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FIXED POINT THEOREMS FOR EXTENDED INTERPOLATIVE MULTI-VALUED NON-SELF MAPPINGS IN b-METRIC SPACE WITH SOME APPLICATION

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ABSTRACT. The study of fixed point theorems for multi-valued non-self mappings on metrically convex metric space was initiated by Assad and Kirk. In this paper, we will prove and extend the fixed point results for an extended interpolative multivalued non-self-contraction mapping on metrically convex b-metric space. We provide an example for verification of the proven theorems. Also, we give some applications to non-linear matrix equations applied to real-world problems.

1. INTRODUCTION

The study of fixed points of non-self mappings originated with the investigation of Halpern [31]. Also, [28] developed a fixed point theorem for non-self mappings in metric spaces with a Takahashi convex structure. The study of fixed point theorems for non-self mapping on metrically convex metric space with boundary condition was introduced by Assad and Kirk in 1972 [9], in which the concept of fixed point theorem for multi-valued non-self mappings developed rapidly after they proved a non-self multi-valued version of Banach contraction principle. Du *et al.* [27] gave the results of fixed point theory for various multivalued non-self-maps. Ćirić and Ume [19] proved the multi-valued non-self-mappings on convex metric spaces. Furthermore, multi-valued non-self almost contractions were launched by Alghamdi *et al.* [4], who proved the existence of fixed points for such type of mappings for metric space. Berinde [13] initiated a group of self-mappings which are recognized as almost contractions. Berinde and Berinde [15] inaugurated fixed point theorems for almost contraction in multi-valued self-mappings. Altun and Minak [7] proved an extension of Assad-Kirk's fixed point theorem for multivalued non-self mappings. Wangwe and Kumar [52] Fixed point theorem for multivalued non-self mappings in partial symmetric spaces. Altun et al. [8] proved the multi-valued non-self almost F-contractions in metric space. Gabeleh and Plebaniak [29] proved a Global optimality result for multivalued non-self mappings in b-metric spaces. Afshari *et al.* [3] gave the results of the existence of fixed points of set-valued mappings in b-metric spaces.

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The concept of b-metric space was studied earlier by different authors such as Bourbaki [18], Vulpe *et al.* [51], Bakhtin [11] and Czerwik [21] gave the generalization of metric space to b-metric spaces and developed Banach's contraction principle [12] to these spaces. Czerwik [21] prove the fixed point theorem for single-valued mapping in b-metric space. Berinde [14] proved the results on sequences of operators and fixed points in quasi-metric spaces. Further, Czerwik [23] proved the results for the fixed point theorem for multivalued mappings in b-metric space. Since then several researchers have been attracted to their research in this direction. Singh *et al.* [49] they proved the stability of iterative procedures for multivalued maps in metric spaces. Czerwik *et al.* [22] proved a Round-off stability of iteration procedures for set-valued operators in b-metric spaces. Aydi *et al.* [10] proved a fixed point theorem for set-valued quasi-contractions in b-metric spaces. Hieu, and Van Dung [32] proved some fixed point results for generalized rational type contraction mappings in partially ordered b-metric spaces. Maria *et al.* [44] proved the results of fixed point theorems on multi-valued mappings in b-metric spaces. Karapinar [37] gave a short survey on the recent fixed point results on b-metric spaces. Abdeljawad *et al.* [1] gave the solutions of the nonlinear integral equation and fractional differential equation using the technique of a fixed point with a numerical experiment in extended b-metric space. Younis *et al.* [54] gave the graphical structure of extended b-metric spaces with an application to the transverse oscillations of a homogeneous bar. I. Ghasab *et al.* [30] proved the triples of (v, u, ϕ)-contraction and (q, p, ϕ)-contraction in b-metric spaces and its application. Berinde and Păcurar [] gave brief early developments in fixed point theory on b-metric spaces. Lael [41] proved the fixed points of multivalued mappings in b-metric spaces and their application to linear systems. Cosentino *et al.* [20] gave a concept of Multi-valued F-contractions in b-metric space.

Karapinar [38] using the concept of Krein *et al.* [40] modified the classical Kannan [35, 36] contraction phenomena to an interpolative Kannan contraction one to maximize the rate of convergence of an operator to a unique fixed point. However, by giving a counter-example, Karapinar and Agarwal [39] pointed out a gap in the paper by [38] about the assumption of the fixed point being unique and came up with a corrected version. They provided a counter-example to verify that the fixed point need not be unique and invalidate the assumption of a unique fixed point. Since then, several results for variants of interpolative mapping proved for single and multivalued in various abstract spaces. Yesilkaya *etal.* [] made a study on some multi-valued interpolative contractions. Debnath [25] gave the results of set-valued Meir-Keeler, Geraghty and Edelstein type fixed point results in b-metric spaces. Aliouche and Hamaizia [6] proved the common fixed point theorems for multivalued mappings in b-metric spaces with an application to integral inclusions. Shagari *et al.* [48] proved interpolative contractions and isolationistic fuzzy set-valued maps with applications. Debnath and de La Sen [26] proved the set-valued interpolative Hardy–Rogers and set-valued Reich–Rus–Ćirić-type contractions in b-metric spaces. Ali *et al.* [5] proved the new generalizations of set-valued interpolative Hardy-Rogers type contractions in b-metric spaces. Iqbal *et al.* [33] proved a fixed point of generalized weak contraction in b-metric spaces. Alansari and Ali [2] proved an interpolative presic type contractions and related results. Pitchaimani and Saravanan [47] established the concept of extended interpolative mapping in b-metric space.

The main results of this paper will generalize and extend the obtained results of Ishak *et al.* [34], Alghamdi *et al.* [4] and Assad and Kirk [9] to a class of extended interpolative non-self contraction mapping on metrically b-metric space. We will be able to extend many other works of the same analogous in the literature. We also provided an illustrative example.

2. Preliminaries

This section gives some definitions, Lemmas and preliminary results which will be useful for developing our main results.

The concept of b-metric was defined by Bakhtin [11] and Czerwik [21] which is as follows:

Definition 2.1. [11,21] *A b-metric space is a triple (X, d, s) consisting of a non-empty set X with a constant $s \geq 1$ together with a function $d : X \times X \to \mathbb{R}^+$, called the b-metric, such that for all $x, y, z \in X$ we have the following properties:*

(BM1) *$d(x, y) = d(x, x) = d(y, y)$ if and only if $x = y$;*

(BM2) *$d(x, y) = d(y, x)$; and*

(BM3) *$d(x, z) \leq s[d(x, y) + d(y, z)]$.*

Then, the triple (X, d, s) is called a b-metric space.

An example which satisfies the properties of b-metrics:

Example 2.1. *[51] consider $X = \mathbb{R}^2$ and $d : X \times X \to \mathbb{R}^+$ defined for $x = (x_1, x_2), y = (y_1, y_2)$ by*

$$d(x, y) \leq \begin{cases} |x_1 - x_2|, \ y_1 = y_2, \\ 2(|x_1 - x_2| + |y_1 - y_2|), \ y_1 \neq y_2. \end{cases}$$

Then, (X, d) is a quasi metric space with constant $s = 2$.

Example 2.2. *[11, 18] Let $X = L^p[0, 1]$ be the collection of all real functions $x(t)$ such that*

$$\int_0^1 |x(t)|^p dt < \infty,$$

where $t \in [0, 1]$ and $0 < p < 1$. For a function $d : X \times X \to \mathbb{R}^+$ defined by

$$d(x, y) \ = \ \Big(\int_0^1 |x(t) - y(t)|^p dt\Big)^{\frac{1}{p}},$$

for each $x, y \in L^p[0, 1]$, the order pair (X, d) forms a b-metric space with $s = 2^{\frac{1}{p}}$.

Example 2.3. *[37] Let $X = \mathbb{R}$, then the function $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ defined as*

$$d(x, y) \ = \ |x - y|^2,$$

is a b-metric on $\mathbb{R}$.

Example 2.4. *[37] Let $X = \{0, 1, 2\}$ and $d : X \times X \to [0, \infty)$ such that*

$$\begin{aligned} d(0, 0) &= d(0, 2) = d(2, 0) = 1, \\ d(1, 2) &= d(2, 1) = \alpha \geq 2, \\ d(0, 0) &= d(1, 1) = d(2, 2) = 0. \end{aligned}$$

Then, we have

$$d(x,y) \leq \frac{\alpha}{2}[d(x,z)+d(z,y)],$$

for all $x,y,z \in X$.

We give the following topological properties of b-metric space from [11,21].

Definition 2.2. *[11, 21]*

(i) *A sequence* $\{x_n\} \subseteq X$ *converge to* $x \in X$ *if*

$$\lim_{n\to\infty} d(x_n,x) = 0.$$

(ii) *A sequence* $\{x_n\}$ *is a Cauchy sequence in* X *if, for every given* $\epsilon > 0$*, there exists a positive integer* $n(\epsilon)$ *such that*

$$d(x_n,x_m) < \epsilon,$$

for all $n,m \geq n(\epsilon)$.

(iii) *A b-metric space* (X,d,s) *is said to be complete if and only if each Cauchy sequence converges to some* $x \in X$, $s \geq 1$.

Lemma 2.1. *[21] Let* (X,d,s) *be a b-metric space and let* $\{x_n\}$ *be a sequence in* X*. If*

$$\lim_{n\to\infty} x_n = y,$$

and

$$\lim_{n\to\infty} x_n = z,$$

then $y = z$.

Lemma 2.2. *[21] Let* (X,d,s) *be a b-metric space and let* $\{x_n\}$ *be a sequence in* X *such that*

$$d(x_n,x_{n+1}) \leq \alpha d(x_{n-1},x_n),$$

for some $\alpha \in \left(0,\frac{1}{s}\right)$ *and each* $n \in \mathbb{N}$*. Then* $\{x_n\}$ *is a Cauchy sequence in* X.

2.1. **b-Hausdorff metric.** Now, we have to introduce some concepts of the Hausdorff metric to convex b-metric spaces from [22,23]. Let (X,d,s) be convex Hausdorff b-metric space, H_b to be a Hausdorff b-metric induced by b-metric d and $CB^b(X)$ be the family of all non-empty closed bounded subset of X. For $A,B \in CB^b(X)$, define a function $H_b : CB^b(X)\times CB^b(X) \to \mathbb{R}^+$ by

$$H_b(A,B) = \max\{\delta_b(A,B),\delta_b(B,A)\}, \tag{1}$$

where

$$\delta_b(A,B) = \sup\{d(a,B) : a \in A\};$$
$$\delta_b(B,A) = \sup\{d(b,A) : b \in B\};$$

and

$$d(x,A) = \inf\{d(x,a) : a \in A\}.$$

Next, we shall observe some properties of multi-valued mapping in Hausdorff b-metric [22,23].

Lemma 2.3. *[22, 23] For any $A, B, C \in CB^b(X)$ and let (X, d, s) be a b-metric space with $x, y \in X$, we have the following;*

(i) $d(x, B) \leq d(x, b)$ *for any* $b \in B$;
(ii) $\delta_b(A, B) \leq H_b(A, B)$;
(iii) $d(x, B) \leq H_b(A, B)$, *for any* $x \in A$;
(iv) $H_b(A, B) = H_b(B, A)$;
(v) $H_b(A, A) = 0$;
(vi) $H_b(A, C) \leq s[H_b(A, B) + H_b(B, C)]$.

Lemma 2.4. *[22, 23] Let (X, d, s) be a b-metric space and for $A, B \in CB^b(X)$. Then, for any $s > 1$ and $a \in A$, there exists $b = b(a) \in B$ such that*

$$d(a, b) = sH_b(A, B).$$

Lemma 2.5. *[22, 23] Let (X, d, s) be a b-metric space, for $A, B \in CB^b(X)$ and $x \in X$. Then, for any $s > 1$, we have*

$$d(x, A) = 0 \Leftrightarrow x \in \bar{A} = A,$$

where $\bar{A}$ denote the closure of the set A.

Assad and Kirk [9] introduced yet another useful remark on the fixed point theorem for multi-valued non-self mapping in complete metrically convex metric space.

Remark 2.1. [9] *If C is a nonempty closed subset of a complete and metrically convex metric spaces (X, d), then for any $x \in C$, $y \notin C$, there exists a point $z \in \partial C$ (the boundary of C) such that*

$$d(x, z) + d(z, y) = d(x, y).$$

Mohammadi *et al.* [45] gave the following definition and theorem for the extended interpolative Ćirić-Reich-Rus type F-contraction mappings in metric space.

Definition 2.3. *[45] Let (X, d, s) be a b-metric space, we say that the multivalued mapping $\mathrm{T} : X \to CB(X)$ is an extended interpolative multivalued Ćirić-Reich-Rus type F-contraction mappings if there exists $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$, $\tau > 0$ and $F \in \mathcal{F}$ such that*

$$\tau + F(H(Tx, Ty)) \;\; \leq \;\; \alpha F(d(x, y)) + \beta F(d(x, Tx)) + (1 - \alpha - \beta)F(d(y, Ty)),$$

for all $x, y \in X \backslash Fix(T)$ with $x \neq Tx$ with $H(Tx, Ty) > 0$.

Theorem 2.1. *[45] Let (X, d, s) be a complete b-metric space and T be an extended interpolative multivalued Ćirić-Reich-Rus type F-contraction. Assume in addition that*

$$F(\inf A) \;\; = \;\; \inf(F(A)).$$

Then T posses a fixed point.

Dass and Gupta [24] proved the following results on metric space:

Theorem 2.2. *[24] Let (X, d) be a complete complex metric space and $T : X \to X$ be a mappings such that*

(i)

$$d(Tx,Ty) \leq \beta d(x,y) + \alpha \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)},$$

for all $x, y \in X$, *where* α, β *are non-negative real with* $\alpha + \beta < 1$ *and*

(ii) *for some* $x_0 \in X$, *the sequence of iterates* $\{T^n x_0\}$ *has a subsequences* $\{T^{n_k} x_0\}$ *with* $z = \lim_{n\to\infty} T^{n_k} x_0$.

Then z *is a unique fixed point of* T.

3. Results

We prove the following theorem

Theorem 3.1. *Let* (X, d, s) *be a complete metrically convex b-metric spaces,* K *a non-empty closed subset of* X *and* $T : K \to CB^b(X)$ *an extended interpolative multi-valued non-self contraction mappings. Assume that the following conditions hold:*

(i) $Tx \in K$ *for each* $x \in \partial K$,

(ii) *there exists* $\beta \in \left(0, \frac{1}{s}\right)$ $\forall$ $x, y \in K$ *and* $H_b(Tx, Ty) > 0$, *such that*

$$sH_b(Tx,Ty) \leq \beta\Big[\alpha d(x,y) + (1-\alpha)\frac{[1+d(x,Tx)]d(y,Ty)}{1+d(x,y)}\Big], \tag{2}$$

for all $x, y \in X$, *where* α, β *are non-negative real with* $\alpha + \beta < 1$. *If* T *satisfies Rothe's type condition that is* $x \in \partial K \implies Tx \subset K$, *then* T *posses a fixed point* z *and* w *in* K, *such that* $d(z, w) > 0$.

Proof. Let x be an arbitrary point in X. If T is an extended multivalued non-self mapping using condition (i) we have $Tx \subset K$ for each $x \in \partial K$. Then, there exist $x_0 \in K$ such that $x_0 \in Tx_0$. Thus T possesses a fixed point in K. On contrary to that, we constructing two sequences $\{x_n\} \in K$ and $\{y_n\} \in K$ in the following way: Let $x_0 \in K$ and $y_1 \in Tx_0$. If $y_1 \in K$, let $x_1 = Tx_0 = y_1$. If $y_1 \notin K$, select a point $x_1 \in \partial K$ using Remark 2.1 we obtain

$$d(x_0,x_1) + d(x_1,y_1) = d(x_0,y_1).$$

Thus, we have $x_1 \in K$ and so using Lemma 2.4, we may choose $y_2 \in Tx_1$ so that

$$d(y_1,y_2) \leq sH_b(Tx_0,Tx_1).$$

Now, if $y_2 \in K$, let $x_2 = y_2$ and if $y_2 \notin K$, then, select $x_2 \in \partial K$ using Remark 2.1, we have

$$d(x_1,x_2) + d(x_2,y_2) = d(x_1,y_2).$$

Therefore, $x_2 \in K$ and using Lemma 2.4, we can choose $y_3 \in Tx_2$ such that

$$d(y_2,y_3) \leq sH_b(Tx_1,Tx_2).$$

Furthermore, if $y_3 \in K$, let $x_3 = y_3$ and if $y_3 \notin K$. Then, select $x_3 \in \partial K$ using Remark 2.1 such that

$$dp(x_2,x_3) + d(x_3,y_3) = d(x_2,y_3).$$

Therefore, $x_3 \in K$ and using Lemma 2.4, we can choose $y_4 \in Tx_3$ such that

$$d(y_3,y_4) \leq sH_b(Tx_2,Tx_3).$$

Continuing the arguments in this style we construct two sequence (x_n) and (y_n), such that for $n = 1, 2, 3\dots,$ we have

(i) $y_{n+1} \in Tx_n$;
(ii) $d(y_{n+1}, y_n) \leq s^n H_b(Tx_n, Tx_{n-1})$;
(iii) $y_{n+1} \in K$, if $y_{n+1} = x_{n+1}$;
(iv) $y_{n+1} \neq x_{n+1}$, whenever $y_{n+1} \notin K$ and $x_{n+1} \in \partial K$ is such that

$$d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}) = d(x_n, y_{n+1}). \tag{3}$$

For $x_n \in \partial K$ (boundary of K), $y_n = Tx_n$ and $Tx_n \in K$, we claim that (x_n) is a Cauchy sequence. Let us define the two sequences in the following ways:

$$P = \{x_i \in (x_n) : x_i = y_i,\ i = 1, 2, 3...\}.$$

$$Q = \{x_i \in (x_n) : x_i \neq y_i,\ i = 1, 2, 3...\}.$$

To develop our results for the b-Hausdorff metric $d(x_n, x_{n+1})$ for $n \geq 2$, we have three cases to investigates:

Case I

If x_n and $x_{n+1} \in P$, then $y_n = x_n$, $y_{n+1} = x_{n+1}$ and y_n, $y_{n+1} \in P$. By Lemma 2.4, we have

$$d(x_n, x_{n+1}) \leq sH_b(Tx_{n-1}, Tx_n).$$

Let $x = x_{n-1}$ and $y = x_n$ in (2), we get

$$\begin{aligned} d(x_n, x_{n+1}) &\leq sH_b(Tx_{n-1}, Tx_n) \\ &\leq \beta\Big[\alpha d(x_{n-1}, x_n) + (1-\alpha)\frac{[1 + d(x_{n-1}, Tx_{n-1})]d(x_n, Tx_n)}{1 + d(x_{n-1}, x_n)}\Big], \\ &\leq \beta\Big[\alpha d(x_{n-1}, x_n) + (1-\alpha)\frac{[1 + d(x_{n-1}, x_n)]d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_n)}\Big], \\ &\leq \beta\Big[\alpha d(x_{n-1}, x_n) + (1-\alpha)d(x_n, x_{n+1})\Big], \\ (s - \beta(1-\alpha))d(x_n, x_{n+1}) &\leq \alpha\beta d(x_{n-1}, x_n), \\ (s + \alpha\beta - \beta))d(x_n, x_{n+1}) &\leq \alpha\beta d(x_{n-1}, x_n), \\ d(x_n, x_{n+1}) &\leq \frac{\alpha\beta}{(s + \alpha\beta - \beta))}d(x_{n-1}, x_n). \end{aligned}$$

Case II

If $x_n \in P$, $x_{n+1} \in Q$, then we have the following: $x_n = y_n = Tx_{n-1}$, $x_{n+1} \neq y_{n+1} = Tx_n$, using (3), we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}), \\ &= d(x_n, y_{n+1})), \\ &= d(y_n, y_{n+1})). \\ &\leq sH_b(Tx_{n-1}, Tx_n)). \end{aligned}$$

Assume $x = x_{n-1}$ and $y = x_n$ in (2), we get

$$\begin{aligned} d(x_n, x_{n+1}) &\leq sH_b(Tx_{n-1}, Tx_n) \\ &\leq \beta\Big[\alpha d(x_{n-1}, x_n) + (1-\alpha)\frac{[1 + d(x_{n-1}, Tx_{n-1})]d(x_n, Tx_n)}{1 + d(x_{n-1}, x_n)}\Big], \\ &\leq \beta\Big[\alpha d(x_{n-1}, x_n) + (1-\alpha)\frac{[1 + d(x_{n-1}, x_n)]d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_n)}\Big], \\ &\leq \beta\Big[\alpha d(x_{n-1}, x_n) + (1-\alpha)d(x_n, x_{n+1})\Big], \\ (s - \beta(1-\alpha))d(x_n, x_{n+1}) &\leq \alpha\beta d(x_{n-1}, x_n), \\ (s + \alpha\beta - \beta))d(x_n, x_{n+1}) &\leq \alpha\beta d(x_{n-1}, x_n), \\ d(x_n, x_{n+1}) &\leq \frac{\alpha\beta}{(s + \alpha\beta - \beta))} d(x_{n-1}, x_n). \end{aligned} \tag{4}$$

Case III

If $x_n \in Q$, $x_{n+1} \in P$, $y_n \neq x_n, x_{n-1} \in P$, $x_{n+1} \in P$, $x_{n-1} = y_{n-1}$, $x_{n+1} = y_{n+1}$, $y_n \in Tx_{n-1}$.

Assume that

$$x_{n+1} = Tx_n, \;\; x_n \neq y_n = Tx_{n-1}.$$

Consequently, we get,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, y_{n+1}), \\ &= d(x_n, y_n) + d(y_n, x_{n+1}), \\ &\leq (x_n, y_n) + d(y_n, y_{n+1}), \\ &\leq d(x_n, y_n) + sH_b(Tx_{n-1}, Tx_n). \end{aligned} \tag{5}$$

Suppose that

$$sH_b(Tx_{n-1}, Tx_n) = \frac{\alpha\beta}{(s + \alpha\beta - \beta))} d(x_{n-1}, x_n) \leq d(x_{n-1}, x_n). \tag{6}$$

Using (4) and (6) in (7), we get

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, y_n) + d(x_{n-1}, x_n), \\ &\leq d(x_{n-1}, x_n) + d(x_n, y_n), \\ &\leq d(y_{n-1}, y_n), \\ d(x_n, x_{n+1}) &\leq sH_b(Tx_{n-2}, Tx_{n-1}). \end{aligned}$$

Let $x = x_{n-2}, y = x_{n-1}$ in (2), we obtain

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \beta\Big[\alpha d(x_{n-2}, x_{n-1}) + (1-\alpha)\frac{[1 + d(x_{n-2}, Tx_{n-2})]d(x_{n-1}, Tx_{n-1})}{1 + d(x_{n-2}, x_{n-1})}\Big], \\ &\leq \beta\Big[\alpha d(x_{n-2}, x_{n-1}) + (1-\alpha)\frac{[1 + d(x_{n-2}, x_{n-1})]d(x_{n-1}, x_n)}{1 + d(x_{n-2}, x_{n-1})}\Big], \\ &\leq \beta\Big[\alpha d(x_{n-2}, x_{n-1}) + (1-\alpha)d(x_{n-1}, x_n)\Big]. \end{aligned} \tag{7}$$

Using Lemma (2.2) and (7) we get

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq \frac{\beta}{s}\Big[\alpha d(x_{n-2}, x_{n-1}) + \alpha(1-\alpha)d(x_{n-2}, x_{n-1})\Big], \\
d(x_n, x_{n+1}) &\leq \frac{\beta}{s}\Big[2\alpha(1-\alpha)d(x_{n-2}, x_{n-1})\Big], \\
d(x_n, x_{n+1}) &\leq \frac{2\alpha\beta(1-\alpha)}{s}d(x_{n-2}, x_{n-1}).
\end{aligned}$$

Combining all three cases for $n \geq 2$, we have the following possible outcome;

$$d(x_n, x_{n+1}) \leq \begin{cases} \frac{\alpha\beta}{(s+\alpha\beta-\beta))}d(x_{n-1}, x_n), \\ \\ \frac{2\alpha\beta(1-\alpha)}{s}d(x_{n-2}, x_{n-1}). \end{cases}$$

Assume that $\alpha = \frac{1}{2}, \beta = \frac{1}{4}$ and $s = 2$, $\delta = \frac{\alpha\beta}{(s+\alpha\beta-\beta))}$ and $\sigma = \frac{2\alpha\beta(1-\alpha)}{s}$ we have

$$d(x_n, x_{n+1}) \leq \begin{cases} \delta\ d(x_{n-1}, x_n), \\ \\ \xi\ d(x_{n-2}, x_{n-1}). \end{cases}$$

$$\xi = \max\{\delta, \xi\}.$$

By induction, it follows that for $n \geq 2$ we have

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq \max\{\delta, \xi\} \max\{d(x_{n-1}, x_n), d(x_{n-2}, x_{n-1})\}, \\
&\leq \xi^{\frac{n-1}{2}} \max\{d(x_{n-1}, x_n), d(x_{n-2}, x_{n-1})\}, \\
&\leq \xi^{\frac{n-1}{2}} z_n,
\end{aligned}$$

where

$$\begin{aligned}
z_n &= \max\{d(x_{n-1}, x_n), d(x_{n-2}, x_{n-1})\}. \\
z_2 &= \max\{d(x_1, x_2), d(x_0, x_1)\}.
\end{aligned}$$

Now, for each $m > n$ using $(BM3)$ we get

$$\begin{aligned}
d(x_n, x_m) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)], \\
&= sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) \\
&\quad + s^3 d(x_{n+2}, x_m) + \ldots, \\
&\leq (s\xi^{\frac{n-1}{2}} + s^2\xi^{\frac{n-1}{2}} + s^3\xi^{\frac{n-1}{2}} + s^3\xi^{\frac{n-1}{2}} + \ldots)z_2, \\
&\leq s^{n+1}\xi^{\frac{n-1}{2}}(1 + s + s^2 + s^3 + \ldots)z_2, \\
&\leq \frac{s^{n+1}\xi^{\frac{n-1}{2}}}{1-s} z_2, \forall n \geq 2.
\end{aligned}$$

This implies that $\{x_n\}, \forall n \in \mathbb{N}$ is a Cauchy sequence. Thus,

$$\lim_{n\to\infty} d(x_n, x_m) = 0.$$

Since (X, d) is complete b-metric space and K is closed implies that there exists $w \in K$ such that

$$w = \lim_{n\to\infty} x_n.$$

For a sequence $\{x_n\}$, there is a subsequence $\{x_{n_k}\}$ such that

$$x_{n_k} = Tx_{n_{k-1}}.$$

We show that $w \in Tw$. By using $(BM3)$ and (2) we get

$$\begin{aligned} d(w, Tw) &\leq s[d(w, Tx_n) + d(Tx_{n,Tw})], \\ &\leq sd(w, Tx_n) + sd(Tx_{n,Tw}), \\ &\leq sd(w, Tx_n) + sH_b(Tx_{n,Tw}). \end{aligned} \tag{8}$$

Let $x = x_n$ and $y = w$ in (2), we have

$$\begin{aligned} sH_b(Tx_n, Tw) &\leq \beta\Big[\alpha d(x_n, w) + (1-\alpha)\frac{[1 + d(x_n, Tx_n)]d(w, Tw)}{1 + d(x_n, w)}\Big], \\ &\leq \beta\Big[\alpha d(w, w) + (1-\alpha)\frac{[1 + d(w, Tw)]d(w, Tw)}{1 + d(w, w)}\Big], \\ &\leq \beta\Big[(1-\alpha)[1 + d(w, Tw]d(w, Tw)\Big]. \end{aligned} \tag{9}$$

Using (9) in (8) we get

$$d(w, Tw) \leq sd(w, Tw) + \beta(1-\alpha)[1 + d(w, Tw)]d(w, Tw).$$

The above inequality is satisfied if and only if

$$d(w, Tw) = 0.$$

Hence, w is a fixed point of T. If z is another fixed point of T such that $z = Tz$. Then

$$d(w, z) = d(Tw, Tz),$$

but $z \neq w$, thus we have $d(z, w) > 0$, which shows that T is a multivalued mapping in K. □

Our second result is as follows:

Theorem 3.2. *Let (X, d, s) be a complete metrically convex b-metric spaces, K a non empty closed subset of X and $T : K \to 2^X$ an extended interpolative multivalued Ćirić-Reich-Rus type multi-valued non-self contraction mappings if there exists $\alpha, \beta \in \left(0, \frac{1}{s}\right)$ with $\alpha + \beta \leq 1$ such that the following conditions are holds:*

(i) *$Tx \in K$ for each $x \in \partial K$,*
(ii) *there exists*

$$sH_b(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + (1 - \alpha - \beta)d(y, Ty),$$

for all $x, y \in K \backslash Fix(T)$ with $x \neq Tx$ with $H(Tx, Ty) > 0$.

Proof. The proof of this theorem follows similar steps of Theorem 3.1. Hence the proof is completed. □

We give an example to demonstrate the application of Theorem 3.1.

Example 3.1. *Let X be a set of real numbers with the usual norm, $K = [0, 1]$, and denote the unit interval of real numbers with $d(x, y) = |x - y|^2$ with $s = 2$ and $T : K \longrightarrow 2^K$ defined by*

$$Tx = \begin{cases} \frac{x}{9}, & 0 \leq x \leq \frac{1}{2}, \\ \\ \frac{x}{9} + \frac{8}{9}, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

$K = [0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$. *Then* T *has two fixed points that are* 0 *and* 1.

Proof. We claim that T has no fixed point in K. Since $K = [0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$ and ∂K (boundary of K) is $\partial K = \{0, \frac{1}{2}, 1\}$. Using condition (i) of Theorem 3.1, we have

$$\begin{aligned}
x \in \partial K &\Rightarrow Tx \in K, \\
0 \in \partial K &\Rightarrow T0 = 0, \\
\frac{1}{2} \in \partial K &\Rightarrow T\frac{1}{2} = \frac{1}{18}. \\
1 \in \partial K &\Rightarrow T1 = 1. \\
\frac{1}{2} \in \partial K &\Rightarrow T\frac{1}{2} = \frac{1}{18} + \frac{8}{9} = \frac{17}{18}.
\end{aligned}$$

We note that $\{0\}$ and $\{0, 1\}$ are bounded sets in K. By Lemma 2.5, if $x \in \{0, 1\}$, then

$$\begin{aligned}
x \in \overline{\{0\}} &\Leftrightarrow d(x, \{0\}) = d(x, x), \\
&\Leftrightarrow d(x, 0) = |x - 0|^2, \\
&\Leftrightarrow x = 0 \Leftrightarrow x \in \{0\}.
\end{aligned}$$

Next,

$$\begin{aligned}
x \in \{0, 1\} &\Leftrightarrow d(x, \{0, 1\}) = d(x, A), \\
&\Leftrightarrow \min\{d(x, 0), \{x, 1\}\}, \\
&\Leftrightarrow \min\{|x - 0|^2, |x - 1|^2\}, \\
&\Leftrightarrow x \in \{0, 1\}.
\end{aligned}$$

Hence, $\{0, 1\}$ is also closed with respect to convex b-metric d.

Now, to show that the contractive condition of Theorem 3.1 is satisfied, we have the following cases to investigate:

Case 1. For $x, y \in \left\{0, \frac{1}{2}\right\}$. Then, we have

$$Tx = \left[0, \frac{x}{9}\right], Ty = \left[0, \frac{y}{9}\right].$$

By (1), we have

$$\begin{aligned}
H_b(Tx, Ty) &= \max\{\delta_b(Tx, Ty), \delta_b(Ty, Tx)\} \\
H_b\Big(\Big[0, \frac{x}{9}\Big], \Big[0, \frac{y}{9}\Big]\Big) &= \max\Big\{\delta_b\Big(\Big[0, \frac{x}{9}\Big], \Big[0, \frac{y}{9}\Big]\Big), \delta_b\Big(\Big[0, \frac{y}{9}\Big], \Big[0, \frac{x}{9}\Big]\Big)\Big\},
\end{aligned}$$

where

$$\begin{aligned}
\delta_b(Tx, Ty) &= \max\{d(a, Ty); a \in Tx\} \\
\delta_b\Big(\Big[0, \frac{x}{9}\Big], \Big[0, \frac{y}{9}\Big]\Big) &= \max\Big\{d\Big(0, \Big[0, \frac{y}{9}\Big]\Big), d\Big(\frac{x}{9}, \Big[0, \frac{y}{9}\Big]\Big)\Big\}, \\
d\Big(0, \Big[0, \frac{x}{9}\Big]\Big) &= \min\Big\{d\Big(0, 0\Big), d\Big(0, \frac{x}{9}\Big)\Big\},
\end{aligned}$$

$$\begin{aligned}
&= \min\left\{0, \frac{x^2}{81}\right\} = 0.\\
d\Big(\frac{x}{9}, \Big[0, \frac{y}{9}\Big]\Big) &= \min\Big\{d\Big(\frac{x}{9}, 0\Big), d\Big(\frac{x}{9}, \frac{y}{9}\Big)\Big\},\\
&= \min\left\{\frac{x^2}{81}, \frac{|x-y|^2}{81}\right\} = \frac{|x-y|^2}{81}.\\
\delta_b(Tx, Ty) &= \max\Big\{0, \frac{|x-y|^2}{81}\Big\} = \frac{|x-y|^2}{81},
\end{aligned}$$

and

$$\begin{aligned}
\delta_b(Ty, Tx) &= \max\{d(a, Tx); a \in Ty\}\\
\delta_b\Big(\Big[0, \frac{y}{9}\Big], \Big[0, \frac{x}{9}\Big]\Big) &= \max\Big\{d\Big(0, \Big[0, \frac{x}{9}\Big]\Big), d\Big(\frac{y}{9}, \Big[0, \frac{x}{9}\Big]\Big)\Big\},\\
d\Big(0, \Big[0, \frac{x}{9}\Big]\Big) &= \min\Big\{d\Big(0, 0\Big), d\Big(0, \frac{x}{9}\Big)\Big\},\\
&= \min\left\{0, \frac{x^2}{81}\right\} = 0.\\
d\Big(\frac{y}{9}, \Big[0, \frac{x}{9}\Big]\Big) &= \min\Big\{d\Big(\frac{y}{9}, 0\Big), d\Big(\frac{y}{9}, \frac{x}{9}\Big)\Big\},\\
&= \min\left\{\frac{y^2}{81}, \frac{|y-x|^2}{81}\right\} = \frac{|y-x|^2}{81}.\\
\delta_b(Ty, Tx) &= \max\Big\{0, \frac{|y-x|^2}{81}\Big\} = \frac{|y-x|^2}{81}.
\end{aligned}$$

$$\begin{aligned}
H_b(Tx, Ty) &= \max\{\delta_p(Tx, Ty), \delta_p(Ty, Tx)\},\\
&= \max\left\{\frac{|x-y|^2}{81}, \frac{|x-y|^2}{81}\right\} = \frac{|y-x|^2}{81}
\end{aligned}$$

Similarly, we calculate

$$\begin{aligned}
d(x, y) &= |x-y|^2.\\
d(x, Tx) &= d\Big(x, \Big[x, \frac{x}{9}\Big]\Big) = \min\Big\{d(x, 0), d\Big(x, \frac{x}{9}\Big)\Big\}.\\
&= \min\Big\{|x-0|^2, \Big|x - \frac{x}{9}\Big|^2\Big\} = \Big|x - \frac{x}{9}\Big|^2 = \Big|\frac{8x}{9}\Big|^2.\\
d(y, Ty) &= d\Big(y, \Big[0, \frac{y}{9}\Big]\Big) = \min\Big\{d(y, 0), d\Big(y, \frac{y}{9}\Big)\Big\}.\\
&= \min\Big\{|y-0|^2, \Big|y - \frac{y}{9}\Big|^2\Big\} = \Big|y - \frac{y}{9}\Big|^2 = \Big|\frac{8y}{9}\Big|^2.
\end{aligned}$$

Using the above equations in (2) we obtain

$$s\frac{|y-x|^2}{81} \leq \beta\Big[\alpha|x-y|^2 + (1-\alpha)\frac{\Big[1 + \Big|\frac{8x}{9}\Big|^2\Big]\Big|\frac{8y}{9}\Big|^2}{1 + |x-y|^2}. \tag{10}$$

By substituting the values of $\alpha = \frac{1}{2}, \beta = \frac{1}{4}$ and $s = 2$, $x = 0, y = \frac{1}{2}$ in the above inequality, we get

$$\frac{s}{324} \leq \frac{437}{3220}\beta.$$

The inequality (10) is satisfied.

Case 2.

Next, we calculate $H_b(Tx,Ty)$ for $x \in \left\{0, \frac{1}{2}\right\}, y \in \left\{\frac{1}{2}, 1\right\}$.

$$Tx = \left[0, \frac{x}{9}\right], Ty = \left[0, \frac{y+8}{9}\right].$$

By (1), we have

$$\begin{aligned} H_b(Tx,Ty) &= \max\{\delta_b(Tx,Ty), \delta_b(Ty,Tx)\} \\ H_b\Big(\Big[0,\frac{x}{9}\Big],\Big[0,\frac{y+8}{9}\Big]\Big) &= \max\Big\{\delta_b\Big(\Big[0,\frac{x}{9}\Big],\Big[0,\frac{y+8}{9}\Big]\Big), \\ &\qquad \delta_b\Big(\Big[0,\frac{y+8}{9}\Big],\Big[0,\frac{x}{9}\Big]\Big)\Big\}, \end{aligned}$$

where

$$\begin{aligned} \delta_b(Tx,Ty) &= \max\{d(a,Ty); a \in Tx\} \\ \delta_b\Big(\Big[0,\frac{x}{9}\Big],\Big[0,\frac{y+8}{9}\Big]\Big) &= \max\Big\{d\Big(0,\Big[0,\frac{y+8}{9}\Big]\Big), d\Big(\frac{x}{9},\Big[0,\frac{y+8}{9}\Big]\Big)\Big\}, \\ d\Big(0,\Big[0,\frac{y+8}{9}\Big]\Big) &= \min\Big\{d\Big(0,0\Big), d\Big(0,\frac{y+8}{9}\Big)\Big\}, \\ &= \min\Big\{0, \frac{(y+8)^2}{81}\Big\} = 0. \\ d\Big(\frac{x}{9},\Big[0,\frac{y+8}{9}\Big]\Big) &= \min\Big\{d\Big(\frac{x}{9},0\Big), d\Big(\frac{x}{9},\frac{y+8}{9}\Big)\Big\}, \\ &= \min\Big\{\frac{x^2}{81}, \frac{|x-y+8|^2}{81}\Big\} = \frac{|x-y+8|^2}{81}. \\ \delta_b(Tx,Ty) &= \max\Big\{0, \frac{|x-y|^2}{81}\Big\} = \frac{|x-y+8|^2}{81}, \end{aligned}$$

and

$$\begin{aligned} \delta_b(Ty,Tx) &= \max\{d(a,Tx); a \in Ty\} \\ \delta_b\Big(\Big[0,\frac{y+8}{9}\Big],\Big[0,\frac{x}{9}\Big]\Big) &= \max\Big\{d\Big(0,\Big[0,\frac{x}{9}\Big]\Big), d\Big(\frac{y+8}{9},\Big[0,\frac{x}{9}\Big]\Big)\Big\}, \\ d\Big(0,\Big[0,\frac{x}{9}\Big]\Big) &= \min\Big\{d\Big(0,0\Big), d\Big(0,\frac{x}{9}\Big)\Big\}, \\ &= \min\Big\{0, \frac{x^2}{81}\Big\} = 0. \\ d\Big(\frac{y+8}{9},\Big[0,\frac{x}{9}\Big]\Big) &= \min\Big\{d\Big(\frac{y+8}{9},0\Big), d\Big(\frac{y+8}{9},\frac{x}{9}\Big)\Big\}, \\ &= \min\Big\{\frac{(y+8)^2}{81}, \frac{|y-x+8|^2}{81}\Big\} = \frac{|y-x+8|^2}{81}. \\ \delta_b(Ty,Tx) &= \max\Big\{0, \frac{|y-x+8|^2}{81}\Big\} = \frac{|y-x+8|^2}{81}. \end{aligned}$$

$$\begin{aligned} H_b(Tx,Ty) &= \max\{\delta_p(Tx,Ty), \delta_p(Ty,Tx)\}, \\ &= \max\left\{\frac{|x-y+8|^2}{81}, \frac{|y-x+8|^2}{81}\right\} = \frac{|x-y+8|^2}{81} \end{aligned}$$

Similarly, we calculate

$$\begin{aligned} d(x,y) &= |x-y|^2, \\ d(x,Tx) &= d\Big(x,\Big[x,\frac{x}{9}\Big]\Big) = \min\Big\{d(x,0), d\Big(x,\frac{x}{9}\Big)\Big\} \\ &= \min\Big\{|x-0|^2, \Big|x-\frac{x}{9}\Big|^2\Big\} = \Big|x-\frac{x}{9}\Big|^2 = \Big|\frac{8x}{9}\Big|^2 \\ d(y,Ty) &= d\Big(y,\Big[0,\frac{y+8}{9}\Big]\Big) = \min\Big\{d(y,0), d\Big(y,\frac{y+8}{9}\Big)\Big\} \\ &= \min\Big\{|y-0|^2, \Big|y-\frac{y+8}{9}\Big|^2\Big\} = \Big|y-\frac{y+8}{9}\Big|^2 = \Big|\frac{8(y-1)}{9}\Big|^2 \end{aligned}$$

Using the above equations in (2) we obtain

$$s\frac{|y-x+8|^2}{81} \leq \beta\Big[\alpha|x-y|^2 + (1-\alpha)\frac{\Big[1+\Big|\frac{8x}{9}\Big|^2\Big]\Big|\frac{8(y-1)}{9}\Big|^2}{1+|x-y|^2}. \tag{11}$$

By substituting the values of $\alpha = \frac{1}{2}, \beta = \frac{1}{4}$ and $s = 2$,$x = \frac{1}{2}, y = 1$ in the above inequality, we get

$$\frac{289}{324}s \leq \frac{1}{8}\beta.$$

The inequality (11) is satisfied.

Case 3.

Furthermore, we calculate $H_b(Tx,Ty)$ for $x,y \in \Big[\frac{1}{2},1\Big]$,we have

$$Tx = \Big[0,\frac{x+8}{9}\Big], Ty = \Big[0,\frac{y+8}{9}\Big].$$

By (1), we obtain

$$\begin{aligned} H_b(Tx,Ty) &= \max\{\delta_b(Tx,Ty), \delta_b(Ty,Tx)\} \\ H_b\Big(\Big[0,\frac{x+8}{9}\Big],\Big[0,\frac{y+8}{9}\Big]\Big) &= \max\Big\{\delta_b\Big(\Big[0,\frac{x+8}{9}\Big],\Big[0,\frac{y+8}{9}\Big]\Big), \delta_b\Big(\Big[0,\frac{y+8}{9}\Big],\Big[0,\frac{x+8}{9}\Big]\Big)\Big\}, \end{aligned}$$

where

$$\begin{aligned} \delta_b(Tx,Ty) &= \max\{d(a,Ty); a \in Tx\} \\ \delta_b\Big(\Big[0,\frac{x+8}{9}\Big],\Big[0,\frac{y+8}{9}\Big]\Big) &= \max\Big\{d\Big(0,\Big[0,\frac{y+8}{9}\Big]\Big), d\Big(\frac{x+8}{9},\Big[0,\frac{y+8}{9}\Big]\Big)\Big\}, \\ d\Big(0,\Big[0,\frac{y+8}{9}\Big]\Big) &= \min\Big\{d\Big(0,0\Big), d\Big(0,\frac{y+8}{9}\Big)\Big\}, \\ &= \min\Big\{0,\frac{(y+8)^2}{81}\Big\} = 0. \\ d\Big(\frac{x+8}{9},\Big[0,\frac{y+8}{9}\Big]\Big) &= \min\Big\{d\Big(\frac{x+8}{9},0\Big), d\Big(\frac{x+8}{9},\frac{y+8}{9}\Big)\Big\}, \\ &= \min\Big\{\frac{(x+8)^2}{81},\frac{|x-y|^2}{81}\Big\} = \frac{|x-y|^2}{81}. \\ \delta_b(Tx,Ty) &= \max\Big\{0,\frac{|x-y|^2}{81}\Big\} = \frac{|x-y|^2}{81}, \end{aligned}$$

and

$$\begin{aligned}
\delta_b(Ty,Tx) &= \max\{d(a,Tx); a\in Ty\}\\
\delta_b\Big(\Big[0,\frac{y+8}{9}\Big],\Big[0,\frac{x+8}{9}\Big]\Big) &= \max\Big\{d\Big(0,\Big[0,\frac{x+8}{9}\Big]\Big), d\Big(\frac{y+8}{9},\Big[0,\frac{x+8}{9}\Big]\Big)\Big\},\\
d\Big(0,\Big[0,\frac{x+8}{9}\Big]\Big) &= \min\Big\{d\Big(0,0\Big), d\Big(0,\frac{x+8}{9}\Big)\Big\},\\
&= \min\left\{0,\frac{(x+8)^2}{81}\right\} = 0.\\
d\Big(\frac{y+8}{9},\Big[0,\frac{x+8}{9}\Big]\Big) &= \min\Big\{d\Big(\frac{y+8}{9},0\Big), d\Big(\frac{y+8}{9},\frac{x+8}{9}\Big)\Big\},\\
&= \min\left\{\frac{(y+8)^2}{81},\frac{|y-x|^2}{81}\right\} = \frac{|y-x|^2}{81}.\\
\delta_b(Ty,Tx) &= \max\Big\{0,\frac{|y-x|^2}{81}\Big\} = \frac{|y-x|^2}{81}.
\end{aligned}$$

$$\begin{aligned}
H_b(Tx,Ty) &= \max\{\delta_p(Tx,Ty),\delta_p(Ty,Tx)\},\\
&= \max\left\{\frac{|x-y|^2}{81},\frac{|y-x|^2}{81}\right\} = \frac{|x-y|^2}{81}
\end{aligned}$$

Similarly, we calculate

$$\begin{aligned}
d(x,y) &= |x-y|^2.\\
d(x,Tx) &= d\Big(x,\Big[0,\frac{x+8}{9}\Big]\Big) = \min\Big\{d(x,0), d\Big(x,\frac{x+8}{9}\Big)\Big\}.\\
&= \min\Big\{|x-0|^2,\Big|x-\frac{x}{9}\Big|^2\Big\} = \Big|x-\frac{x+8}{9}\Big|^2 = \Big|\frac{8(x-1)}{9}\Big|^2.\\
d(y,Ty) &= d\Big(y,\Big[0,\frac{y+8}{9}\Big]\Big) = \min\Big\{d(y,0), d\Big(y,\frac{y+8}{9}\Big)\Big\}\\
&= \min\Big\{|y-0|^2,\Big|y-\frac{y+8}{9}\Big|^2\Big\} = \Big|y-\frac{y+8}{9}\Big|^2 = \Big|\frac{8(y-1)}{9}\Big|^2.
\end{aligned}$$

Using the above equations in (2) we obtain

$$s\frac{|y-x|^2}{81} \leq \beta\Big[\alpha|y-x|^2+(1-\alpha)\frac{\Big[1+\Big|\frac{8(x-1)}{9}\Big|^2\Big]\Big|\frac{8(y-1)}{9}\Big|^2}{1+|y-x|^2}. \tag{12}$$

By substituting the values of $\alpha=\frac{1}{2},\beta=\frac{1}{4}$ and $s=2$, $x=\frac{1}{2}, y=1$ in the above inequality, we get

$$\frac{1}{324}s \leq \frac{1}{8}\beta.$$

The inequality (12) is satisfied.

Which is a contraction of our claims. Hence, T has two fixed points which are 0 and 1 that satisfy Equation (2) and Rothe's boundary condition of Theorem 3.1. Hence the proof is completed. □

4. An Application to Non-linear Matrix Equation in b-Metric Spaces

In this section, we establish Thomson b-metric for illustration of Theorem 3.1. In 1963, Thompson [50] introduced the results on certain contraction mappings in a partially ordered vector space. Nussbaum [46] proved Hilbert's projective metric and iterated non-linear maps. Lim [43] gave a solution on solving the nonlinear matrix equation $X = Q + \sum_{i=1}^{m} = M_i X^{\delta_i} M_i^{\star}$ via a contraction principle. Berzig, M. and Samet [17] solved systems of nonlinear matrix equations involving Lipshitzian mappings. Liao *et al.* [42] proved the Thompson metric method for solving a class of non-linear matrix equations.

We define Thomson metric [50] for $s \geq 2$. The $s \times s$ Hermitian positive definite matrices. The Thomson matrices are defined by

$$d(A,B) \quad \leq \quad \max\Big\{\log M\Big(A/B\Big), \log M\Big(B/A\Big)\Big\},$$

where

$$M\Big(A/B\Big) \quad = \quad \inf\{\lambda > 0 : A \leq \lambda B\} = \lambda \max\Big(B^{-\frac{1}{2}} A B^{-\frac{1}{2}}\Big),$$

is the maximum eigenvalue of $B^{-\frac{1}{2}} A B^{-\frac{1}{2}}$. Here $X \leq Y$ means that $Y - X$ is positive semi-definite. Thomson metric [50] and Nussbaum [46], they sowed that $\mathcal{P}(s)$ is a complete metric space for the Thomson metric d defined by

$$d(A,B) \quad = \quad \Big|\Big|\log\Big(B^{-\frac{1}{2}} A B^{-\frac{1}{2}}\Big)\Big|\Big|,$$

where $\|.\|$ implies spectral norm. Also, the Thomson metric exists on an open normal convex cone of real Banach space. In particular the open convex cone of positive definite operators of Hilbert space. It is invariant under the matrix inversion and congruence transformations.

$$d(A,B) \quad = \quad d(A^{-1}, B^{-1}) = d(MAM^{\star}, MBM^{\star}),$$

for any singular matrix M. The useful results of the Thomson metric on the non-positive curvature property are given by

$$d(X^r, Y^r) \quad \leq \quad r d(X,Y), \; r \in [0,1]. \tag{13}$$

For an invariant property of the metric, we have

$$d(MX^rM, MY^rM) \quad \leq \quad |r| d(X,Y), \; r \in [0,1]. \tag{14}$$

We have the following lemma from [43].

Lemma 4.1. *For all $A, B, C, D \in \mathcal{P}(s)$, we have*

$$d(A+B, C+D) \quad \leq \quad \max\{d(A,C), d(B,D)\}. \tag{15}$$

In particular

$$d(A+B, C+D) \quad \leq \quad d(B,C). \tag{16}$$

By Lemma 4.1 we define the Thomson b-metric as

$$d(A+B, C+D) \quad \leq \quad d(B,C) = |B-C|^2. \tag{17}$$

Berzing and Samet [17] considered the solution to the system of nonlinear matrix equations.

$$X^r = Q_i + \sum_{j=1}^{m} (A_j^\star T_{ij}(X_j) A_j)^{\delta_{ij}}, i = 1, 2, 3, \ldots \tag{18}$$

where $r_i \geq 1, 0 < |\delta_{ij} \leq 1, Q_i \geq 0$ and A_i are nonsingular matrices and $(X_1, X_2, \ldots, X_m) \in (\mathcal{P}(s))^m$ with $s \geq 2$.

Liao *et al.* [42] showed that (18) can be reduced to

$$X^r = Q + (A^\star T(X) A), \tag{19}$$

for $m = 1, i = 1, \delta_{11}$ and $X \in \mathcal{P}(s)$.

Consider the system of the non-linear matrix equation

$$T : \begin{cases} X_1 = & I_n + A_1^\star (X_1^{\frac{1}{3}} + B_1)^{\frac{1}{2}} A_1 + A_2^\star (X_2^{\frac{1}{4}} + B_2)^{\frac{1}{3}} A_2 + A_3^\star (X_3^{\frac{1}{5}} + B_3)^{\frac{1}{4}} A_3, \\ X_2 = & I_n + A_1^\star (X_1^{\frac{1}{5}} + B_1)^{\frac{1}{4}} A_1 + A_2^\star (X_2^{\frac{1}{3}} + B_2)^{\frac{1}{2}} A_2 + A_3^\star (X_3^{\frac{1}{4}} + B_3)^{\frac{1}{3}} A_3, \\ X_3 = & I_n + A_1^\star (X_1^{\frac{1}{4}} + B_1)^{\frac{1}{3}} A_1 + A_2^\star (X_2^{\frac{1}{5}} + B_2)^{\frac{1}{4}} A_2 + A_3^\star (X_3^{\frac{1}{3}} + B_2)^{\frac{1}{3}} A_3, \end{cases} \tag{20}$$

where A_i are $s \times s$ non singular matrices.

Solving (20) is equivalent to find the values $(X_1, X_2, X_3) \in (\mathcal{P}(s))^3$ for $s = 3$ solution to

$$X_i^r = Q_i + \sum_{j=1}^{m} (A_j^\star T_{ij}(X_j) A_j)^{\delta_{ij}}, i = 1, 2, 3 \textit{ and } m = 3. \tag{21}$$

Define

$$T_{ij} = \left(X_j^{\theta_{ij}} + B_j\right)^{\gamma_{ij}}, \tag{22}$$

where

$$\theta = \theta_{ij} = \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{3} & \frac{1}{4} \end{pmatrix},$$

$$\gamma = \gamma_{ij} = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{2} \end{pmatrix},$$

Now, we are equipped to prove the following theorem

Theorem 4.1. *Suppose the following hypothesis holds:*

(i)

$$X_i^r = Q_i + \sum_{j=1}^{m} (A_j^\star T_{ij}(X_j) A_j)^{\delta_{ij}}, i = 1, 2, 3,$$

with $Q_i = I_n, \delta_{ij} = 1$.

(ii) *There exists a metric function* $d : [0,1] \times [0,1] \times \mathbb{R}^s \to \mathbb{R}$, *such that*

$$\left\| d\left(X_j^{\theta_{ij}} + B_j\right)^{\gamma_{ij}}, d\left(Y_j^{\theta_{ij}} + B_j\right)^{\gamma_{ij}} \right\| \leq \varpi \|X - Y\|,$$

where

$$\|X - Y\| = d(X, Y) = |X - Y|^2,$$

and

$$d(X,Y) = sH_b(TX,TY) \leq \frac{\beta}{s}\Big[\alpha d(X,Y) + (1-\alpha)\frac{[1+d(X,TX)]d(Y,TY)}{1+d(X,Y)}\Big],$$

(iii)

$$\varpi = \frac{\beta}{s}\|\frac{\delta_{ij}\theta_{ij}\gamma_{ij}}{r_i}\|^2 \leq 1.$$

Then, equation (20) has a solution.

Proof. Let a mapping $T_{ij} : (\mathcal{P}(s))^s \to (\mathcal{P}(s))^s$ defined by

$$T_{ij}X_i = \left(Q_i + \sum_{j=1}^{m}(A_j^\star T_{ij}(X_j)A_j)^{\delta_{ij}}\right)^{\frac{1}{r_i}}, i,j = 1,2,3,$$

and

$$H_b(TX,TY) \leq d(T_{ij}X, T_{ij}Y),$$

for all $X, Y \in \mathcal{P}(s)$.

Using conditions $(i), (ii)$ and (iii), gives

$$\begin{aligned}
||d(T_{ij}X,T_{ij}Y)||^2 &\leq \Bigg|\Bigg|d\Bigg(\Big(Q_i + \sum_{j=1}^{m}(A_j^\star T_{ij}(X_j)A_j)^{\delta_{ij}}\Big)\Bigg)^{\frac{1}{r_i}}, \\
&\quad d\Bigg(\Big(Q_i + \sum_{j=1}^{m}(A_j^\star T_{ij}(Y_j)A_j)^{\delta_{ij}}\Big)\Bigg)^{\frac{1}{r_i}}\Bigg|\Bigg|^2, \\
&\leq \Bigg|\Bigg|\frac{1}{r_i}d\Big(Q_i + \sum_{j=1}^{m}(A_j^\star T_{ij}(X_j)A_j)^{\delta_{ij}}\Big), \\
&\quad \frac{1}{r_i}d\Big(Q_i + \sum_{j=1}^{m}(A_j^\star T_{ij}(Y_j)A_j)^{\delta_{ij}}\Big)\Bigg|\Bigg|^2, \\
&\leq \Bigg|\Bigg|\frac{\delta_{ij}}{r_i}d\Big(\sum_{j=1}^{m}(A_j^\star T_{ij}(X_j)A_j)\Big), \frac{\delta_{ij}}{r_i}d\Big(\sum_{j=1}^{m}(A_j^\star T_{ij}(Y_j)A_j)\Big)\Bigg|\Bigg|^2, \\
&\leq \Bigg|\Bigg|\frac{\delta_{ij}}{r_i}\Big(d\Big(\sum_{j=1}^{m}(A_j^\star T_{ij}(X_j)A_j)\Big), d\Big(\sum_{j=1}^{m}(A_j^\star T_{ij}(Y_j)A_j)\Big)\Big)\Bigg|\Bigg|^2, \\
&\leq \Bigg|\Bigg|\frac{\delta_{ij}}{r_i}\Big(d\Big(X_j^{\theta_{ij}} + B_j\Big)^{\gamma_{ij}}, d\Big(Y_j^{\theta_{ij}} + B_j\Big)^{\gamma_{ij}}\Big)\Bigg|\Bigg|^2, \\
&\leq \Bigg|\Bigg|\frac{\delta_{ij}\gamma_{ij}}{r_i}\Big(d\Big(X_j^{\theta_{ij}} + B_j\Big), d\Big(Y_j^{\theta_{ij}} + B_j\Big)\Big)\Bigg|\Bigg|^2, \\
&\leq \Bigg|\Bigg|\frac{\delta_{ij}\gamma_{ij}}{r_i}d\Big(X_j^{\theta_{ij}}, Y_j^{\theta_{ij}}\Big)\Bigg|\Bigg|^2, \\
&\leq \Bigg|\Bigg|\frac{\delta_{ij}\gamma_{ij}\theta_{ij}}{r_i}d\Big(X_j, Y_j\Big)\Bigg|\Bigg|^2, \\
&\leq \frac{\beta}{s}\|\frac{\delta_{ij}\theta_{ij}\gamma_{ij}}{r_i}\|^2\|X-Y\|^2, \\
&\leq \|\frac{\delta_{ij}\theta_{ij}\gamma_{ij}}{r_i}\|^2\frac{\beta}{s}\Big[\alpha d(X,Y) + (1-\alpha)\frac{[1+d(X,TX)]d(Y,TY)}{1+d(X,Y)}\Big],
\end{aligned}$$

$$
\begin{aligned}
d(T_{ij}X, T_{ij}Y) &\leq \frac{\beta}{s}\|\frac{\delta_{ij}\theta_{ij}\gamma_{ij}}{r_i}\|^2 d(X,Y),\\
d(T_{ij}X, T_{ij}Y) &\leq \varpi d(X,Y).
\end{aligned}
$$

Hence Theorem 4.1 satisfied, thus Theorem 3.1 verified. □

5. Conclusions

The main contribution of this study to fixed point theory is the fixed point result given in Theorem 3.1. This theorem provides the extended interpolative non-self contraction mapping on metrically b-metric space. This paper, inspired by the results obtained by Ishak *et al.* [34], Alghamdi *et al.* [4] and Assad and Kirk [9]. We also provided an illustrative example to support the results and an application to the non-linear matrix equations.

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