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EXISTENCE RESULTS FOR FRACTIONAL FISHER-KOLMOGOROFF STEADY STATE PROBLEM

SALAH A. KHAFAGY[1,*], S.H. RASOULI[2] AND HASSAN M. SERAG[1]

Abstract. In the present paper, we study the existence results of positive weak solution for the fractional Fisher-Kolmogoroff steady state problem. We establish a condition under which the system under consideration has a positive weak solution. Also, we consider the case in which there is no positive weak solution. We use the method of sub-supersolutions to establish our existence results.

1. Introduction

Many authors are interested in the study of existence results of weak solutions for linear [31], semilinear [15] and nonlinear systems [19]- [24] by using different methods. In [18, 20], the sub-super solutions method is used to give necessary conditions for the existence of positive weak solutions. Liu [28] has obtained existence and uniqueness of solution to semilinear fractional elliptic equation by using the Stampacchia's theorem. Mountain Pass Theorems for non-local elliptic operators are used in [33]. Also, the Browder theorem approach was used to demonstrate the existence and uniqueness of the positive weak solution for a quasilinear weighted (p, q)-Laplacian system, see [26].

In [2], the authors have been studied the diffusive logistic equation

$$\begin{cases} -\Delta u = au - bu^2 & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \tag{1}$$

where a and b are real numbers with $b > 0$ (see [16, 29]). They proved that $u_1 = 1$ is a supersolution and when $a > \lambda_1$, $u_2 = \epsilon\phi_1$ is a subsolution for $\epsilon > 0$ small, where λ_1 is the first eigenvalue and ϕ_1 is the corresponding eigenfunction of the Laplacian operator Δ.

In this paper, we study the existence and nonexistence results of positive weak solution for the fractional Fisher-Kolmogoroff steady state problem

$$\begin{cases} (-\Delta)^s u = \lambda u(1-u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{2}$$

[1]Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City (11884), Cairo, Egypt

[2]Department of Mathematics, Faculty of Basic Sciences, Babol Noshirnani University of Technology, Babol, Iran

[*]Corresponding author

E-mail address: salahabdelnaby.211@azhar.edu.eg, serraghm@yahoo.com, s.h.rasouli@nit.ac.ir.
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where Ω is a bounded domain in $\mathbb{R}^n$ with $n > 2s$, λ is a positive parameter and $(-\Delta)^s$ is the fractional Laplacian operator of order s with $s \in (0,1)$ defined by [17]

$$(-\Delta)^s u(x) = C(n,s)\ P.V. \int\limits_{R^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy, \tag{3}$$

where $P.V.$ stands for Cauchy principal value and $C(n,s)$ is a normalization constant, with precise value $C(n,s) = 2^{2s} s\Gamma((n+2s)/2)/(\pi^{\frac{n}{2}}\Gamma(n-1))$. We establish a conditions under which system (2) has a positive weak solution. Also, we consider the case in which there is no positive weak solution. We will use the method of sub–supersolutions to establish our results (see e.g. [10] and [11]). Due to the appearance of fractional Laplacian operator in (2); the extensions are challenging and nontrivial. There are very few and sparse works till now pledging with the existence of solution for fractional Laplacian systems by using the sub–supersolutions method.

The Fisher-Kolmogoroff model is one of the most famous fundamental models in mathematical biology and ecology [25, 30], which was proposed, on the one hand, by Fisher [13] who interpreted the spread of an advantageous gene in a population; and on the other hand, by Kolmogorov et al [27] who additionally obtained the basic analytical results for this equation.. The Fisher-Kolmogoroff problem is givin by

$$\frac{\partial u}{\partial t} = D \triangle u + \rho u(1-u), \tag{4}$$

where D, ρ are positive constants. It describes a reaction-diffusion process describing the behavior of the concentration $u(x,t)$ of molecules of type x diffusing with diffusion constant D and reacting according to $Y + X \leftrightarrow 2x$, ρ represents the reaction rate coefficient, and the concentration of molecules of type Y is assumed constant [4].

Recently, systems involving fractional Laplace operators has been attracted the intersts of many scientists. This is naturally due to such operators are now experiencing impressive applications in different fields. In particular, they appear in many subjects of science such as finance [8], population dynamics [35], probability [3, 6], phase transitions [14, 34], material science [1], optimization [12], water waves [9].

This paper covers the following sections: In section 2, we provide a suitable functional framework for problem (2). Section 3 is devoted to derive the existence of positive weak solution for system (2) via sub–supersolutions method. Also, we consider the nonexistence result.

2. Technical results

In this section, we interpret appropriate function spaces which are imperative for our analysis. We define the fractional-order Sobolev space by (see [17, 37]).

$$W^{s.2}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy < \infty\}, \tag{5}$$

equipped with the norm

$$\|u\|_{W^{s.2}(\mathbb{R}^n)} = (\int\limits_{\mathbb{R}^n} |u|^2 dx + \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy)^{1/2}. \tag{6}$$

Let

$$W_0^{s.2}(\mathbb{R}^n) = \{u \in W^{s.2}(\mathbb{R}^n) : u = 0 \text{ a.e. } u \in \mathbb{R}^n \backslash \Omega\}, \tag{7}$$

be a closed linear subspace of $W^{s.2}(\mathbb{R}^n)$, and its norm is given by

$$\|u\|_{W_0^{s.2}(\mathbb{R}^n)} = (\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy)^{1/2}, \tag{8}$$

which is equivalent to the norm given by (6).

Now, we introduce some technical results [5,36] concerning the fractional Laplacian eigenvalue problem

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega. \end{cases} \tag{9}$$

Lemma 2.1. *There exists the first eigenvalue $\lambda_1 > 0$ and precisely one corresponding eigenfunction $\phi_1 > 0$ a.e. in Ω of the eigenvalue problem (9). Moreover, it is characterized by*

$$\lambda_1 = \inf_{u \in W_0^{s,2}(\mathbb{R}^n)\backslash\{0\}} \frac{\int\limits_{\mathbb{R}^n} |(-\Delta)^{s/2} u|^2 dx}{\int\limits_{\Omega} u^2 dx}. \tag{10}$$

3. Existence and nonexistence results

In this section, existence and nonexistence results of positive weak solution for system (2) are derived. We shall prove the existence by constructing a positive weak subsolution $\underline{u} \in W_0^{s,2}(\mathbb{R}^n)$ and a positive supersolution $\overline{u} \in W_0^{s,2}(\mathbb{R}^n)$ of (2) such that $\underline{u} \leq \overline{u}$. That is, $\underline{u}$ satisfies $\underline{u} = 0$ in $\mathbb{R}^n \backslash \Omega$ and

$$\int\limits_{\mathbb{R}^n} (-\Delta)^{s/2} \underline{u} (-\Delta)^{s/2} \zeta dx \leq \lambda \int\limits_{\Omega} \underline{u}(1-\underline{u})\zeta dx, \tag{11}$$

and $\overline{u}$ satisfies $\overline{u} = 0$ in $\mathbb{R}^n \backslash \Omega$ and

$$\int\limits_{\mathbb{R}^n} (-\Delta)^{s/2} \overline{u} (-\Delta)^{s/2} \zeta dx \geq \lambda \int\limits_{\Omega} \overline{u}(1-\overline{u})\zeta dx, \tag{12}$$

for all test function $\zeta \in W_0^{s,2}(\Omega)$ with $\zeta \geq 0$.

Then the following result holds:

Lemma 3.1. *(see [7, 21]) Suppose there exist a weak subsolution $\underline{u}$ and a weak supersolution $\overline{u}$ of (2) such that $\underline{u} \leq \overline{u}$; then there exists a weak solution u of (2) such that $\underline{u} \leq u \leq \overline{u}$.*

Definition 3.2. *A function $u \in W_0^{s,2}(\Omega)$ is said to be a weak solution of (2) if for every $\zeta \in W_0^{s,2}(\Omega)$, we have*

$$\int\limits_{\mathbb{R}^n} (-\Delta)^{s/2} u (-\Delta)^{s/2} \zeta dx = \lambda \int\limits_{\Omega} u(1-u)\zeta dx. \tag{13}$$

Our main results of this paper are the following theorems.

Theorem 3.3. *If $\lambda \leq \lambda_1$, then system (2) has no positive weak solution.*

Proof. Suppose $u \in W_0^{s,2}(\Omega)$ be a positive weak solution of (2). We shall prove the theorem by arriving at a contradiction. Multiplying (2) by u, we have

$$\int_{\mathbb{R}^n} |(-\Delta)^{s/2}u|^2 dx = \lambda \int_{\Omega} u(1-u)u dx. \tag{14}$$

But from (10), we obtain

$$\lambda_1 \int_{\Omega} u^2 dx \leq \int_{\mathbb{R}^n} |(-\Delta)^{s/2}u|^2 dx. \tag{15}$$

Combining (14) and (15), we have

$$\lambda_1 \int_{\Omega} u^2 dx \leq \lambda \int_{\Omega} u(1-u)u dx,$$

and so

$$(\lambda_1 - \lambda) \int_{\Omega} u^2 dx \leq 0.$$

Hence $\lambda_1 \leq \lambda$. The proof complete.

Theorem 3.4. *If $\lambda > \lambda_1$, system (2) has a positive weak solution.*

Proof. Suppose $\lambda > \lambda_1$ be fixed. Let λ_1 be the first eigenvalue of the fractional eigenvalue problem (5) and ϕ_1 the corresponding eigenfunction satisfying $\phi_1 > 0$ in Ω with $\|\phi_1\|_\infty = 1$. Then we have

$$\begin{cases} (-\Delta)^s \phi_1 = \lambda_1 \phi_1 & \text{in } \Omega, \\ \phi_1 = 0 & \text{in } \mathbb{R}^n \backslash \Omega. \end{cases} \tag{16}$$

Let $e(x)$ be the positive weak solution of [32]

$$\begin{cases} (-\Delta)^s e(x) = 1 & \text{in } \Omega, \\ e(x) = 0 & \text{in } \mathbb{R}^n \backslash \Omega. \end{cases} \tag{17}$$

We denote $\overline{u} = Ae(x)$, where the constant $A > 0$ is sufficiently large and to be chosen later. We shall verify that $\overline{u}$ is the weak supersolution of (2). To do this, let $\zeta \in W_0^{s,2}(\Omega)$ with $\zeta \geq 0$. Then, we have

$$\int_{\Omega} (-\Delta)^s \overline{u} \zeta dx = A \int_{\Omega} \zeta dx.$$

A simple calculations show that $Ae - (Ae)^2$ bounded above by $\frac{1}{4}$. So, we choose A large enough such that $A \geq \lambda/4$. Then, we have

$$\begin{aligned} \int_{\Omega} (-\Delta)^s \overline{u} \zeta dx &\geq \frac{\lambda}{4} \int_{\Omega} \zeta dx \\ &\geq \lambda \int_{\Omega} (Ae - (Ae)^2) \zeta dx \\ &= \lambda \int_{\Omega} \overline{u}(1-\overline{u}) \zeta dx. \end{aligned}$$

So, equation (12) is satisfy and $\overline{u}$ is the weak supersolution of (2).

Next, we construct a weak subsolution $\underline{u}$ of system (2). Let $\underline{u} = \frac{1-\mu}{2\|\phi_1\|_\infty}\phi_1$ where $\mu = \sqrt{\frac{\lambda_1}{\lambda}} \in (0,1)$. A simple calculations show that $\underline{u} \le \frac{1}{2}$ and $\lambda_1 < \lambda[1 - (\frac{1-\mu}{2})]$. Now, we are in a position enaples us to verify that $\underline{u}$ is the positive weak subsolution of (2).

A calculation shows that

$$\begin{aligned}
\int_\Omega (-\Delta)^s \underline{u}\zeta dx &= \frac{1-\mu}{2\|\phi_1\|_\infty}\int_\Omega (-\Delta)^s \phi_1 \zeta dx \\
&= \frac{1-\mu}{2\|\phi_1\|_\infty}\int_\Omega \lambda_1\phi_1\zeta dx \\
&\le \frac{1-\mu}{2\|\phi_1\|_\infty}\int_\Omega \lambda[1-(\frac{1-\mu}{2})]\phi_1\zeta dx \\
&\le \lambda\frac{1-\mu}{2\|\phi_1\|_\infty}\int_\Omega [1-(\frac{1-\mu}{2\|\phi_1\|_\infty}\phi_1)]\phi_1\zeta dx \\
&= \lambda\int_\Omega \frac{1-\mu}{2\|\phi_1\|_\infty}\phi_1[1-\frac{1-\mu}{2\|\phi_1\|_\infty}\phi_1]\zeta dx \\
&= \lambda\int_\Omega \underline{u}(1-\underline{u})\zeta dx.
\end{aligned}$$

Hence $\underline{u}$ is a weak subsolution for system (2). Moreover, since $\underline{u} \le 1/2$, we can choose A large such that $\underline{u} \le \overline{u}$. Thus, there exists a weak solution u of (2) with $\underline{u} \le u \le \overline{u}$. This completes the proof of Theorem 3.4.

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