DOI: 10.28919/ejma.2023.3.21

Eur. J. Math. Appl. (2023)3:21
URL: http://ejma.euap.org

BIJECTIONS OF PLANE HUSIMI GRAPHS AND CERTAIN COMBINATORIAL STRUCTURES

YVONNE WAKUTHII KARIUKI[1] AND ISAAC OWINO OKOTH[2,*]

Abstract. Plane Husimi graphs are combinatorial structures obtained when we replace edges in plane trees with complete graphs such that the resultant structures are connected and cycle-free. The formula that counts these structures is known to enumerate other combinatorial structures. In this paper, we construct bijections between the set of plane Husimi graphs and the sets of plane trees, dissections of convex polygons, sequences satisfying certain properties, standard Young tableaux, Deutsch paths and restricted lattice paths.

1. Introduction

Plane trees (or ordered trees) have been generalised by many authors. One way of generalising them is by considering graphs obtained when the edges of the trees are replaced by complete graphs or polygons such that the properties of trees are not lost, i.e. cycle-freeness and connectedness. They are commonly referred to as tree-like structures. We now define the main object of our study.

Definition 1.1 ([12]). *A* plane Husimi graph *is a rooted block graph drawn in the plane such that its blocks are ordered.*

In Figure 1, we show a plane Husimi graph on 11 vertices with 6 blocks.

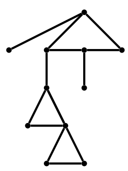


Figure 1. Example of a plane Husimi graph.

Plane Husimi graphs as well as other related structures have been extensively studied in [1, 2, 4, 5, 7, 9–13, 16] among other papers. A block b attached to a vertex i in a plane Husimi

[1]Department of Mathematics, Kibabii University, Bungoma, Kenya

[2]Department of Pure and Applied Mathematics, Maseno University, Maseno, Kenya

*Corresponding author

E-mail addresses: ykariuki@kibu.ac.ke, ookoth@maseno.ac.ke.

Key words and phrases. plane Husimi graphs; plane trees; polygon dissections; valid sequences; standard Young tableaux; Deutsch paths; restricted lattice paths.

Received 10/08/2023.

graph is called a *block child* of i and the outdegree of a vertex j is the number of its block children [13]. Okoth [12] enumerated plane Husimi graphs according to number of vertices, number of blocks, block types and number of leaves. In the same paper, the author derived that the number of plane Husimi graphs on $n+1$ vertices with k blocks is given by

$$\frac{1}{n+1}\binom{n+k}{k}\binom{n-1}{k-1}. \tag{1}$$

Richard Stanley, in his illustrious book [17, Exercise 6.39] asks for combinatorial interpretations for various structures enumerated by little and large Schröder numbers. We note that when we sum over all k in Equation (1), we obtain the little Schröder numbers. It is known that little Schröder numbers count plane trees with a given number of endpoints and no vertices of outdegree 1 (see [17, Exercise 6.39, (b)]). In Section 2, we construct a bijection between the set of plane Husimi graphs with $n+1$ vertices and k blocks, and the set of plane trees on $n+1$ vertices with k internal vertices and no vertex of outdegree 1. The sequence of Equation (1) is also listed in Neil Sloane's celebrated Online Encyclopaedia of Integer Sequences [15] as A0033282. Among the listed structures of the sequence include number of dissections of a regular $(n+2)$-gon with $k-1$ diagonals, number of standard Young tableaux of shape $(k,k,1^{n-k})$ among others. A bijection between the set of these dissections and Young tableaux was obtained by Stanley in [18]. An equivalent bijection between the set of plane Husimi graphs and these dissections is established in Section 3, and the connection to standard Young tableaux is obtained in Section 4.

Consider a sequence $(s_1, s_2, \ldots, s_{n+k})$ of integers which satisfy the following properties:

(i) Either $s_i \geq 1$ or $s_i = -1$, for all $i = 1, 2, \ldots, n+k$
(ii) The number of $s_i \geq 1$ is k and the remaining ones are all equal to -1.
(iii) The sum $s_1 + s_2 + \cdots + s_{n+k} = 0$.
(iv) The partial sum $p_i = s_1 + s_2 + \cdots + s_i \geq 0$ for all $i = 1, 2, \ldots, n+k$.

We shall call sequences satisfying the aforementioned properties as *valid sequences*. Stanley [18] constructed a bijection between the set of these sequences and standard Young tableaux of type $(k,k,1^{n-k})$. This shows that Equation (1) counts valid sequences of length $n+k$. A bijection between the set of plane Husimi graphs and the set of valid sequences is given in Section 5.

Deutsch paths, due to Emeric Deutsch as stated in [14], are Dyck paths which start at the origin and use up steps of length greater than or equal to one and unit down steps. We establish a bijection between the set of these paths and the sets of plane Husimi graphs and valid sequences in Section 6. Lattice paths that use unit horizontal step and unit vertical steps that begin at the origin and lie weakly below the line $y = mx$ have been studied by various authors, for example [6, 8]. We shall refer to these lattice paths as *restricted lattice paths*. We construct a bijection between plane Husimi graphs and restricted lattice paths in which horizontal paths are allowed to have steps of length at least 1. The bijection is achieved in Section 7 as well as their relation to valid sequences.

2. Plane trees

In the sequel, we prove our first result which is a generalization of the known bijection between the sets of plane trees and binary trees.

Theorem 2.1. *There is a bijection between the set of plane trees with $n+1$ vertices and k internal vertices such that each internal vertex has at least one child and the set of plane Husimi graphs on $n+1$ vertices with k blocks.*

Proof. Let H be a plane Husimi graph on $n+1$ vertices with k blocks. Let the leftmost block child of the root be of size d and let the plane Husimi graphs attached at the vertices of leftmost block child be $H_1, H_2, \ldots, H_d$ as shown in the left of Figure 2.

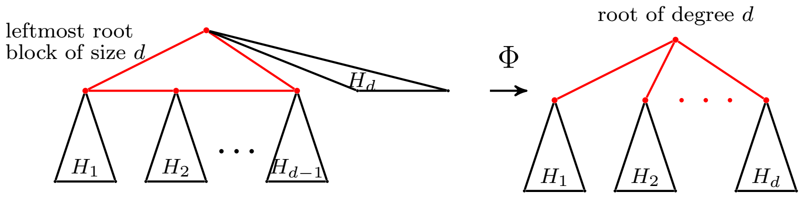


FIGURE 2. The map Φ.

We define a map Φ as follows: The image $\Phi(H)$ is a graph with an identified vertex (root) such that $H_1, H_2, \ldots, H_d$ are it's subgraphs attached in this order from left to right to the children of the root. See Figure 2. The corresponding plane tree on $n+1$ vertices is thus obtained by recursively applying Φ to transform $H_1, H_2, \ldots, H_d$ to trees. Each block contributes an internal vertex. In Figure 3, we get a depiction of the procedure.

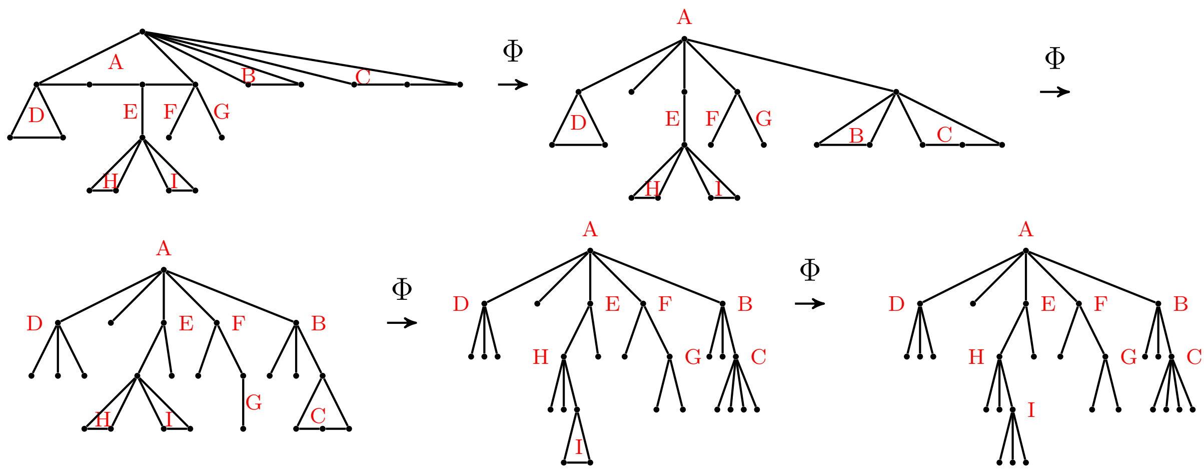


FIGURE 3. Example of obtaining a plane tree from a plane Husimi graph.

We obtain the reverse procedure Φ^{-1}: Consider a plane tree on $n+1$ vertices with k internal vertices such that each internal vertex has at least two children. For each internal vertex with i children, we create a block of size i consisting of the internal child and its children except the rightmost child. Delete the rightmost edge and adjoin the rightmost child to its root. The structure obtained is a plane Husimi graph on n vertices with k blocks since each internal vertex contributes a block. This procedure is illustrated in Figure 4.

□

We note that in the bijection Φ described in the proof of Theorem 2.1, block size responds to outdegree in the plane tree. Okoth in [12, Theorem 4.1] showed that the number of plane

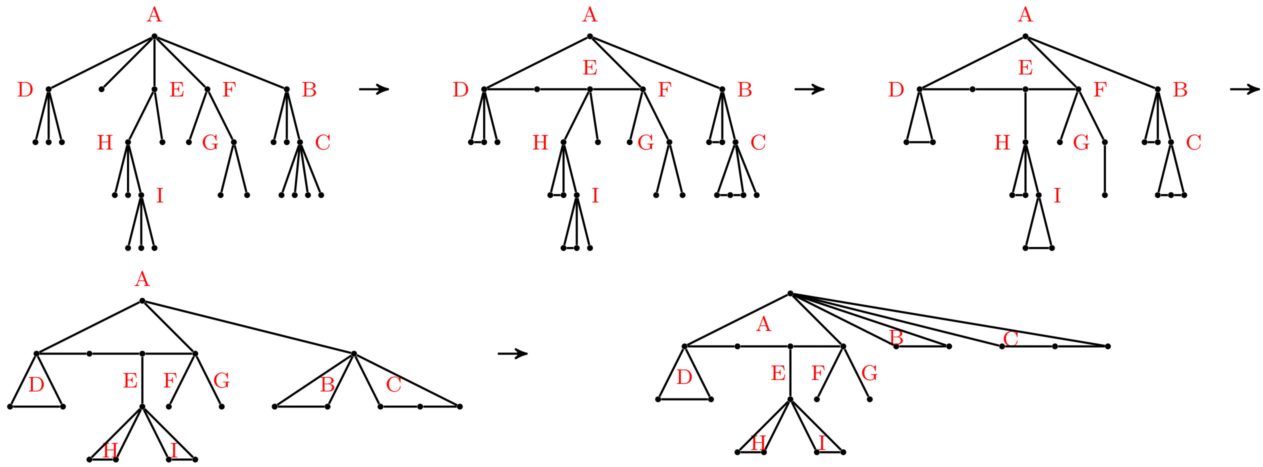


FIGURE 4. Example of obtaining a plane Husimi graph from a plane tree.

Husimi graphs on n vertices with k blocks such that n_i blocks are of size $i \geq 2$ is given by

$$\frac{1}{n}\binom{n+k-1}{k}\binom{k}{n_2, n_3, \ldots}.$$

We thus obtain the following corollary:

Corollary 2.2. *There are*

$$\frac{1}{n}\binom{n+k-1}{k}\binom{k}{n_2, n_3, \ldots}$$

plane trees on n vertices with k internal vertices such that there are n_i vertices with outdegree i for every $i \geq 2$ and no vertices of outdegree 1.

We also note that if the blocks are all of size 2, then the bijection described in this section is the rotation correspondence between plane trees and binary trees obtained in [3].

3. Polygon dissections

Theorem 3.1. *There is a bijection between the set of dissections of regular polygons on $n+2$ vertices with k regions and the set of plane Husimi graphs on $n+1$ vertices with k blocks.*

Proof. Consider a regular $(n+2)$-gon with $k-1$ diagonals. The diagonals divide the polygon into k regions. Label the upper left vertex as 1. Label the remaining vertices as 2 through to $n+2$ in the anticlockwise direction. Let $v_1, v_2, \ldots, v_m$, where $v_i \leq v_{i+1}$ and $m \geq 3$, be vertices of a region. Now, create edges $v_1v_2, v_2v_3, \ldots, v_{m-2}v_{m-1}, v_{m-1}v_1$. Do this for all the regions. The graph obtained is a plane Husimi graph whose vertices have been labelled $1, 2, \ldots, n+1$ in preorder, i.e., visit the root, leftmost subgraph, second leftmost subgraph and so on. The labels can thus be dropped. Since a region in the polygon corresponds to a block in the plane Husimi graph, the graph is on $n+1$ vertices with k blocks. In Figure 5, we give an illustration of this procedure.

We obtain the reverse procedure: Consider a plane Husimi graph H on $n+1$ vertices with k blocks. Traverse H in preorder and label a vertex i if it is the i^{th} vertex to be traversed. We obtain the corresponding regular $(n+2)$-gon with k regions as follows: Draw a regular polygon with $n+2$ vertices labelled $1, 2, \ldots, n+2$ in anticlockwise direction such that vertex 1 is the

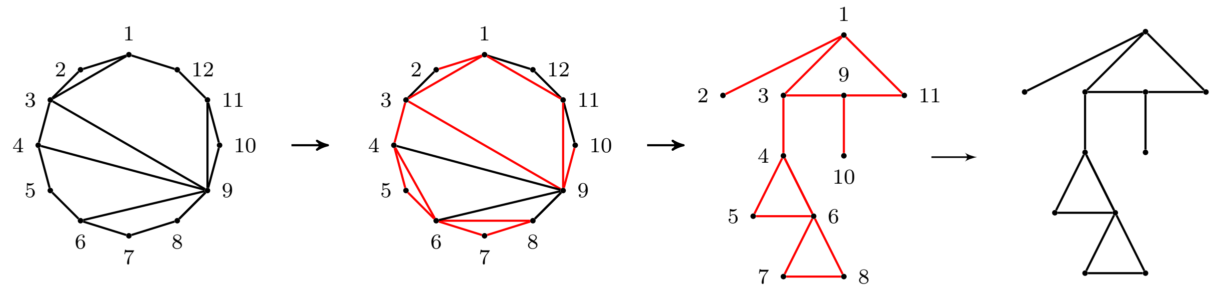


FIGURE 5. Obtaining a plane Husimi graph from a polygon dissection.

upper left vertex. Inscribe the plane Husimi graph in the polygon such that an edge ij in the Husimi graph is a straight line in the polygon. Note that vertex $n+2$ is isolated.

Let m be the vertex incident to vertex 1 that has the largest label. Delete this edge and draw straight lines from 1 and m to $n+2$. We now move around the polygon, starting at vertex 1, in anticlockwise direction and add straight lines inside the polygon recursively as follows:

(i) Let u and v be vertices with the smallest and largest label in a block respectively. Moreover, let w be a vertex with the largest label that is incident to u. The vertex w belongs to another block and uw is a diagonal line dissecting the polygon. Delete edge uv and draw a new edge vw.

(ii) For $i = 1, 2, \ldots, n$, draw the lines from i to $i+1$ if they are not already there.

We describe the reverse procedure in Figure 6. □

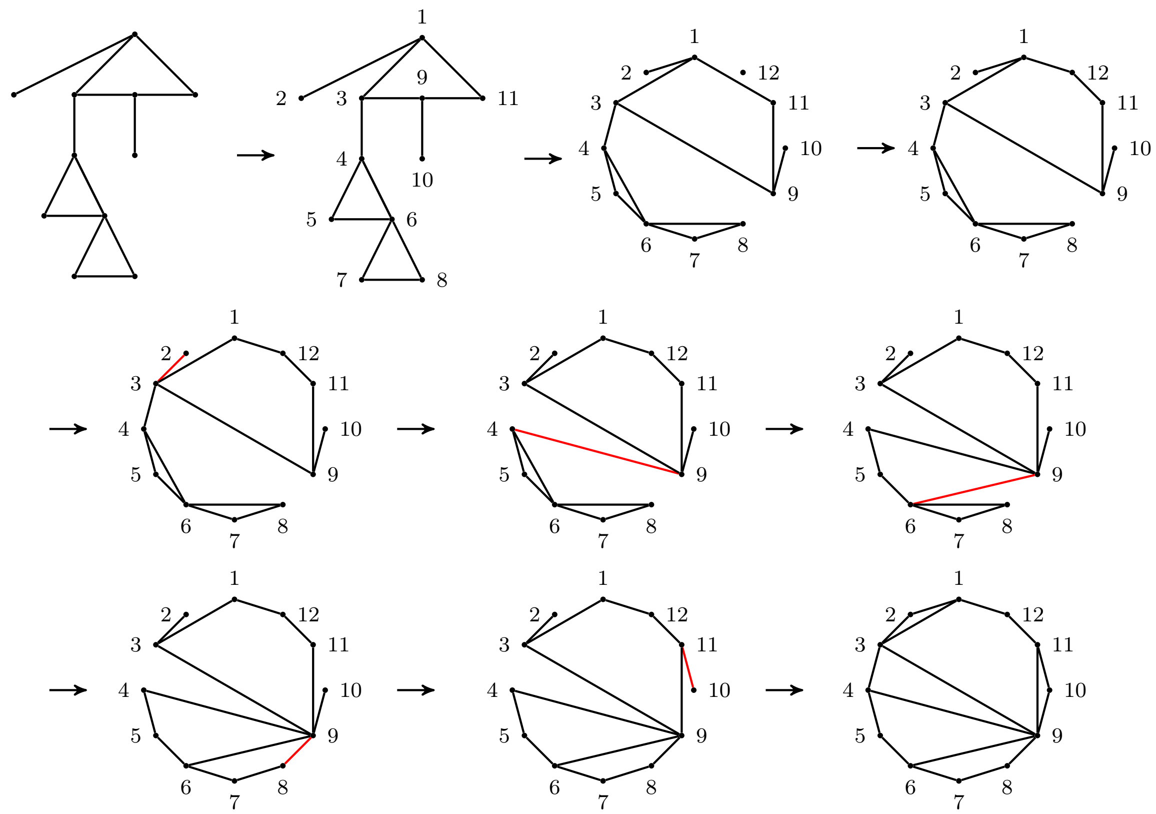


FIGURE 6. Obtaining a polygon dissection from plane Husimi graph.

In [13], it was proved that the number of plane Husimi graphs on $n+1$ vertices and k blocks with root degree d is given by

$$\frac{d}{k}\binom{n+k-d-1}{k}\binom{n-1}{k-1}. \tag{2}$$

We note that the root degree in the plane Husimi graph corresponds to the number of regions which are adjacent to vertex 1 in the polygon dissection thus the following result:

Corollary 3.2. *There are*

$$\frac{d}{k}\binom{n+k-d-1}{k}\binom{n-1}{k-1}$$

different dissections of a regular $(n+2)$-gon with k diagonals such that a fixed vertex is incident to d regions.

4. Standard Young tableaux

Theorem 4.1. *There is a bijection between the set of plane Husimi graphs on $n+1$ vertices with k blocks and the set of standard Young tableaux of type $(k, k, 1^{n-k})$.*

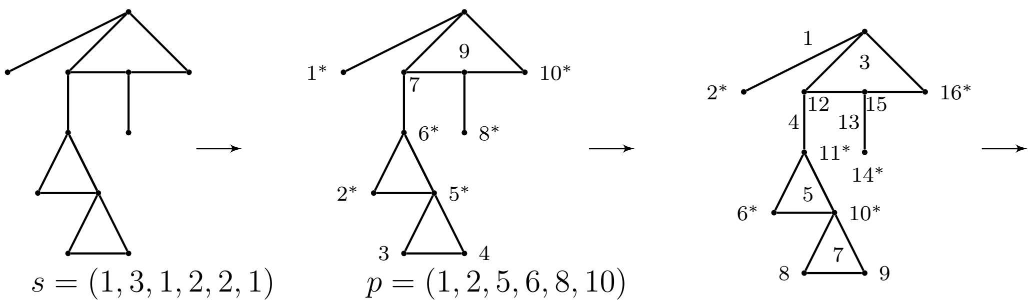
Proof. Consider a plane Husimi H on $n+1$ vertices with k blocks. We obtain a standard Young tableau by the following procedure:

(i) We traverse H by preorder and obtain a sequence $s = (s_1, s_2, \ldots, s_k)$ where s_i is one less than the size of the i^{th} block to be traversed. The sequence has length k since each block contributes an entry in the sequence.
(ii) Get the partial sum sequence $p = (p_1, p_1+1, p_2+1, \ldots, p_{k-1})$ where $p_j = s_1+s_2+\cdots+s_j$ for $j = 1, 2, \ldots, k$.
(iii) We traverse H again but this time, label the vertices of H (except the root) as $1, 2, \ldots, n$ starting at the vertex where we first encounter a leaf. Note that a vertex is given a label if it is encountered for the last time.
(iv) Mark the vertices with labels in the sequence p with an asterisk, $*$. Remove the labels except the asterisks.
(v) We traverse the plane Husimi graph again and record an entry i in the:
   (a) first row of the Young tableau if the i^{th} item to be visited is a block.
   (b) second row, if the i^{th} item to be visited is a vertex with an asterisk.
   (c) first column, if the i^{th} item to be visited is a vertex without an asterisk.

In Figure 7, we get a standard Young tableau from a plane Husimi graph using the procedure described above.

We provide the reverse procedure: Consider a sequence $(1, 2, \ldots, n+k)$. Perform the following procedure given a standard Young tableau:

(i) Change all the entries in the sequence that are in the second row to x_i^* where $i = 1, 2, \ldots, k$ and all the entries of first column (except of row 1 and row 2) to x_j where $j = 1, 2, \ldots, n-k$. Rename the remaining integers as $1, 2, \ldots, k$ in this order.
(ii) Change entry 1 of the sequence to the number of x_i's and x_i^* that are appear before x_2^*.
(iii) Change entry $2 \leq j \leq k-1$ of the sequence to the number of x_i^*'s and x_i's that appear from x_j^* to x_{j+1}^* but not inclusive of x_{j+1}^*.



$s = (1, 3, 1, 2, 2, 1)$ $p = (1, 2, 5, 6, 8, 10)$

1	3	4	5	7	13
2	6	10	11	14	16
8					
9					
12					
15					

FIGURE 7. Obtaining a Young tableau from plane Husimi graph.

(iv) Change entry k of the sequence to the number of x_i's and x_i^*'s that are encountered after x_k^* and inclusive of x_k^*.

(v) Change all the x_j's and x_j^*'s to x.

(vi) Starting at the first entry in the sequence, draw a block size of i if the entry is $i-1$ such that one vertex is identified as the root. Then move to the child in the block that appears on the far left. If the second entry is $j \geq 1$ then attach a block of size $j+1$. Otherwise, if the entry is x then move to the second child in the block. If there is no available vertex to attach a block then move back to the root until you find a vertex to attach a block. Continue until are the entries in the sequence are encountered.

The graph obtained is a plane Husimi graph on $n+1$ vertices with k blocks since each entry greater than 0 contributes a block and x contributes a vertex. An example to explain this procedure is given in Figure 8.

□

5. Valid sequences

Theorem 5.1. *There is a bijection between the set of plane Husimi graphs on $n+1$ vertices with k blocks and the set of valid sequences with k entries greater than 0 and n entries which are equal to -1.*

Proof. Let H be a plane Husimi graph on $n+1$ vertices with k blocks. We obtain a sequence of length $n+k$ as follows: We traverse H using preorder traversal or depth first search by visiting a block before its vertices. We write $i-1$ if a block of size $i > 1$ is visited. Otherwise, record -1 if a vertex visited is a leaf or it is a non-leaf visited for the last time on the way back to the root. We don't record a value for the root. We note the following:

(i) The length of the sequence is $n+k$ since each vertex in the graph (except the root) and each block contributes an entry in the sequence. Let the sequence be $(s_1, s_2, \ldots, s_{n+k})$.

(ii) Either $s_i \geq 1$ or $s_i = -1$.

1	3	4	5	7	13
2	6	10	11	14	16
8					
9					
12					
15					

$$\longrightarrow \quad (1, x_1^*, 2, 3, 4, x_2^*, 5, x_1, x_2, x_3^*, x_4^*, x_3, 6, x_5^*, x_4, x_6^*)$$

$$\longrightarrow \quad (1, x, 3, 1, 2, x, 2, x, x, x, x, x, 1, x, x, x) \quad \longrightarrow$$

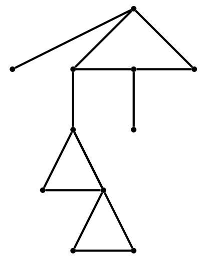


FIGURE 8. Obtaining a plane Husimi graph from a Young tableau.

(iii) Exactly k entries are greater than 0 and exactly n entries are equal to $s_i = -1$.

(iv) The sum $s_1 + s_2 + \cdots + s_{n+k} = 0$ since each vertex (except the root) contributes 1 as we record a contribution from the block and -1 if we record a contribution of the vertex.

(v) The partial sum $p_i = s_1 + s_2 + \cdots + s_i \geq 0$ for all i since a record of -1 is done when a vertex is visited is leaf or is a non-leaf encountered at the last time on the way back to root. This is done after the recording of the size of the block to which the vertex belongs.

The sequence obtained is thus a valid sequence described in Section 1.

We now obtain the reverse procedure: Consider a valid sequence of length $n+k$ with k entries greater than 0 and n entries equal to -1. We obtain a plane Husimi graph by the following procedure. Starting at the first entry in the sequence, draw a block of size i if the entry is $i-1$ such that one vertex is identified as the root. Then move to the child in the block on the far left. If the second entry is $j \geq 1$ then attach a block of size $j+1$. Otherwise, if the entry is -1 then move to the second child in the block. If there is no available vertex to attach a block then move back to the root until you find a vertex to attach a block. The graph obtained is a plane Husimi graph on $n+1$ vertices with k blocks since each entry greater than 0 contributes a block and -1 contributes a vertex. This bijection is described in Figure 9.

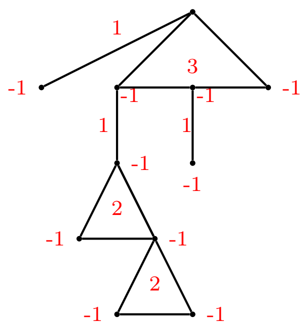
□

6. DEUTSCH PATHS

As stated in Section 1, Deutsch paths are special kinds of lattice paths. We relate them to plane Husimi graphs and valid sequences.

Theorem 6.1. *There is a bijection between the set plane Husimi graphs with $n+1$ vertices with k blocks and the set of Deutsch paths with n down steps and k up steps.*

Proof. Let H be a plane Husimi graph on $n+1$ vertices with k blocks. We obtain a Deutsch path with k up steps and n down steps as follows: We traverse H using preorder traversal. A block is visited before the subtrees attached to its vertices. Draw an up step of length $i-1$ if



$\leftrightarrow (1, -1, 3, 1, 2, -1, 2, -1, -1, -1, -1, -1, 1, -1, -1, -1)$

FIGURE 9. Bijection plane Husimi graph and valid sequences

a block of size $i > 1$ is visited. Otherwise, draw a unit down step if the vertex visited is a leaf or is a non-leaf vertex traversed for the last time on the way back to the root.

For the reverse procedure, start at the origin of the Deutsch path and draw a block of size i if the length of the up step is $i-1$, such that one vertex is identified as the root. Then move to the child in the block that appears on the far left. If the next path is an up step of length $j \geq 1$ then attach a block of size $j+1$. Otherwise, if the next path is a unit down step then move to the second child in the block. If there is no available vertex to attach a block then move back to the root until you find a vertex to attach a block. The graph obtained is a plane Husimi graph on $n+1$ vertices with k blocks since each up step contributes a block and each down step contributes a vertex. The bijection is described in Figure 10.

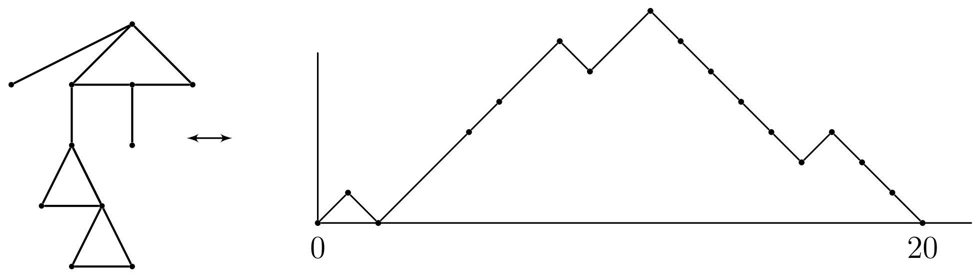


FIGURE 10. Bijection between plane Husimi graphs and Deutsch paths.

□

From the bijection constructed, the degree of the root of plane Husimi graphs corresponds to number of hills in the Deutsch path thus Equation (2) gives the number of Deutsch path with n down steps, k up steps and d hills.

For the rest of this section, we prove the following result:

Theorem 6.2. *There is a bijection between the set of valid sequences of length $n+k$ and the set of Deutsch paths with n down steps and k up steps.*

Proof. Consider a valid sequence of length $n+k$. There are n entries which are equal to -1 and k entries which are greater than 0. Let the sequence be $(s_1, s_2, \ldots, s_{n+k})$. Starting at the beginning of the sequence, for each entry $s_i > 0$, draw up step of length s_i and for each entry $s_i = -1$, draw a unit down step. We obtain a Deutsch path with n down steps and k up steps.

To reverse the procedure, start at the origin of the Deutsch path and record the length of each up step and -1 for each down step. We now show that we obtain a valid sequence of length $n + k$:

(i) The length of the sequence is $n + k$ since each up step (k in number) contributes an entry greater than 0 and each down step (n in number) contributes an entry -1 to the sequence. Let the sequence be $(s_1, s_2, \ldots, s_{n+k})$.

(ii) The sum $s_1 + s_2 + \cdots + s_{n+k} = 0$ since the sum of the lengths of up steps is n and that of the down steps is n. Since each down step is given an entry -1, then the total sum is 0.

(iii) The partial sum $p_i = s_1 + s_2 + \cdots + s_i \geq 0$ for all i since a record of -1 is done when a down step is visited. Since the path does not go below the x-axis, the sum will always be non-negative.

See Figure 11 for an example. □

$$(1, -1, 3, 1, 2, -1, 2, -1, -1, -1, -1, -1, 1, -1, -1, -1) \quad \longleftrightarrow$$

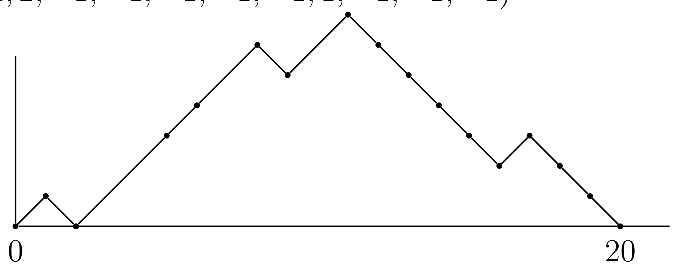


Figure 11. Bijection between valid sequences and Deutsch paths.

7. Restricted lattice paths

In this section, we construct a bijection which relates restricted lattice paths with plane Husimi graphs as well as the bijection between the set of these paths and the set of valid sequences.

Theorem 7.1. *There is a bijection between the set of plane Husimi graphs on $n + 1$ vertices with k blocks and the set of restricted lattice paths with k horizontal steps and n vertical steps.*

Proof. Let H be a plane Husimi graph on $n + 1$ vertices with k blocks. We obtain a restricted lattice path with k horizontal steps and n vertical steps as follows: We traverse H using preorder traversal. Again a block is visited before its vertices. Draw a horizontal step of length $i - 1$ if the block traversed is of size $i > 1$. Otherwise, draw a unit vertical step if the vertex visited is a leaf or is a non-leaf visited for the last time on the way back to the root. The path does not cross the line $y = x$.

For the reverse procedure, start at the origin of the restricted lattice path and draw a block of size i if the length of the horizontal step is $i - 1$, such that one vertex is identified as the root. Then move to the child of the block on the far left. If the next path is a horizontal step of length $j \geq 1$ then attach a block of size $j + 1$. Otherwise, if the next path is a unit vertical step then move to the second child in the block. If there is no available vertex to attach a block then move back towards the root until you find a vertex to attach a block. The graph obtained is a plane Husimi graph on $n + 1$ vertices with k blocks since each horizontal step contributes a block and each vertical step contributes a vertex. Figure 12 is an illustration of the bijection.

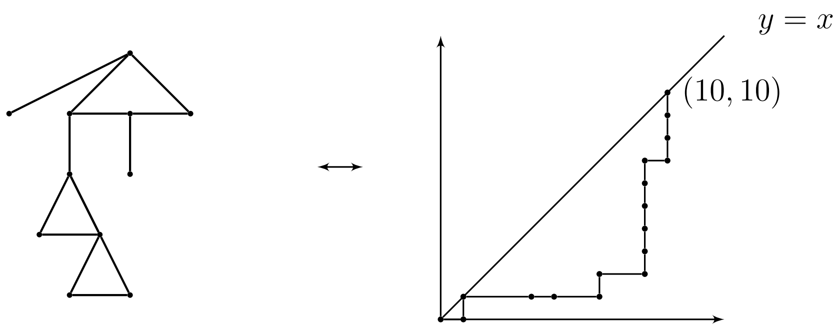


FIGURE 12. Bijection between plane Husimi graphs and restricted lattice paths.

□

We also prove:

Theorem 7.2. *There is a bijection between the set of valid sequences of length $n + k$ and the set of restricted lattice paths with k horizontal steps and n vertical steps.*

Proof. We construct a bijection between the set of valid sequences of length $n+k$ and restricted lattice paths with n vertical steps and k horizontal steps as follows: Consider a valid sequence of length $n + k$. There are n entries which are equal to -1 and k entries which are greater than 0. Let the sequence be $(s_1, s_2, \ldots, s_{n+k})$. Starting at the beginning of the sequence, for each entry $s_i > 0$, draw a horizontal step of length s_i and for each entry $s_i = -1$, draw a unit vertical step. We obtain a restricted path with n vertical steps and k horizontal steps. Since $s_1 + s_2 + \cdots + s_i$ for $i \geq 1$, then the path lie weakly below the line $y = x$.

To reverse the procedure, start at the origin of the restricted lattice path and record the length of each horizontal step and -1 for each vertical step. We show that sequence obtained is a valid sequence of length $n + k$:

(i) The length of the sequence is $n+k$ since each horizontal step (k in number) contributes an entry greater than 0 and each vertical step (n in number) contributes an entry -1 to the sequence. Let the sequence be $(s_1, s_2, \ldots, s_{n+k})$.
(ii) The sum $s_1 + s_2 + \cdots + s_{n+k} = 0$ since the sum of the lengths of horizontal steps is n and that of the vertical steps is n. Since each vertical step is recorded as -1 in the sequence, then the total sum is 0.
(iii) The partial sum $p_i = s_1 + s_2 + \cdots + s_i \geq 0$ for all i since a record of -1 is done when a vertical step is visited. Since the path stays weakly below the line $y = x$, the sum will always be non-negative.

See Figure 13 for an example.

$$(1, -1, 3, 1, 2, -1, 2, -1, -1, -1, -1, -1, 1, -1, -1) \longleftrightarrow$$

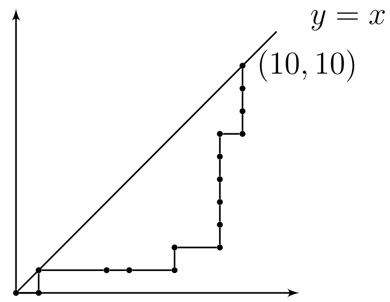


FIGURE 13. Bijection between valid sequences and restricted lattice paths.

□

REFERENCES

[1] F. Bergeron, G. Labelle, P. Leroux, Combinatorial species and tree-like structures, Encyclopaedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1997.

[2] M. Bóna, M. Bousquet, G. Labelle, P. Leroux, Enumeration of m-ary cacti, Adv. Appl. Math. 24 (2000), 22-56.

[3] P. Flajolet, R. Sedgewick, Analytic combinatorics, Cambridge University Press, Cambridge, 2009.

[4] I.P. Goulden, D.M. Jackson, The combinatorial relationship between trees, cacti and certain connection coefficients for the symmetric group, Eur. J. Comb. 13 (1992), 357-365.

[5] F. Harary, G.E. Uhlenbeck, On the number of Husimi trees, Proc. Nat. Acad. Sci. 39 (1953), 315-322.

[6] K. Humphreys, A history and a survey of lattice path enumeration, J. Stat. Plan. Inference, 140 (2010), 2237-2254.

[7] K. Husimi, Note on Mayers' theory of cluster integrals, J. Chem. Phys. 18 (1950), 682-684.

[8] C. Krattenthaler, Lattice path enumeration, arXiv preprint, arXiv:1503.05930, 2015.

[9] P. Leroux, Enumerative problems inspired by Mayer's theory of cluster integrals, Electron. J. Comb. 11(R32) (2004), 1.

[10] J.E. Mayer, Equilibrium Statistical Mechanics: The international encyclopedia of physical chemistry and chemical physics, Pergamon Press, Oxford, 1968.

[11] I.O. Okoth, Combinatorics of oriented trees and tree like structures, PhD thesis, Stellenbosch University, 2015.

[12] I.O. Okoth, On noncrossing and plane tree-like structures, Commun. Adv. Math. Sci. 4 (2021), 89-99.

[13] C. A. Onyango, I. O. Okoth, D. M. Kasyoki, Enumeration of plane and d-ary tree-like structures, Ann. Math. Comput. Sci. 17 (2023), 10-25.

[14] H. Prodinger, Deutsch paths and their enumeration, Open J. Discr. Appl. Math. 4 (2021), 12-18.

[15] N.J.A. Sloane, The on-line encyclopedia of integer sequences (OEIS), Available online at `http://oeis.org`.

[16] C. Springer, Factorizations, trees, and cacti, in: Proceedings of the Eighth International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC), University of Minnesota, 1996, 427-438.

[17] R.P. Stanley, Enumerative Combinatorics, Cambridge University Press, Cambridge, 1999.

[18] R.P. Stanley, Polygon dissections and standard Young tableaux, J. Comb. Theory Ser. A. 76 (1996), 175-177.