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# NORM ESTIMATES FOR OPERATORS IN NORM-ATTAINABLE C*-ALGEBRAS

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Abstract. Norm estimates for various types of Banach algebra operators have been studied over decades with interesting results obtained. However, it still remains an open problem to determine the norm of an operator in a general Banach space setting. In this note, we consider norm attaining operators in C*-algebras and establish their lower bound and upper bound norm estimates.

## 1. Introduction

The theory of norm-attaining operators appeared in the second half of the twentieth century (see [6], [7]- [11] and the references therein). It started with the classical Bishop-Phelps Theorem [1] which established that the set of norm attaining functionals for a Banach space is dense in the dual space. A question on whether there could be an extension of this result for operators was raised. Lindenstrauss [5] endeavored to answer this question by conducting a study which resulted in a first counter example in addition to obtaining other positive results. Other several researchers have also tackled the notion of norm-attainability with interesting results obtained (see [2] and [4]). In this paper we characterize norms of norm attaining operators in $C^*$-algebras and establish these norms via the lower bound and upper bound norm estimates.

## 2. Preliminaries

We provide some useful preliminary concepts which are useful in the sequel in this section.

**Definition 2.1.** *( [15]) A nonnegative function $\|.\| : V \to \mathbb{R}$ with real values is a norm if the following conditions are satisfied:*

(i). $\|x\| \geq 0$ *and* $\|x\| = 0$ *iff* $x = 0 \ \ \forall x \in V$
(ii). $\|x + y\| \leq \|x\| + \|y\| \ \ \forall x, y \in V$
(iii). $\|\lambda x\| = |\lambda|\|x\| \ \ \forall \lambda \in \mathbb{C}$ *and* $x \in V$.

*The ordered pair $(V, \|.\|)$ is then known as a normed space. For details on normed spaces see [2], [12], [13] and [14].*

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**Definition 2.2.** *( [3]) A $C^*$-Algebra is a Banach algebra $A$ having an involution $*$ (that is a conjugate linear map of $A$ onto itself satisfying $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for $x, y \in A$) which satisfies $\|x^*x\| = \|x\|^2$ for all $x \in A$.*

**Definition 2.3.** *( [8]) Let $H$ be a complex Hilbert space and $B(H)$ be the $C^*$-algebra of all bounded linear operators on $H$. An operator $A \in B(H)$ is said to be norm-attaining if there exists $x \in H$ with $\|x\| = 1$ such that $\|Ax\| = \|A\|$.*

**Remark 2.4.** *For details on norm-attainability criterion and characterization of norm-attainable operators, see [9] and [10].*

## 3. Main Results

We characterize norms of norm attaining operators in $C^*$-algebras. The set of all norm attaining operators is denoted by $NA(H)$. We note that $NA(H)$ is a norm-attainable $C^*$-algebra. We begin with the following result.

**Theorem 3.1.** *Let $S_o \in NA(H)$ and $S_o = S_o^*$. Then $\mu \in \mathbb{R}$ exists in order for $S^*(\varphi_o(S_o g)) = \mu\varphi_o(g)$ and $\|S_o\| = \mu$.*

*Proof.* Let $\Gamma_{o_1}, \Gamma_{o_2} \in [NA(H)]^*$ be defined as $\Gamma_{o_1}(f_o) = \langle f_o, \varphi_{o_1}(g)\rangle$,
$\Gamma_{o_2}(f_o) = \frac{1}{\|S_o\|}\langle S_o f_o, \varphi_{o_2}(g)\rangle = \frac{1}{\|S_o\|}\langle f_o, S^*(\varphi_{o_2}(S_o g))\rangle$. Then $\|\Gamma_{o_1}\| = 1$ (since $\|\varphi_{o_1}(g)\| = 1$) and $\Gamma_{o_1}(g) = \|g\|$, so $\Gamma_{o_1}$ norms $g$. Similarly $\|\varphi_{o_2}(S_o g)\| = 1$ implies that $\|\Gamma_{o_2}\| \leq 1$, but using $\|S_o g\| = \|S_o\|\|g\|$ we have $\Gamma_{o_2}(g) = \|g\|$ which means that $g$ is normed by $\Gamma_{o_2}$. Now, since $(H)$ is smooth, then $\Gamma_{o_1} = \Gamma_{o_2}$. Therefore the result holds with $\mu = \|S_o\|$ as required. □

**Corollary 3.2.** *Let $NA(H)_1 = L^q(X, \mu)$, $NA(H)_2 = L^p(Y, W)$ where $q, p \in (1, \infty)$ with $h$ being a solution in Proposition 3.1 . Let $S \in NA(H)$ in order for $S : NA(H)_1 \to NA(H)_2$. Then $S$ has $h$ as its critical point and $\beta = \alpha^q\|g\|_p^{q-p}$. Moreover $\|S\| = \alpha^q$.*

*Proof.* We first note that if $h$ satisfies theorem 3.1, then we have
$\|Sh\|_p = \langle Sh, \varphi_{o_2}(Sh)\rangle = \langle h, S^*\varphi_{o_2}(Sh)\rangle = \langle h, \lambda\varphi_{o_1}(h)\rangle = \lambda\|h\|_q$. Substituting $\varphi_{L^q}(h) = \|h\|_p^{-(q-1)}\text{sgn}(h)|h|^{q-1}$ into
$S^*(\varphi_{o_2}(Sh)) = \lambda\varphi_{o_1}(h)$ and multiplication by $\|Sh\|_p^{p-1}$ gives
$S^*((Sh)|Sh|^{q-1}) = \lambda\|Sh\|_p^{p-1}\|h\|_p^{-(q-1}(h)|h|^{q-1} = \lambda^p\|h\|_q^{p-q}(h)|h|^{q-1}$ as required. □

**Proposition 3.3.** *Let $S \in NA(H)$ be bounded and $\|S\| < 1$. Then $(I - S)$ is bounded with an inverse equal to $\sum\limits_{k=0}^{\infty} S^k$ i.e $\lim_{n\to\infty} \sum\limits_{k=0}^{n} S^k$ is norm convergent to $(I - S)^{-1}$.*

*Proof.* Given $\|S\| < 1$ then $\sum\limits_{k=0}^{n}\|S\|^k$ is a geometric series which is convergent and $\sum\limits_{k=0}^{n}\|S^k\| < \sum\limits_{k=0}^{n}\|S\|^k$ then implies the sequence $\sum\limits_{k=0}^{n}\|S^k\|$ is norm cauchy and therefore converges to a bounded operator as the range of $S$ is a Banach space. Now as a power series, $(I - S)\sum\limits_{k=0}^{\infty} S^k = \sum\limits_{k=0}^{\infty}(S^k - S^{k+1}) = 1$ since it is norm convergent. Likewise, $(\sum\limits_{k=0}^{\infty} S^k)(1 - S) = \sum\limits_{k=0}^{\infty}(S^k - S^{k+1}) = 1$ which shows that the series is equal to $(1 - S)^{-1}$ hence completing the proof. □

**Proposition 3.4.** *Let $S \in M_n$ be norm attaining and $\psi : M_n \to \mathbb{C}^n$ be a contractive linear mapping so that $\psi(S) = \|S\|$. Then $\exists$ a vector $\xi \in \mathbb{C}^n$ where $\|\xi\| = 1$ and $\|S\xi\| = \|S\|$ with $\langle S\xi, \xi\rangle = \overline{\varphi(I)}\|S\|$.*

*Proof.* Let $S \neq 0$ and $\|S\| = 1$ (or else $S$ can be replaced by $\frac{S}{\|S\|}$). By Theorem ??, a matrix $C \in M_n$ exists so that we have $\psi(Q) = trace\ (CQ)\ \forall Q \in M_n$. Letting $C = P_oV$ where $P_o \in M_n^+$ and $V$ is unitary, $trace\ Q = \psi(V^*) \leq \|V^* = 1 = \psi(S)\| = trace\ (P_oVS)$. Let $Q = V^*P_o^{\frac{1}{2}}$ and $A = SP_o^{\frac{1}{2}}$. Then $trace\ (Q^*A) = trace\ (P_oVS) = 1$, $trace\ (Q^*Q) = trace\ (P_o) \leq 1$, $trace\ (A^*A) = trace\ (P_o^{\frac{1}{2}}S^*SP_o^{\frac{1}{2}}) \leq trace\ (P_o) \leq 1$ and therefore $trace\ (Q-A)^*(G-A) \leq 0$. Hence $Q = A$, $trace\ (P_o) = trace\ (Q^*G) = trace\ (Q^*A) = 1$ and $trace\ (SP_o) = trace\ (V^*P_o) = \overline{\phi(I)}$. Since $P_o = P_o^{\frac{1}{2}}Q^*QP_o^{\frac{1}{2}} = P_o^{\frac{1}{2}}Y^*YP_o^{\frac{1}{2}} = P_o^{\frac{1}{2}}S^*SP_o^{\frac{1}{2}}$, then it means range $P_o$ is an $m$-dimensional linear subspace $u \in \mathbb{C}^n : \|Su\| = \|u\|$ with $1 \leq m \leq n$. If $B$ is a matrix whose order is $n \times m$ so that $B^*B = I_k$ where $BB^*$ is a projection onto range $P$, we have $P_oBB^* = P_o = BB^*P_o$ where $B^*P_oB$ is a matrix belonging to a compact convex set $\mathcal{D} = \{R \in M_k^+ : trace\ R = 1, trace\ (B^*SBR) = \overline{\psi}\}$ found through the intersection between three real hyperplanes and the $k \times k$ hermitian matrices, $M_k^+$. $\mathcal{D}$ contains a rank-1 matrix $xx^*$ where $x \in \mathbb{C}^k$ with the properties $\langle B^*SBx, x\rangle = \overline{\psi(I)}$ and $\|x\|^2 = 1 = tr\ (xx^*)$. If we take $y = Bx$, all the required conditions for $y$ are obtained. ☐

**Proposition 3.5.** *Let $S_1, S_2 \in NA(H)$ be nonzero $k \times k$ matrices. Then the statements which follow are similar.*

(i). *$\exists$ unit vectors $x, \xi \in \mathbb{C}^k$ such that $\|S_1x\| = \|S_1\|$, $\|S_2\xi\| = \|S_2\|$ and $\frac{\langle S_1x, x\rangle}{\|S_1\|} = \frac{\langle S_2\xi, \xi\rangle}{\|S_2\|}$.*

(ii). *$\exists$ unit vectors $x, y \in \mathbb{C}^n$ such that $\|S_1x\| = \|S_1\|$, $\|S_2\xi\| = \|S_2\|$ and $\|S_1x\| + \|S_2\xi\| \leq \|(S_1 + \mu I)x\| + \|(S_2 - \mu I)\xi\|\ \forall \mu \in \mathbb{C}$.*

(iii). $\|S_1\| + \|S_2\| \leq \|S_1 + \mu I\| + \|S_2 - \mu I\|\ \forall \mu \in \mathbb{C}$

*Proof.* $(i) \Rightarrow (ii)$:Suppose $x$ and $\xi$ satisfy condition $(i)$. Then

$$\begin{aligned}
\|S_1x\| + \|S_1\xi\| &= \left\|\begin{pmatrix} \langle S_1x, x\rangle \\ \{\|S_1x\|^2 - |\langle S_1x, x\rangle|^2\}^{\frac{1}{2}} \end{pmatrix}\right\| + \left\|\begin{pmatrix} \langle S_2\xi, \xi\rangle \\ \{\|S_2y\|^2 - |\langle S_2\xi, \xi\rangle|^2\}^{\frac{1}{2}} \end{pmatrix}\right\| \\
&= \| \begin{pmatrix} \langle (S_1 + \mu I)x, x\rangle \\ \{\|(S_1 + \mu I)x\|^2 - |\langle (S_1 + \mu I)x, x\rangle|^2\}^{\frac{1}{2}} \end{pmatrix} \\
&\quad + \begin{pmatrix} \langle (S_2 - \mu I)\xi, \xi\rangle \\ \{\|(S_2 - \mu I)\xi\|^2 - |\langle (S_2 - \mu I)\xi, \xi\rangle|^2\}^{\frac{1}{2}} \end{pmatrix} \| \\
&\leq \left\|\begin{pmatrix} \langle (S_1 + \mu I)x, x\rangle \\ \{\|(S_1 + \mu I)x\|^2 - |\langle (S_1 + \mu I)x, x\rangle|^2\}^{\frac{1}{2}} \end{pmatrix}\right\| \\
&\quad + \left\|\begin{pmatrix} \langle (S_2 - \mu I)\xi, \xi\rangle \\ \{\|(S_2 - \mu I)\xi\|^2 - |\langle (S_2 - \mu I)\xi, \xi\rangle|^2\}^{\frac{1}{2}} \end{pmatrix}\right\| \\
&= \|(S_1 + \mu I)x\| + \|(S_2 - \mu I)\xi\|
\end{aligned}$$

$(ii) \Rightarrow (iii)$ as the vectors $x$ and $\xi$ are unit. $(iii) \Rightarrow (i)$:Considering $(M_k \times M_k, u)$ which is a linear normed space with $u(X_1, X_2) = \|X_1\| + \|X_2\|$, we have a contractive linear functional $\psi$ with respect to $u$ on span $\{(S_1, S_2), (I, -I)\}$ defined by $\psi(S_1, S_2) = \|S_1\| + \|S_2\|$ and $g(I, -I) = 0$ if and only if $(iii)$ holds. Now $\psi$ is extended to a contractive linear functional $\Psi$ on $M_k \times M_k$ by Hahn Banach Theorem to get $\|S_1\| + \|S_2\| = \Psi(S_1, S_2) \leq |\Psi(S_1, 0)| + |\Psi(0, S_2)| \leq \|S_1\| + \|S_2\|$ which gives $\Psi(S_1, 0) = \|S_1\|$ and $(0, S_2) = \|S_2\|$. By Proposition 3.4 and given that $X_1 \mapsto \Psi(X_1, 0)$ is contractive we have a unit vector $x \in \mathbb{C}^n$ giving $\|S_1x\| = \|S_1\|$ and $\frac{\langle Sx, x\rangle}{\|S_1\|} = \overline{\Psi(I, 0)}$.

Likewise, we have a complex unit vector $\varsigma$ such that $\|S_2\varsigma\| = \|S_2\|$ and $\frac{\langle S_2\varsigma,\varsigma\rangle}{\|S_2\|} = \overline{F(0,I)}$. Finally, since $\Psi(I,-I) = \Psi(I,0) - \Psi(0,I)$ then $F(I,0) = F(0,I) = 0$ and therefore $(i)$ is true. □

**Lemma 3.6.** *Let $S_1, S_2 \in NA(H)$. Then $\sup\{\|U^*S_1U + V^*S_2V\|\} = min\{\|S_1 + \mu I\| + \|S_2 - \mu I\|\}$ where $U$,$V$ are unitaries and $\mu \in \mathbb{C}$. Additionally the above equality is equivalent to $\sup\{\|S_1X + S_2X\| : X \in NA(H), \|X\| \leq 1\}$ and further equivalent to $\sup\{\|s + t\| : s \in \Gamma(S_1), t \in \Gamma(S_2)\}$.*

*Proof.* Let $\beta \in \mathbb{C}$ and $a \in [0,\infty)$. Suppose $x, x' \in H$ are unit vectors where $\langle x, x'\rangle = 0$ giving $S_1x = \beta + ax'$ with $x$ uniquely determining $x'$ for all $a \neq 0$. Then
$\begin{pmatrix} \beta \\ a \end{pmatrix} = \begin{pmatrix} \langle S_1x, x\rangle \\ \{\|S_1x\|^2 - |\langle S_1x, x\rangle|^2\}^{\frac{1}{2}} \end{pmatrix}$ is a vector in $\mathbb{C} \times \mathbb{R}$ whose length is $\|S_1x\|$. Let

(1)

$$\Phi(S_1) = \begin{pmatrix} \langle S_1x, x\rangle \\ \{\|S_1x\|^2 - |\langle S_1x, x\rangle|^2\}^{\frac{1}{2}} \end{pmatrix} \subseteq \mathbb{C} \times [0,\infty) \tag{2}$$

It should be noted that $\Gamma(S_1 + \mu I) = \{u + \begin{pmatrix} \mu \\ 0 \end{pmatrix} : u \in \Gamma(S_1)\}$ since

$$\{\|S_1x\|^2 - |\langle S_1x, x\rangle|^2\}^{\frac{1}{2}} = \{\|(S_1 + \mu I)x\| - |\langle (S_1 + \mu I)x, x\rangle|^2\}^{\frac{1}{2}}.$$

Hence $\Gamma(S_1) + \Gamma(S_2) = \Gamma(S_1 + \mu I) + \Gamma(S_2 - \mu I)$ which therefore means
$\sup\{\|x + x'\| : u \in \Gamma(S_1), v \in \Gamma(S_2)\}$
$= \sup\{\|x + x\prime\| : x \in \Gamma(S_1 + \mu I), x\prime \in \Gamma(S_2 - \mu I)\} \forall \mu \in \mathbb{C}$. Now taking $e_1, e_2 \in H$ be unit vectors such that $\langle e_1, e_2\rangle = 0$ and letting $x \in \Gamma(S_1)$, $x' \in \Gamma(S_2)$, then we have $U, V \in NA(H)$ as unitary operators to give $x = \begin{pmatrix} \langle U^*S_1Ue_1, e_1\rangle \\ \langle U^*S_1Ue_1, e_2\rangle \end{pmatrix}$ and $x' = \begin{pmatrix} \langle V^*S_2Ve_1, e_1\rangle \\ \langle V^*S_2Ve_1, e_2\rangle \end{pmatrix}$. This gives

$$\|x + x'\| = \|(U^*S_1U + V^*S_2V)e_1\| \leq \|U^*S_1U + V^*S_2V\|$$

so that
$\sup\{\|x + x'\| : x \in \Gamma(S_1), x' \in \Gamma(S_2)\} \leq \sup\{\|U^*S_1U\| + \|V^*S_2V\|\}$. It is clear that if $A \in NA(H)$ is a contraction with $\mu \in \mathbb{C}$, then

$$\begin{aligned} \|S_1A + AS_2\| &\leq \|(S_1 + \mu I)A\| + \|A(S_2 - \mu I)\| \\ &\leq \|S_1 + \mu I\| + \|S_2 - \mu I\| \end{aligned}$$

and therefore

$$\begin{aligned} \sup\{\|U^*S_1U + V^*S_2V\|\} &= \sup\{\|S_1UV^* + UV^*S_2\|\} \\ &\leq \sup\{\|S_1A + AS_2\| : \|X\| \leq 1\} \\ &\leq \min\{\|S_1 + \mu I\| + \|S_2 - \mu I\| : \mu \in \mathbb{C}\} \end{aligned}$$

It is now sufficient to show that,

$$\min\{\|S_1 + \mu I\| + \|S_2 - \mu I\|\} \leq \sup\{\|u + v\| : u \in \Gamma(S_1), v \in \Gamma(S_2)\} \tag{3}$$

If $S_1$ or $S_2$ is a scalar operator, the result holds. Suppose none of the two is a scalar. We start with the finite-dimensional case by letting $\|S_1 + \mu_0 I\| + \|S_2 - \mu_0 I\|\| \leq \|S_1 + \mu I\| + \|S_2 - \mu I\| \forall \mu \in \mathbb{C}$. Since $\Gamma(S_1) + \Phi(S_2) = \Gamma(S_1 + \mu I) + \Gamma(S_2 - \mu I)$, we may take $\mu_0 = 0$ to simplify the work.

By Proposition 3.5, we have unit vectors $\xi, \zeta \in \mathbb{C}^n$ such that $\|S_1\xi\| = \|S_1\|, \|S_2\zeta\| = \|\zeta\|$ and $\frac{\langle S_1 x, x\rangle}{\|S_1\|} = \frac{\langle T\zeta,\zeta\rangle}{\|S_2\|}$. Let
$x = \begin{pmatrix} \langle S_1\xi,\xi\rangle \\ \{\|S_1\xi\|^2 - |\langle S_1\xi,\xi\rangle|^2\}^{\frac{1}{2}} \end{pmatrix} \in \Gamma(S_1)$, $x^{'} = \begin{pmatrix} \langle S_2\zeta,\zeta\rangle \\ \{\|S_2\zeta\|^2 - |\langle S_2\zeta,\zeta\rangle|^2\} \end{pmatrix} \in \Gamma(S_2)$ so that we have $\|x + x^{'}\| = \|x\| + \|x^{'}\| = \|S_1\xi\| + \|S_2\zeta\| = \|S_1\| + \|S_2\|$ as required.
We next look at the infinite-dimensional case. Assume inequality (3) is false. Then a real number $\varepsilon > 0$ exists which gives
$\sup\{\|x + x^{'}\| : x \in \Gamma(S_1), x^{'} \in \Phi(S_2)\} < \|S_1 + \mu I\| + \|S_2 - \mu I\| - \varepsilon$ for all complex numbers $\mu$. Therefore infinitely many complex numbers $\alpha_1, ..., \alpha_m$ can be found to give $\{\mu \in \mathbb{C} : |\mu| \leq \|S_1\| + \|S_2\|\} \subseteq \bigcup_{i=1}^{m}\{\alpha \in \mathbb{C} : |\alpha - \alpha_i| < \frac{\varepsilon}{4}\}$. Choosing unit vectors $\varsigma_1, ..., \varsigma_m$ and $\eta_1, ..., \eta_m$ in $H$ we have $\|(S_1 + \alpha_i I)\varsigma_i\| > \|S_1 + \alpha_i I\| - \frac{\varepsilon}{4}$ and $\|(S_2 - \alpha_i I)\eta_i\| > \|S_2 - \alpha_i I\| - \frac{\varepsilon}{4}$ for all $i = 1, ...m$. Letting $H_o$ be the finite-dimensional subspace of $H$ whose spanning vectors are $\xi_1, ..., \xi_m$, $\eta_1, ..., \eta_m$ and $S_1\xi_1, ..., S_1\xi_m$, $S_2\eta_1, ..., S_2\eta_m$ while $S_1', S_2'$ are compressions of $S_1, S_2$ respectively with $I'$ also being a compression of $I$ on $H_o$, we get

$$\min\{\|S_1 + \mu I^{'}\| + \|S_2 - \mu I^{'}\|\}$$
$$= \sup\{\|x + x^{'}\| : x \in \Gamma(S_1), x^{'} \in \Gamma(S_2)\} \tag{4}$$
$$\leq \sup\{\|x + x^{'}\| : x \in \Gamma(S_1), x^{'} \in \Gamma(S_2)\} \tag{5}$$

by applying the finite dimensional case. Moreover, for each $\mu \in \mathbb{C}$ with $|\mu| \leq \|S_1\| + \|S_2\|$, there exists $i$ so that $|\mu - \alpha_i| < \frac{\varepsilon}{4}$ and hence

$$\begin{aligned} \|S_1' + \mu I'\| &> \|S_1' + \alpha_i I'\| - \frac{\varepsilon}{4} \\ &\geq \|(S_1' + \alpha_i I')x_i\| - \frac{\varepsilon}{4} \\ &= \|(S_1 + \alpha_i I)x_i\| - \frac{\varepsilon}{4} \\ &> \|S_1 + \alpha_i I\| - \frac{\varepsilon}{2} \end{aligned}$$

Similarly, $\|S_2' - \mu I'\| > \|S_2 - \alpha_i I\| - \frac{\varepsilon}{2}$ so that
$\|S_1' + \mu I'\| + \|S_2' - \mu I'\| > \sup\{\|x + x^{'}\| : x \in \Gamma(S_1), x^{'} \in \Gamma(S_2)\}$. Likewise when $|\mu| > \|S_1\| + \|S_2\|$ we get

$$\begin{aligned} \|S_1' + \mu I'\| + \|S_2' - \mu I'\| &\geq \|2\mu I'\| - \|S_1'\| + \|S_2'\| \\ &> \|S_1\| + \|S_2\| \\ &> \sup\{\|x + x^{'}\| : x \in \Gamma(S_1), x^{'} \in \Gamma(S_2)\}. \end{aligned}$$

Hence this contradicts inequality (5) which therefore means that inequality (3) is true. □

**Theorem 3.7.** *Let $S, T \in NA(H)$ and $\mu_0 \in \mathbb{C}$. Then $\|S - \mu_0 I\| + \|T - \mu_0 I\| \leq \|S - \mu I\| + \| - \mu IT\| \forall \mu \in \mathbb{C}$. Moreover, if $\delta_1 = \|S - \mu_0 I\|$ and $\delta_2 = \|T - \mu_0 I\|$, then*

$$\sup\{\|S - U^*TU\| : U \text{ unitary}\} = \delta_1 + \delta_2 \tag{6}$$

$$\|g(S) + U^*h(T)U\| \leq \max_{z \in \Gamma(\mu_0;\delta_1)} |g(z)| + \max_{z \in \Gamma(\mu_0;\delta_2)} |h(z)| \tag{7}$$

*for every $U$ and every pair $g(t)$ and $h(t)$ of polynomials.*

*Proof.* Suppose $S, T \in NA(H)$ and that the hypotheses are satisfied by $\delta_1, \delta_2, \mu_o$. Then by Proposition 3.6 the pair $(S, -T)$ gives

$$\begin{aligned}\sup\{\|S - U^*TU\| : U\text{unitary}\} &= \|S - \mu_0 I\| + \|T - \mu_0 I\| \\ &= \delta_1 + \delta_2\end{aligned}$$

as claimed. Using the Von Neumann inequality gives

$$\begin{aligned}\|g(S) + U^*h(T)U\| &\leq g(S) + \|h(T)\| \\ &\leq \max_{z_0 \in \Gamma(\mu_0;\delta_1)} |g(z_0)| + \max_{z_0 \in \Gamma(\mu_0;\delta_2)} |h(z_0)|\end{aligned}$$

□

**Proposition 3.8.** *Let $S_o, S_1 \in NA(H)$ and $E(S_o, S_1)$ set of complex numbers $\beta_0$ satisfying $\|S_o - \beta_0 I\| + \|S_1 - \beta_0 I\| \leq \|S_o - \beta I\| + \|S_1 - \beta I\|\ \forall \beta \in \mathbb{C}$, then $E(S_o, S_1)$ is either a closed line segment or a single point.*

*Proof.* Being a set of complex numbers, $(S_o, S_1)$ is clearly a compact set. We now prove the convexity of $E(S_o, S_1)$. Letting $\beta_1, \beta_2 \in E(S_o, S_1)$ and $\beta_0 = t\beta_1 + (1-t)\beta_2$ with $t \in (0, 1)$ gives $\|S_o - \beta_0 I\| + \|S_1 - \beta_0 I\| \leq t\{\|S_o - \beta_1\| + \|S_1 - \beta_1 I\|\} + (1-t)\{\|S_o - \beta_2 I\| + \|S_1 - \beta_2 I\|\} \leq \|S_o - \beta I\| + \|S_1 - \beta I\|\ \forall \beta \in \mathbb{C}$. Therefore $\beta_0 \in E(S_o, S_1)$. Suppose that $E(S_o, S_1)$ does not include any disk $D_o(\beta_0; \delta) = \{\beta \in \mathbb{C} : |\beta - \beta_0| \leq \delta\}$ with $\delta > 0$. Incase $D_o(\beta_0; \delta) \subseteq E(S_o, S_1)$, we further assume that $\beta_0 = 0$ with $(S_o, S_1)$ in the place of $(S_o - \beta_0 I, S_1 - \beta_0 I)$. Hence

$$\|S_o\| + \|S_1\| = \|S_o - \beta I\| + \|S_1 - \beta I\|\ \forall \beta \in D_o(0; \delta) \tag{8}$$

Since $D_o(0; \delta) \setminus \{0\}$ is a set which is connected and $g : D_o(0; \delta) \setminus \{0\} \to \mathbb{R}$ is a continuous function defined by $g(\beta) = \|S_o - \beta I\| - \|S_o + \beta I\|$ and given that $-g(\beta) = g(-\beta)$, there must exist $\beta' \neq 0$ to give $g(\beta\prime) = 0$ that is $\|S_o - \beta' I\| = \|S_o + \beta' I\|$. By equality (8), it is true that $\|S_1 - \beta' I\| = \|S_1 + \beta' I\|$. But

$$\begin{aligned}2\|S_o - \beta' I\|^2 &= \|S_o - \beta' I\|^2 + \|S_o + \beta' I\|^2 \\ &\geq \|(S_o - \beta' I)^*(S_o - \beta' I) + (S_o + \beta' I)^*(S_o + \beta' I)\| \\ &= 2\|S_o^* S_o + |\beta'|^2 I\| \\ &= 2\{\|S_o\|^2 + |\beta'|^2\} \\ &> 2\|S_o\|^2\end{aligned}$$

which leads to $\|S_o - \beta' I\| > \|S_o\|$. Similarly, $\|S_o - \beta' I\| > \|S_o\|$ and therefore we get $\|S_o\| + \|S_1\| < \|S_o - \beta' I\| + \|S_1 - \beta' I\|$ which contradicts equality (8). Hence $E(S_o, S_1)$ is either a closed line segment or a single point.

□

**Proposition 3.9.** *Let $S_1, S_2, S_3 \in NA(H)$ with $S_1$ and $S_2$ positive. Then $|\langle S_3 x, y\rangle|^2 \leq \langle S_1 x, x\rangle\langle S_2 y, y\rangle\ \forall x, y \in H$ if and only if $\begin{pmatrix} S_1 & S_3^* \\ S_3 & S_2 \end{pmatrix}$ is positive in $NA(H \oplus H)$.*

*Proof.* Assume that $\begin{pmatrix} S_1 & S_3^* \\ S_3 & S_2 \end{pmatrix}$ is a positive operator in $NA(H \oplus H)$. Then $\forall x, y \in H$, the Schwarz inequality for positive operators gives

$\left|\left\langle \begin{pmatrix} S_1 & S_3^* \\ S_3 & S_2 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix} \right\rangle\right|^2$
$\leq \left\langle \begin{pmatrix} S_1 & S_3^* \\ S_3 & S_2 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} S_1 & S_3^* \\ S_3 & S_2 \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix} \right\rangle$. Simplification of these inner products gives the required result.
Conversely, suppose the result is true. Then given any $x, y \in H$, we get

$$\begin{aligned}
\left\langle \begin{pmatrix} S_1 & S_3^* \\ S_3 & S_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle &= \langle S_1x, x\rangle + \langle S_3^*y, x\rangle \\
&\quad + \langle S_3x, y\rangle + \langle S_2y, y\rangle \\
&= \langle S_1x, x\rangle + \langle S_2y, y\rangle + 2Re\langle S_3x, y\rangle \\
&\geq 2\langle S_1x, x\rangle^{\frac{1}{2}}\langle Ty, y\rangle^{\frac{1}{2}} + 2Re\langle S_3x, x\rangle \\
&\geq 2|\langle S_3x, y\rangle| + 2Re\langle S_3x, y\rangle \\
&\geq 2|\langle S_3x, y\rangle| - 2|\langle S_3x, y\rangle| \\
&= 0
\end{aligned}$$

Hence $\begin{pmatrix} S_1 & S_3^* \\ S_3 & S_2 \end{pmatrix}$ is positive.

□

**Lemma 3.10.** *Let $S_1, S_2, S_3 \in NA(H)$ and $S_1, S_2$ be positive with $S_2S_3 = S_3S_1$. If $\begin{pmatrix} S_1 & S_3^* \\ S_3 & S_2 \end{pmatrix} \in NA(H \oplus H)$ is positive , then $\begin{pmatrix} g(S_1)^2 & S_3^* \\ S_3 & h(S_1)^2 \end{pmatrix}$ is also positive for continuous non-negative functions $g$ and $h$ on $[0, \infty)$ which satisfies the condition that $g(t)h(t) = t$ for $t$ in the interval $[0, \infty)$.*

*Proof.* Suppose $S_1$ and $S_2$ are both invertible, then for any continuous function $h$ on $[0, \infty)$, $h(A)S_3 = S_3h(S_1)$ since $S_2S_3 = S_3S_1$. Similarly, since $t \in [0, \infty)$ implies $g(t)h(t) = t$, then $g(C)h(C) = C$ for any operator $C \in NA(H)$ which is positive. This implies that $h(S_2)S_2^{-\frac{1}{2}}S_3g(S_1)S_1^{-\frac{1}{2}} = S_3$. Therefore,
$$\begin{pmatrix} g(S_1)^2 & S_3^{*^2} \\ S_3 & h(S_1) \end{pmatrix} = \begin{pmatrix} g(S_1)S_1^{-\frac{1}{2}} & 0 \\ 0 & h(S_2)S_2^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} S_1 & S_3^* \\ S_3 & S_2 \end{pmatrix} \begin{pmatrix} g(S_1)S_1^{-\frac{1}{2}} & 0 \\ 0 & h(S_2)S_2^{-\frac{1}{2}} \end{pmatrix}$$
which together with the fact that $\begin{pmatrix} S_1 & S_3^* \\ S_3 & S_2 \end{pmatrix}$ is positive completes the proof. □

**Lemma 3.11.** *Let $S \in NA(H)$. Then $\begin{pmatrix} |S| & S^* \\ S & |S^*| \end{pmatrix}$ is a positive operator in $NA(H \oplus H)$ where $|S| = (S^*S)^{\frac{1}{2}}$ and $|S^*| = (SS^*)^{\frac{1}{2}}$.*

*Proof.* On $H \oplus H$, let $A = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$. Then $A$ is self-adjoint and $A^2 = \begin{pmatrix} S^*S & 0 \\ 0 & SS^* \end{pmatrix}$. Since the square root of a positive operator is unique, we get

$A = \begin{pmatrix} |S| & 0 \\ 0 & |S^*| \end{pmatrix}$. This therefore means that by the spectral theorem $|A| + |A|$ is positive due to $A$ being self-adjoint. Hence $\begin{pmatrix} |S| & S^* \\ S & |S^*| \end{pmatrix}$ is positive in $NA(H \oplus H)$. □

**Theorem 3.12.** *Let $S \in NA(H)$ and $f$ and $h$ be as in Proposition 3.10. Then $|\langle S\pi, \pi\rangle| \leq \|f(|S|)\pi\|\|h(|S^*|)y\|$ for each $\pi$ and $y$ in $H$.*

*Proof.* As $S|S|^2 = |S^*|^2 S$, it follows that $S|S| = |S^*|S$ and hence by Lemmas 3.10 and 3.11, we have $\begin{pmatrix} f(|S|)^2 & S^* \\ S & h(|S^*|)^2 \end{pmatrix}$ being positive in $NA(H \oplus H)$. Therefore from Proposition 3.9 the result follows. □

**Lemma 3.13.** *Let $S_o, S_1 \in NA(H)$. Then $\|S_o + S_1\| = \|S_o\| + \|S_1\|$ is equivalent to $\|S_o\|\|S_1\| \in \overline{W(S_o^* S_1)}$.*

*Proof.* Let $\|S_o + S_1\| = \|S_o\| + \|S_1\|$. Then a sequence of vectors $\{y_n\}_n$ for each $n$ exists with $\|y_n\| = 1$ such that $\lim_{n\to\infty} \|S_o y_n + S_1 y_n\| = \|S_o\| + \|S_1\|$. But

$$\begin{aligned} \|S_o y_n + S_1 y_n\| &\leq \|S_o y_n\| + \|S_1 y_n\| \\ &\leq \|S_o\|\|y_n\| + \|S_1\|\|y_n\| \\ &\leq \|S_o\| + \|S_1\| \end{aligned}$$

which means that $\lim_{n\to\infty}(\|S_o y_n\| + \|S_1 y_n\|) = \|S_o\| + \|S_1\|$. Hence it can be deduced that $\lim_{n\to\infty} \|S_o y_n\| = \|S_1\|$ and $\lim_{n\to\infty} \|S_o y_n\| = \|S_o\|$. Thus the identity
$\|S_o y_n + S_1 y_n\|^2 = \|S_o y_n\|^2 + \|S_1 y_n\|^2 + 2Re(\langle S_o^* S_2 y_n, y_n\rangle)$ shows that
$\lim_{n\to\infty} Re(\langle S_o^* S_1 y_n, y_n\rangle) = \|S_o\|\|S_1\|$ and since
$|\langle S_o^* S_1 y_n, y_n\rangle| = (Re(\langle S_o^* S_1 y_n, y_n\rangle))^2 + (Im(\langle S_o^* S_1 y_n, y_n\rangle))^2)^{\frac{1}{2}}$ and

$$\begin{aligned} |\langle S_o^* S_2 y_n, y_n\rangle| &\leq \|S_o^* S_1 y_n\| \\ &\leq \|S_o\|\|S_1\| \end{aligned}$$

then we have $\lim_{n\to\infty} |\langle S_o^* S_1 y_n, y_n\rangle| = \|S_o\|\|S_1\|$. Thus $\lim_{n\to\infty} Im(\langle S_o^* S_1 y_n, y_n\rangle) = 0$ which implies that $\lim_{n\to\infty}\langle S_o^* S_1 y_n, y_n\rangle = \|S_o\|\|S_1\|$ meaning $\|S_o\|\|S_1\| \in \overline{W(S_o^* S_1)}$. Conversely, assume that $\|S_o\|\|S_1\| \in \overline{W(S_o^* S_1)}$ and consider $\{y_n\}_n \in H$, which gives $\lim_{n\to\infty}\langle S_o^* S_1 y_n, y_n\rangle = \|S_o\|\|S_1\|$. Then since

$$\begin{aligned} |\langle S_o^* S_1 y_n, y_n\rangle| &\leq \|S_o y_n\|\|S_1\| \\ &\leq \|S_o\|\|S_1\| \end{aligned}$$

it follows that $\lim_{n\to\infty} \|S_o y_n\| = \|S_o\|$. Similarly we have $\lim_{n\to\infty} \|S_1 y_n\| = \|S_1\|$ and since

$$\|S_o y_n + S_1 y_n\|^2 = \|S_o y_n\|^2 + \|S_1 y_n\|^2 + 2Re(\langle S_o^* S_1 y_n, y_n\rangle)$$

and $\lim_{n\to\infty} Re(\langle S_o^* S_1 y_n, y_n\rangle) = \|S_o\|\|S_1\|$, then $\lim_{n\to\infty} \|S_o y_n + S_1 y_n\| = \|S_o\| + \|S_1\|$. Hence $\|S_o + S_1\| = \|S_o\| + \|S_1\|$. □

**Theorem 3.14.** *Let $S_o, S_1 \in NA(H)$. Then the statements which follow are similar.*

(i). $\exists\, \beta \in \mathbb{C}$ *with* $|\beta| = 1$ *in order for* $\|S_o + \beta S_1\| = \|S_o\| + \|S_1\|$

(ii). $\exists\, \beta \in \mathbb{C}$ *with* $|\beta| = 1$ *in order for* $\beta\|S_o\|\|S_1\| \in \overline{W(S_1^* S_o)}$

(iii). $\exists\, \beta \in \mathbb{C}$ *with* $|\beta| = 1$ *in order for* $\beta\|S_o\|\|S_1\| \in \sigma_{ap}(S_1^* S_o)$
(iv). $w(S_o^* S_1) = \|S_o^* S_1\| = \|S_o\|\|S_1\|$
(v). $r(S_o^* S_1) = \|S_o^* S_1\| = \|S_o\|\|S_1\|$

*Proof.* $(i) \Leftrightarrow (ii)$ is as a result of Lemma 3.13. $(iii) \Leftrightarrow (ii) \Leftrightarrow (iv)$ and $(v) \Leftrightarrow (iii)$ are obvious. $(iv) \Leftrightarrow (v)$ is as a result of the fact that for any operator $A \in H$, $r(A) = \|A\|$ if and only if $w(A) = \|A\|$. □

**Proposition 3.15.** *Let* $S_o, S_1 \in NA(H)$. *Then* $\|S_o\|\|S_1\| \in W(S_o^* S_1)$ *and* $0 \in \sigma_{ap}(\|S_1\|S_o - \|S_o\|S_1)$ *is equivalent to either* $S_o$ *or* $S_1$ *being isometric.*

*Proof.* Suppose $\|S_o\|\|S_1\| \in W(S_o^* S_1)$. Then we have a sequence $\{y_n\}_{n=1}^{\infty}$ of vectors for all $n$ with $\|y_n\| = 1$ so that $\lim_{n\to\infty}\langle S_o^* S_1 y_n, y_n\rangle = \|S_o\|\|S_1\|$. Therefore $\lim_{n\to\infty} Re\langle S_o^* S_1 y_n, y_n\rangle = \|S_o\|\|S_1\|$ and as

$$\begin{aligned} |\langle S_o^* S_1 y_n, y_n\rangle| &\leq \|S_o y_n\|\|S_1 y_n\| \\ &\leq \|S_o\|\|S_1\|\|y_n\|^2 \\ &\leq \|S_o\|\|S_1\| \end{aligned}$$

then $\lim_{n\to\infty}\|S_1 y_n\| = \|S_1\|$. Similarly, we have $\lim_{n\to\infty}\|S_o y_n\| = \|S_o\|$. But

$$\begin{aligned} \|(\|S_1\|S_o y_n - \|S_o\|S_1 y_n)\|^2 &= \|S_1\|^2\|S_o y_n\|^2 + \|S_o\|^2\|S_1 y_n\|^2 \\ &\quad -2\|S_o\|\|S_1\|Re(\langle S_o^* S_1 y_n, y_n\rangle) \end{aligned}$$

which implies that $\lim_{n\to\infty}\|(\|S_1\|S_o y_n - \|S_o\|S_1 y_n)\| = 0$ i.e $0 \in \sigma_{ap}(\|S_1\|S_o - \|S_o\|S_1)$.
For the converse, suppose $S_1$ is isometric. Then $0 \in \sigma_{ap}(S_o - \|S_o\|S_1)$ means a sequence $\{y_n\}_n \subseteq H$ exists with $\|y_n\| = 1$ so that
$\lim_{n\to\infty}\|S_o y_n - \|S_o\|S_1 y_n\| = 0$. From $\|S_o y_n - \|S_o\|S_1 y_n\| \geq |\|S_o y_n\| - \|S_o\||$, we have $\lim_{n\to\infty}\|S_o y_n\| = \|S_o\|$. But since $\lim_{n\to\infty}\langle(S_o y_n - \|S_o\|S_1 y_n), S_o y_n\rangle = 0$ we deduce that $\lim_{n\to\infty}\langle S_o^* S_1 y_n, y_n\rangle = \|S_o\|$ and hence $\|S_o\| \in \overline{W(S_o^* S_1)}$. □

## 4. Conclusion

Determining norms of operators still remains an open problem particularly in the general Banach space setting. Norm estimates for various types of Banach algebra operators have been studied over decades with interesting results obtained. In this note, we considered norm attaining operators in C*-algebras and established their lower bound and upper bound norm estimates.

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