din et al. [4] studied the behavior of the solutions of the following fourth-order system of rational difference equations of the form

$$\mu_{n+1} = \frac{\alpha\mu_{n-3}}{\beta + \gamma\omega_n\omega_{n-1}\omega_{n-2}\omega_{n-3}}, \qquad \omega_{n+1} = \frac{\alpha_1\mu_{n-3}}{\beta_1 + \alpha_1\mu_n\mu_{n-1}\mu_{n-2}\mu_{n-3}}.$$

Touafek and Elsayed [5] investigated the periodic nature and gave the form of the solutions to the following systems of rational difference equations

$$\mu_{n+1} = \frac{\omega_n}{\mu_{n-1}(\pm 1 \pm \omega_n)}, \qquad \omega_{n+1} = \frac{\mu_n}{\omega_{1-1}(\pm 1 \pm \mu_n)}.$$

El-Metwally [6] found the solutions form for the following systems of rational difference equations

$$\mu_{n+1} = \frac{\mu_{n-1}\omega_n}{\pm\mu_{n-1} \pm \omega_{n-2}}, \qquad \omega_{n+1} = \frac{\omega_{n-1}\mu_n}{\pm\omega_{n-1} \ \pm \mu_{n-2}}.$$

Kara and Yazlik, [7] investigated the following higher order system of nonlinear difference equations

$$\mu_n = \frac{\mu_{n-k}\omega_{n-k-l}}{\omega_{n-l}(a_n + b_n\mu_{n-k}\omega_{n-k-l})}, \qquad \omega_n = \frac{\omega_{n-k}\mu_{n-k-l}}{\mu_{n-l}(\alpha_n + \beta_n\omega_{n-k}\mu_{n-k-l})}.$$

Elsayed et al. [8] investigated the periodic nature of the solutions of the following systems of rational difference equations

$$\mu_{n+1} = \frac{1 \pm z_n}{\omega_{n-1}}, \ \omega_{n+1} = \frac{1 \pm \mu_n}{z_{n-1}}, \ z = \frac{1 \pm \omega_n}{\mu_{n-1}}.$$

The dynamics of positive solutions for a system of rational difference equations of the following form

$$\mu_{n+1} = \frac{\alpha\mu_{n-1}^2}{\beta + \gamma\omega_{n-2}}, \qquad \omega_{n+1} = \frac{\alpha_1\omega_{n-1}^2}{\beta_1 + \gamma_1\mu_{n-2}}.$$

have been investigated by the authors in [9].

Buyuk and Taskara [10] investigated the form of the solutions of the following rational difference equation system

$$\mu_n = \frac{\nu_{n-1}\nu_{n-3}}{\mu_{n-2} + 2\nu_{n-3}}, \omega_n = \frac{\mu_{n-1}\mu_{n-3}}{-\omega_{n-2} + 6\mu_{n-3}}, \nu_n = \frac{\omega_{n-1}\omega_{n-3}}{\nu_{n-2} + 14\omega_{n-3}}.$$

In addition, other related results to the difference equations and nonlinear system of rational difference equations can be found in Ref. [11]- [16] and the references cited therein.

2. The System: $\mu_{n+1} = \frac{\mu_n\omega_{n-1}}{\omega_{n-1}+\mu_{n-1}}, \quad \omega_{n+1} = \frac{\omega_n\mu_{n-1}}{\mu_{n-1}+\omega_{n-1}}$

In this section, we obtain the form of the solutions of the system of the difference equations:

$$\mu_{n+1} = \frac{\mu_n\omega_{n-1}}{\omega_{n-1} + \mu_{n-1}}, \quad \omega_{n+1} = \frac{\omega_n\mu_{n-1}}{\mu_{n-1} + \omega_{n-1}}. \tag{2}$$

Theorem 2.1. *Assume that* $\{\mu_n, \omega_n\}$ *is a solution of system (2). Then for* $n = 1, 2, ...,$ *we have*

$$\begin{aligned}
\mu_{6n} =& \frac{a(abcd)^{2n}}{\{(a+c)(b+d)(cb+ad)\}^{2n}},\\
\mu_{6n+1} =& \frac{ad(abcd)^{2n}}{(b+d)\{(a+c)(b+d)(cb+ad\}^{2n}},\\
\mu_{6n+2} =& \frac{acd(bc+ad)(abcd)^{2n}}{\{(a+c)(b+d)(cb+ad\}^{2n+1}},\\
\mu_{6n+3} =& \frac{c(abcd)^{2n+1}}{\{(a+c)(b+d)(cb+ad)\}^{2n+1}},\\
\mu_{6n+3} =& \frac{c(abcd)^{2n+1}}{\{(a+c)(b+d)(cb+ad)\}^{2n+1}},\\
\mu_{6n+4} =& \frac{bc(abcd)^{2n+1}}{(b+d)\{(a+c)(b+d)(cb+ad\}^{2n+1}},\\
\mu_{6n+5} =& \frac{(bc+ad)(abcd)^{2n+2}}{d\{(a+c)(b+d)(cb+ad\}^{2n+2}},\\
\omega_{6n} =& \frac{c(abcd)^{2n}}{\{(a+c)(b+d)(cb+ad)\}^{2n}},\\
\omega_{6n+1} =& \frac{bc(abcd)^{2n}}{(b+d)\{(a+c)(b+d)(cb+ad)\}^{2n}},\\
\omega_{6n+2} =& \frac{(bc+ad)(abcd)^{2n+1}}{d\{(a+c)(b+d)(cb+ad)\}^{2n+1}},\\
\omega_{6n+3} =& \frac{a(abcd)^{2n+1}}{\{(a+c)(b+d)(cb+ad\}^{2n+1}},\\
\omega_{6n+4} =& \frac{(ad)^{2n+2}(bc)^{2n+1}}{(b+d)\{(a+c)(b+d)(cb+ad\}^{2n+1}},\\
\omega_{6n+5} =& \frac{(bc+ad)(abcd)^{2n+2}}{b\{(a+c)(b+d)(cb+ad)\}^{2n+2}},
\end{aligned}$$

where $\mu_{-1} = b,\ \mu_0 = a,\ \omega_{-1} = d$ *and* $\omega_0 = c$.

Proof. By using mathematical induction, we can prove the following: For $n > 0$ the result holds. Assume that the result holds for $n-1$, as follows:

$$\begin{aligned}
\mu_{6n-1} =& \frac{(bc+ad)(abcd)^{2n+1}}{d\{(a+c)(b+d)(cb+ad\}^{2n+1}},\\
\mu_{6n-2} =& \frac{bc(abcd)^{2n}}{(b+d)\{(a+c)(b+d)(cb+ad\}^{2n}},\\
\mu_{6n-2} =& \frac{bc(abcd)^{2n}}{(b+d)\{(a+c)(b+d)(cb+ad\}^{2n}},\\
\mu_{6n-3} =& \frac{c(abcd)^{2n}}{\{(a+c)(b+d)(cb+ad)\}^{2n}},\\
\mu_{6n-4} =& \frac{acd(bc+ad)(abcd)^{2n-1}}{\{(a+c)(b+d)(cb+ad\}^{2n}},
\end{aligned}$$

$$\mu_{6n-5} = \frac{ad(abcd)^{2n-1}}{(b+d)\{(a+c)(b+d)(cb+ad\}^{2n-1}},$$

$$\mu_{6n-5} = \frac{ad(abcd)^{2n-1}}{(b+d)\{(a+c)(b+d)(cb+ad\}^{2n-1}},$$

$$\omega_{6n-1} = \frac{(bc+ad)(abcd)^{2n+1}}{b\{(a+c)(b+d)(cb+ad)\}^{2n+1}},$$

$$\omega_{6n-2} = \frac{(ad)^{2n+1}(bc)^{2n}}{(b+d)\{(a+c)(b+d)(cb+ad\}^{2n}},$$

$$\omega_{6n-3} = \frac{a(abcd)^{2n}}{\{(a+c)(b+d)(cb+ad\}^{2n}},$$

$$\omega_{6n-4} = \frac{(bc+ad)(abcd)^{2n}}{d\{(a+c)(b+d)(cb+ad)\}^{2n}},$$

$$\omega_{6n-5} = \frac{bc(abcd)^{2n-1}}{(b+d)\{(a+c)(b+d)(cb+ad)\}^{2n-1}}.$$

Then, from equation (2), it follows that

$$\begin{aligned}\mu_{6n-1} = \frac{\mu_{6n-2}\omega_{6n-3}}{\omega_{6n-3}+\mu_{6n-3}} &= \frac{\left(\frac{bc(abcd)^{2n}}{(b+d)\{(a+c)(b+d)(cb+ad)\}^{2n}}\right)\left(\frac{a(abcd)^{2n}}{\{(a+c)(b+d)(cb+ad)\}^{2n}}\right)}{\left(\frac{a(abcd)^{2n}}{\{(a+c)(b+d)(cb+ad)\}^{2n}}\right)+\left(\frac{c(abcd)^{2n}}{\{(a+c)(b+d)(cb+ad)\}^{2n}}\right)} \\ &= \frac{(abc)(abcd)^{2n}}{(b+d)\{(a+c)(b+d)(cb+ad)\}^{2n}(a+c)}.\frac{d.(bc+ad)}{d.(bc+ad} \\ &= \frac{(bc+ad)(abcd)^{2n+1}}{d\{(a+c)(b+d)(cb+ad)\}^{2n+1}},\end{aligned}$$

and,

$$\begin{aligned}\omega_{6n-1} = \frac{\omega_{6n-2}\mu_{6n-3}}{\mu_{6n-3}+\omega_{6n-3}} &= \frac{\left(\frac{(ad)^{2n+1}(bc)^{2n}}{(b+d)\{(a+c)(b+d)(cb+ad)\}^{2n}}\right)\left(\frac{c(abcd)^{2n}}{\{(a+c)(b+d)(cb+ad)\}^{2n}}\right)}{\left(\frac{c(abcd)^{2n}}{\{(a+c)(b+d)(cb+ad)\}^{2n}}\right)+\left(\frac{a(abcd)^{2n}}{\{(a+c)(b+d)(cb+ad)\}^{2n}}\right)} \\ &= \frac{(adc)(abcd)^{2n}}{\{(a+c)(b+d)(cb+ad)\}^{2n}(b+b)(c+a)}.\frac{b.(bc+ad)}{b.(bc+ad} \\ &= \frac{(bc+ad)(abcd)^{2n+1}}{b\{(a+c)(b+d)(cb+ad)\}^{2n+1}}.\end{aligned}$$

We also see that,

$$\begin{aligned}\mu_{6n+1} = \frac{\mu_{6n}\omega_{6n-1}}{\omega_{6n-1}+\mu_{6n-1}} &= \frac{\left(\frac{a(abcd)^{2n}}{\{(a+c)(b+d)(cb+ad)\}^{2n}}\right)\left(\frac{(bc+ad)(abcd)^{2n+1}}{b\{(a+c)(b+d)(cb+ad)\}^{2n+1}}\right)}{\left(\frac{(bc+ad)(abcd)^{2n+1}}{b\{(a+c)(b+d)(cb+ad)\}^{2n+1}}\right)+\left(\frac{(bc+ad)(abcd)^{2n+1}}{d\{(a+c)(b+d)(cb+ad)\}^{2n+1}}\right)} \\ &= \frac{a(abcd)^{2n}(bc+ad).d}{\{(a+c)(b+d)(cb+ad)\}^{2n}.(bc+ad).(d+b)} \\ &= \frac{ad(abcd)^{2n}}{(b+d)\{(a+c)(b+d)(cb+ad\}^{2n}},\end{aligned}$$

and,

$$\begin{aligned}\omega_{6n+1} = \frac{\omega_{6n}\mu_{6n-1}}{\mu_{6n-1}+\omega_{6n-1}} &= \frac{\left(\frac{c(abcd)^{2n}}{\{(a+c)(b+d)(cb+ad)\}^{2n}}\right)\left(\frac{(bc+ad)(abcd)^{2n+1}}{d\{(a+c)(b+d)(cb+ad)\}^{2n+1}}\right)}{\left(\frac{(bc+ad)(abcd)^{2n+1}}{d\{(a+c)(b+d)(cb+ad)\}^{2n+1}}\right)+\left(\frac{(bc+ad)(abcd)^{2n+1}}{b\{(a+c)(b+d)(cb+ad)\}^{2n+1}}\right)} \\ &= \frac{c(abcd)^{2n}(bc+ad).b}{\{(a+c)(b+d)(cb+ad)\}^{2n}(cb+ad)(d+b)}\end{aligned}$$

$$=\frac{bc(abcd)^{2n}}{(b+d)\{(a+c)(b+d)(cb+ad)\}^{2n}}.$$

Hence, we can easily proof the other relations. The proof has been done. □

Example 2.1 Figure 1 illustrates the behavior of the solutions of the system of difference equations (2) when the initial conditions $\mu_{-1}=4.5,\ \mu_0=-4.5,\ \omega_{-1}=-3.1,\ \omega_0=8.2$.

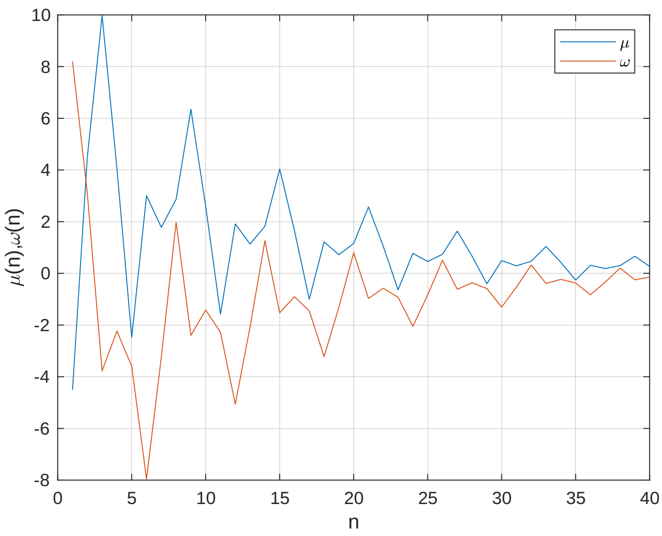

FIGURE 1. Sketch the behavior of the solution of (2)

3. The System: $\mu_{n+1}=\frac{\mu_n\omega_{n-1}}{-\omega_{n-1}-\mu_{n-1}},\quad \omega_{n+1}=\frac{\omega_n\mu_{n-1}}{-\mu_{n-1}-\omega_{n-1}}$

In this section, we get the solutions to the following system of difference equations:

$$\mu_{n+1}=\frac{\mu_n\omega_{n-1}}{-\omega_{n-1}-\mu_{n-1}},\quad \omega_{n+1}=\frac{\omega_n\mu_{n-1}}{-\mu_{n-1}-\omega_{n-1}}. \tag{3}$$

Theorem 3.1. *Assume that $\{\mu_n,\omega_n\}$ is a solution of system (3). Then for $n=1,2,\ldots,$ we have*

$$\mu_{6n}=\frac{a(abcd)^{2n}}{\{(-a-c)(-b-d)(-cb-ad)\}^{2n}},$$
$$\mu_{6n+1}=\frac{ad(abcd)^{2n}}{(-b-d)\{(-a-c)(-b-d)(-cb-ad\}^{2n}},$$
$$\mu_{6n+2}=\frac{acd(-bc-ad)(abcd)^{2n}}{\{(-a-c)(-b-d)(-cb-ad\}^{2n+1}},$$
$$\mu_{6n+3}=\frac{c(abcd)^{2n+1}}{\{(-a-c)(-b-d)(-cb-ad)\}^{2n+1}},$$
$$\mu_{6n+4}=\frac{bc(abcd)^{2n+1}}{(-b-d)\{(-a-c)(-b-d)(-cb-ad\}^{2n+1}},$$
$$\mu_{6n+5}=\frac{(-bc-ad)(abcd)^{2n+2}}{d\{(-a-c)(-b-d)(-cb-ad\}^{2n+2}},$$
$$\omega_{6n}=\frac{c(abcd)^{2n}}{\{(-a-c)(-b-d)(-cb-ad)\}^{2n}},$$

$$\omega_{6n+1} = \frac{bc(abcd)^{2n}}{(-b-d)\{(-a-c)(-b-d)(-cb-ad)\}^{2n}},$$
$$\omega_{6n+2} = \frac{(-bc-ad)(abcd)^{2n+1}}{d\{(-a-c)(-b-d)(-cb-ad)\}^{2n+1}},$$
$$\omega_{6n+3} = \frac{a(abcd)^{2n+1}}{\{(-a-c)(-b-d)(-cb-ad\}^{2n+1}},$$
$$\omega_{6n+4} = \frac{(ad)^{2n+2}(bc)^{2n+1}}{(-b-d)\{(-a-c)(-b-d)(-cb-ad\}^{2n+1}},$$
$$\omega_{6n+5} = \frac{(-bc-ad)(abcd)^{2n+2}}{b\{(-a-c)(-b-d)(-cb-ad)\}^{2n+2}},$$

where $\mu_{-1} = b,\ \mu_0 = a,\ \omega_{-1} = d$ *and* $\omega_0 = c$

Proof. The proof follows the form of the proof of the Theorem (2.1) and so will be omitted. □

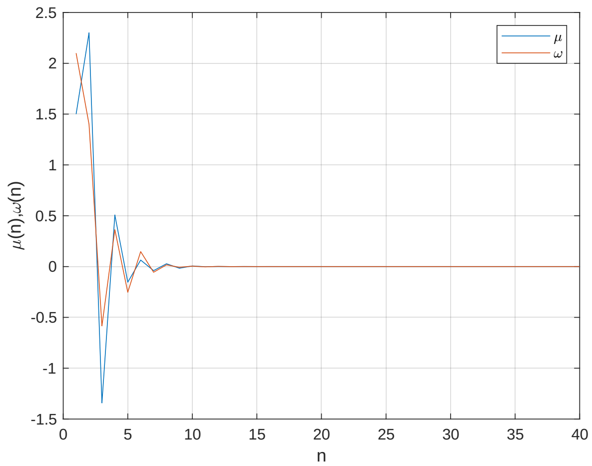

FIGURE 2. Sketch the behavior of the solution of (3)

Example 3.1 Figure 2 demonstrates the behavior of equation (3) when the initial conditions $\mu_{-1} = 2.3,\ \mu_0 = 1.5,\ \omega_{-1} = 1.4,\ \omega_0 = 3.1.$

4. THE SYSTEM: $\mu_{n+1} = \frac{\mu_n \omega_{n-1}}{\omega_{n-1}-\mu_{n-1}}$, $\omega_{n+1} = \frac{\omega_n \mu_{n-1}}{-\mu_{n-1}+\omega_{n-1}}$

In this section, we investigate the solutions of the system of two difference equations

$$\mu_{n+1} = \frac{\mu_n \omega_{n-1}}{\omega_{n-1}-\mu_{n-1}}, \quad \omega_{n+1} = \frac{\omega_n \mu_{n-1}}{-\mu_{n-1}+\omega_{n-1}}. \tag{4}$$

Theorem 4.1. *Assume that* $\{\mu_n, \omega_n\}$ *is a solution of system (4). Then for* $n = 1, 2, ...,$ *we have*

$$\mu_{6n} = \frac{a(abcd)^{2n}}{\{(c-a)(d-b)\}^{2n}\{(ad-cb)(cb-ad)\}^{n}},$$
$$\mu_{6n+1} = \frac{ad(abcd)^{2n}}{(d-b)\{(c-a)(d-b)\}^{2n}\{(ad-cb)(cb-ad)\}^{n}},$$
$$\mu_{6n+2} = \frac{acd(abcd)^{2n}}{\{(c-a)(d-b)\}^{2n+1}\{(ad-cb)(cb-ad)\}^{n}},$$
$$\mu_{6n+3} = \frac{c(abcd)^{2n+1}}{(cb-ad)\{(c-a)(d-b)\}^{2n+1}\{(ad-cb)(cb-ad)\}^{n}},$$

$$\mu_{6n+4} = \frac{bc(a-c)(abcd)^{2n+1}}{(cb-ad)\{(a-c)(b-d)\}^{2n+2}\{(ad-cb)(cb-ad)\}^{n}},$$
$$\mu_{6n+5} = \frac{acb(abcd)^{2n+1}}{(cb-ad)\{(c-a)(d-b)\}^{2n+2}\{(ad-cb)(cb-ad)\}^{n}},$$
$$\omega_{6n} = \frac{c(abcd)^{2n}}{\{(c-a)(d-b)\}^{2n}\{(ad-cb)(cb-ad)\}^{n}},$$
$$\omega_{6n+1} = \frac{cb(abcd)^{2n}}{(d-b)\{(c-a)(d-b)\}^{2n}\{(ad-cb)(cb-ad)\}^{n}},$$
$$\omega_{6n+2} = \frac{(abcd)^{2n+1}}{d\{(c-a)(d-b)\}^{2n+1}\{(ad-cb)(cb-ad)\}^{n}},$$
$$\omega_{6n+3} = \frac{a(abcd)^{2n+1}}{(cb-ad)\{(c-a)(d-b)\}^{2n+1}\{(ad-cb)(cb-ad)\}^{n}},$$
$$\omega_{6n+4} = \frac{(ad)^{2n+2}(cb)^{2n+1}}{(cb-ad)\{(a-c)(b-d)\}^{2n+2}\{(ad-cb)(cb-ad)\}^{n}},$$
$$\omega_{6n+5} = \frac{(abcd)^{2n+1}}{b(cb-ad)\{(c-a)(d-b)\}^{2n+2}\{(ad-cb)(cb-ad)\}^{n}},$$

where $\mu_{-1} = b,\ \mu_0 = a,\ \omega_{-1} = d$ *and* $\omega_0 = c$.

Proof. For $n > 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1$, as follows:

$$\mu_{6n-6} = \frac{a(abcd)^{2n-1}}{\{(c-a)(d-b)\}^{2n-1}\{(ad-cb)(cb-ad)\}^{n-1}},$$
$$\mu_{6n-5} = \frac{ad(abcd)^{2n-1}}{(d-b)\{(c-a)(d-b)\}^{2n-1}\{(ad-cb)(cb-ad)\}^{n-1}},$$
$$\mu_{6n-4} = \frac{acd(abcd)^{2n-1}}{\{(c-a)(d-b)\}^{2n}\{(ad-cb)(cb-ad)\}^{n-1}},$$
$$\mu_{6n-3} = \frac{c(abcd)^{2n}}{(cb-ad)\{(c-a)(d-b)\}^{2n}\{(ad-cb)(cb-ad)\}^{n-1}},$$
$$\mu_{6n-2} = \frac{bc(a-c)(abcd)^{2n}}{(cb-ad)\{(a-c)(b-d)\}^{2n+1}\{(ad-cb)(cb-ad)\}^{n-1}},$$
$$\mu_{6n-1} = \frac{acb(abcd)^{2n}}{(cb-ad)\{(a-c)(b-d)\}^{2n+1}\{(ad-cb)(cb-ad)\}^{n-1}},$$
$$\omega_{6n-6} = \frac{c(abcd)^{2n-1}}{\{(c-a)(d-b)\}^{2n-1}\{(ad-cb)(cb-ad)\}^{n-1}},$$
$$\omega_{6n-5} = \frac{cb(abcd)^{2n-1}}{(d-b)\{(c-a)(d-b)\}^{2n-1}\{(ad-cb)(cb-ad)\}^{n-1}},$$
$$\omega_{6n-4} = \frac{(abcd)^{2n}}{d\{(c-a)(d-b)\}^{2n}\{(ad-cb)(cb-ad)\}^{n-1}},$$
$$\omega_{6n-3} = \frac{a(abcd)^{2n}}{(cb-ad)\{(c-a)(d-b)\}^{2n}\{(ad-cb)(cb-ad)\}^{n-1}},$$
$$\omega_{6n-2} = \frac{(ad)^{2n+1}(cb)^{2n}}{(cb-ad)\{(a-c)(b-d)\}^{2n+1}\{(ad-cb)(cb-ad)\}^{n-1}},$$

$$\omega_{6n-1} = \frac{(abcd)^{2n}}{b(cb-ad)\{(c-a)(d-b)\}^{2n+1}\{(ad-cb)(cb-ad)\}^{n-1}},$$

Now it follows from equation (4) that

$$\begin{aligned}
\mu_{6n-3} &= \frac{\mu_{6n-4}\omega_{6n-5}}{\omega_{6n-5}-\mu_{6n-5}} \\
&= \frac{\left(\frac{acd(abcd)^{2n-1}}{\{(c-a)(d-b)\}^{2n}\{(ad-cb)(cb-ad)\}^{n-1}}\right)\left(\frac{cb(abcd)^{2n-1}}{(d-b)\{(c-a)(d-b)\}^{2n-1}\{(ad-cb)(cb-ad)\}^{n-1}}\right)}{\left(\frac{cb(abcd)^{2n-1}}{(d-b)\{(c-a)(d-b)\}^{2n-1}\{(ad-cb)(cb-ad)\}^{n-1}}\right)-\left(\frac{ad(abcd)^{2n-1}}{(d-b)\{(c-a)(d-b)\}^{2n-1}\{(ad-cb)(cb-ad)\}^{n-1}}\right)} \\
&= \frac{acd(abcd)^{2n-1}.cb(abcd)^{2n-1}}{\{(c-a)(d-b)\}^{2n}\{(ad-cb)(cb-ad)\}^{n-1}.(cb-ad).(abcd)^{2n-1}} \\
&= \frac{c(abcd)^{2n}}{(cb-ad)\{(c-a)(d-b)\}^{2n}\{(ad-cb)(cb-ad)\}^{n-1}},
\end{aligned}$$

and,

$$\begin{aligned}
\omega_{6n-3} &= \frac{\omega_{6n-4}\mu_{6n-5}}{-\mu_{6n-5}+\omega_{6n-5}} \\
&= \frac{\left(\frac{(abcd)^{2n}}{d\{(c-a)(d-b)\}^{2n}\{(ad-cb)(cb-ad)\}^{n-1}}\right)\left(\frac{ad(abcd)^{2n-1}}{(d-b)\{(c-a)(d-b)\}^{2n-1}\{(ad-cb)(cb-ad)\}^{n-1}}\right)}{\left(\frac{-ad(abcd)^{2n-1}}{(d-b)\{(c-a)(d-b)\}^{2n-1}\{(ad-cb)(cb-ad)\}^{n-1}}\right)+\left(\frac{cb(abcd)^{2n-1}}{(d-b)\{(c-a)(d-b)\}^{2n-1}\{(ad-cb)(cb-ad)\}^{n-1}}\right)} \\
&= \frac{(abcd)^{2n}.(ad).(abcd)^{2n-1}}{d.\{(c-a)(d-b)\}^{2n}\{(ad-cb)(cb-ad)\}^{n-1}(-ad+cb).(abcd)^{2n-1}} \\
&= \frac{a(abcd)^{2n}}{(cb-ad)\{(c-a)(d-b)\}^{2n}\{(ad-cb)(cb-ad)\}^{n-1}}.
\end{aligned}$$

We can also prove the other relation. The proof is complete. □

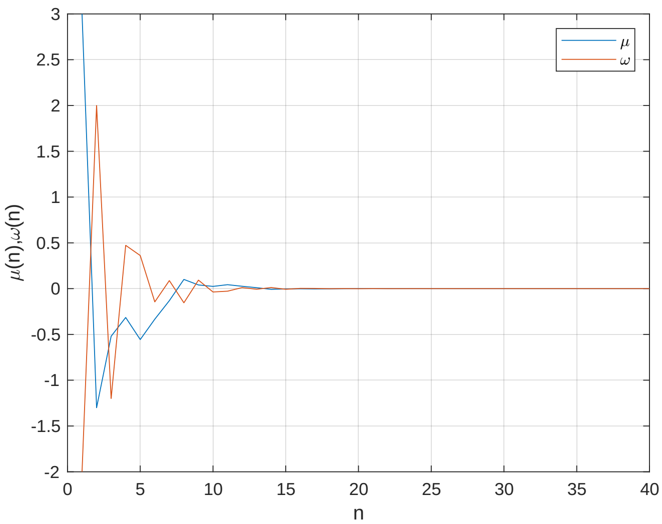

FIGURE 3. Sketch the behavior of the solution of (4)

Example 4.1 In Figure 3 shows the behavior of the solutions of equation (4) where $\mu_{-1} = -1.3,\ \mu_0 = 3,\ \omega_{-1} = 2,\ \omega_0 = -2.$

5. The System: $\mu_{n+1}=\frac{\mu_n\omega_{n-1}}{\omega_{n-1}-\mu_{n-1}}, \quad \omega_{n+1}=\frac{\omega_n\mu_{n-1}}{\mu_{n-1}-\omega_{n-1}}$

In this section, we study the solutions to the following system of two difference equations:

$$\mu_{n+1}=\frac{\mu_n\omega_{n-1}}{\omega_{n-1}-\mu_{n-1}}, \quad \omega_{n+1}=\frac{\omega_n\mu_{n-1}}{\mu_{n-1}-\omega_{n-1}}. \tag{5}$$

Theorem 5.1. *Assume that $\{\mu_n,\omega_n\}$ is a solution of system (2.4). Then for $n=1,2,...$, we have*

$$\begin{aligned}
\mu_{6n}=&\frac{a(abcd)^{2n}}{\{(c-a)(d-b)(ad+cb)\}^{2n}},\\
\mu_{6n+1}=&\frac{ad(abcd)^{2n}}{(d-b)\{(c-a)(d-b)(ad+cb)\}^{2n}},\\
\mu_{6n+2}=&\frac{acd(ad+cb)(abcd)^{2n}}{\{(c-a)(d-b)(ad+cb)\}^{2n+1}},\\
\mu_{6n+3}=&\frac{c(abcd)^{2n+1}}{\{(c-a)(d-b)(ad+cb)\}^{2n+1}},\\
\mu_{6n+4}=&\frac{cb(abcd)^{2n+1}}{(d-b)\{(a-c)(d-b)(ad+cb)\}^{2n+1}},\\
\mu_{6n+5}=&\frac{acb(ad+cb)(abcd)^{2n+1}}{\{(a-c)(d-b)(ad+cb)\}^{2n+2}},\\
\omega_{6n}=&\frac{c(abcd)^{2n}}{\{(c-a)(d-b)(ad+cb)\}^{2n}},\\
\omega_{6n+1}=&\frac{cb(abcd)^{2n}}{(b-d)\{(c-a)(b-d)(ad+cb)\}^{2n}},\\
\omega_{6n+2}=&\frac{acb(ad+cb)(adcb)^{2n}}{\{(c-a)(d-b)(ad+cb)\}^{2n+1}},\\
\omega_{6n+3}=&\frac{a(abcd)^{2n+1}}{\{(c-a)(d-b)(ad+cb)\}^{2n+1}},\\
\omega_{6n+4}=&\frac{(ad)^{2n+2}(cb)^{2n+1}}{(d-b)\{(c-a)(d-b)(ad+cb)\}^{2n+1}},\\
\omega_{6n+5}=&\frac{adc(ad+cb)(abcd)^{2n+1}}{\{(a-c)(d-b)(ad+cb)\}^{2n+2}},
\end{aligned}$$

where $\mu_{-1}=b,\ \mu_0=a,\ \omega_{-1}=d$ and $\omega_0=c$.

Proof. For $n>0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$, as follows

$$\begin{aligned}
\mu_{6n-6}=&\frac{a(abcd)^{2n-1}}{\{(c-a)(d-b)(ad+cb)\}^{2n-1}},\\
\mu_{6n-5}=&\frac{ad(abcd)^{2n-1}}{(d-b)\{(c-a)(d-b)(ad+cb)\}^{2n-1}},\\
\mu_{6n-4}=&\frac{acd(ad+cb)(abcd)^{2n-1}}{\{(c-a)(d-b)(ad+cb)\}^{2n}},\\
\mu_{6n-3}=&\frac{c(abcd)^{2n}}{\{(c-a)(d-b)(ad+cb)\}^{2n}},
\end{aligned}$$

$$\mu_{6n-2} = \frac{cb(abcd)^{2n}}{(d-b)\{(a-c)(d-b)(ad+cb)\}^{2n}},$$
$$\mu_{6n-1} = \frac{acb(ad+cb)(abcd)^{2n}}{\{(a-c)(d-b)(ad+cb)\}^{2n+1}},$$
$$\omega_{6n-6} = \frac{c(abcd)^{2n-1}}{\{(c-a)(d-b)(ad+cb)\}^{2n-1}},$$
$$\omega_{6n-5} = \frac{cb(abcd)^{2n-1}}{(b-d)\{(c-a)(b-d)(ad+cb)\}^{2n-1}},$$
$$\omega_{6n-4} = \frac{acb(ad+cb)(adcb)^{2n-1}}{\{(c-a)(d-b)(ad+cb)\}^{2n}},$$
$$\omega_{6n-3} = \frac{a(abcd)^{2n}}{\{(c-a)(d-b)(ad+cb)\}^{2n}},$$
$$\omega_{6n-2} = \frac{(ad)^{2n+2}(cb)^{2n}}{(d-b)\{(c-a)(d-b)(ad+cb)\}^{2n}},$$
$$\omega_{6n-1} = \frac{adc(ad+cb)(abcd)^{2n}}{\{(a-c)(d-b)(ad+cb)\}^{2n+1}},$$

Now, from Equation (5) that

$$\begin{aligned}
\mu_{6n+5} &= \frac{\mu_{6n+4}\omega_{6n+3}}{\omega_{6n+3}-\mu_{6n+3}} \\
&= \frac{\left(\frac{cb(abcd)^{2n+1}}{(d-b)\{(a-c)(d-b)(ad+cb)\}^{2n+1}}\right)\left(\frac{a(abcd)^{2n+1}}{\{(c-a)(d-b)(ad+cb)\}^{2n+1}}\right)}{\left(\frac{a(abcd)^{2n+1}}{\{(c-a)(d-b)(ad+cb)\}^{2n+1}}\right)-\left(\frac{c(abcd)^{2n+1}}{\{(c-a)(d-b)(ad+cb)\}^{2n+1}}\right)} \\
&= \frac{acb(abcd)^{2n+1}.\{(c-a)(d-b)(ad+cb)\}^{2n+1}}{(d-b)\{(a-c)(d-b)(ad+cb\}^{2n+1}.(a-c)\{(c-a)(d-b)(ad+cb\}^{2n+1}} \\
&= \frac{acb(abcd)^{2n+1}.(ad+cb)}{\{(a-c)(d-b)(ad+cb)\}^{2n+2}},
\end{aligned}$$

and,

$$\begin{aligned}
\omega_{6n+5} &= \frac{\omega_{6n+4}\mu_{6n+3}}{\mu_{6n+3}-\omega_{6n+3}} \\
&= \frac{\left(\frac{(ad)^{2n+2}(cb)^{2n+1}}{(d-b)\{(c-a)(d-b)(ad+cb)\}^{2n+1}}\right)\left(\frac{c(abcd)^{2n+1}}{\{(c-a)(d-b)(ad+cb)\}^{2n+1}}\right)}{\left(\frac{c(abcd)^{2n+1}}{\{(c-a)(d-b)(ad+cb)\}^{2n+1}}\right)-\left(\frac{a(abcd)^{2n+1}}{\{(c-a)(d-b)(ad+cb)\}^{2n+1}}\right)} \\
&= \frac{(ab)^{2n+2}.(cb)^{2n+1}.c(abcd)^{2n+1}}{(d-b)\{(d-b)(c-a)(ad+cb)\}^{2n+1}.(c-a)(abcd)^{2n+1}} \\
&= \frac{acd(abcb)^{2n+1}.(ad+cb)}{\{(d-b)(c-a)(ad+cb)\}^{2n+2}}.
\end{aligned}$$

Hence, we can easily proof the other relations. The proof has been done. □

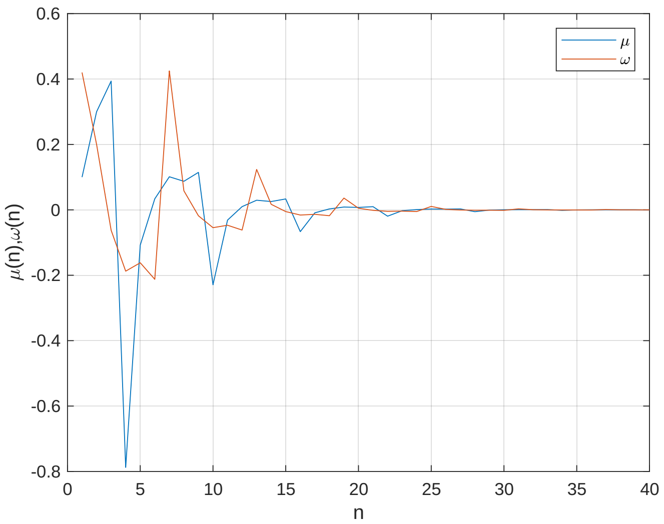

FIGURE 4. Sketch the behavior of the solution of (5)

Example 5.1 In Figure 4, we assume a numerical example for equation (5) where $\mu_{-1} = 0.3$, $\mu_0 = 0.1$, $\omega_{-1} = 0.2$, $\omega_0 = 0.42$.

6. Conclusion

Scientists have shown great interest in trying to understand systems of difference equations. It is known that finding the system of difference equations that can be solved is only one of the challenges. Our interest in this work centered on obtaining the formulas for the solutions of the system (1). We have tried to formulate the solutions to the valid cases of the system (1). We provided some numerical examples for each case with different initial values to show numerical simulations of the system solutions.

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